HIERARCHIES OF FORCING AXIOMS, THE CONTINUUM HYPOTHESIS AND SQUARE PRINCIPLES

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Abstract. I analyze the hierarchies of the bounded and the weak bounded forcing axioms, with a focus on their versions for the class of subcomplete forcings, in terms of implications and consistency strengths. For the weak hierarchy, I provide level-by-level equiconsistencies with an appropriate hierarchy of partially remarkable cardinals. I also show that the subcomplete forcing axiom implies Larson's ordinal reflection principle at ω_2, and that its effect on the failure of weak squares is very similar to that of Martin's Maximum.

1. Introduction

The motivation for this work is the wish to explore forcing axioms for subcomplete forcings in greater detail. Subcomplete forcing was introduced by Jensen in [18]. It is a class of forcings that do not add reals, preserve stationary subsets of ω_1, but may change cofinalities to be countable, for example. Most importantly, subcomplete forcing can be iterated, using revised countable support. Examples of subcomplete forcings include all countably closed forcings, Namba forcing (assuming CH), Příkrý forcing (see [19] for these facts), generalized Příkrý forcing (see [24]), and the Magidor forcing to collapse the cofinality of a measurable cardinal of sufficiently high Mitchell order to ω_1 (see [10]).

Since subcomplete forcings can be iterated, they naturally come with a forcing axiom, the subcomplete forcing axiom, SCFA, formulated in the same way as Martin's axiom or the proper forcing axiom PFA. The overlap between proper forcings and subcomplete forcings is minimal, though. Proper forcings can add reals, which subcomplete forcings cannot. Subcomplete forcings can change cofinalities of regular cardinals to ω, or change the cofinality of a regular cardinal to ω_1 without collapsing cardinals, which proper forcings cannot (see [13]). The class of proper forcings contains all ccc forcings, while no nontrivial subcomplete forcing is ccc (see [24]). It was shown by Jensen in [17] that the subcomplete forcing axiom can be forced over a model with a supercompact cardinal, using essentially the same construction as for PFA. Since subcomplete forcing does not add reals, however, the resulting model will satisfy the continuum hypothesis. So unlike PFA, which implies that $2^\omega = \omega_2$, SCFA is compatible with CH, and even with $\diamondsuit$, since subcomplete forcing preserves $\diamondsuit$, see [17, §4, Lemma 4]. Interestingly, though, Jensen showed

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that SCFA implies the failure of $\Box_\kappa$, for every uncountable cardinal κ, and that it has other consequences similar to PFA, or Martin's Maximum, like the singular cardinals hypothesis.

I originally wanted to explore the extent of the failure of weak square principles under the subcomplete forcing axiom, and to find other uses for the arguments establishing these failures. It soon turned out that these arguments make it possible to determine the consistency strength of the bounded subcomplete forcing axioms BSCFA($\le\kappa$), saying that given a collection of ω_1 many maximal antichains, each having size at most κ, of a subcomplete complete Boolean algebra, there is a filter meeting them all, in the cases $\kappa = \omega_1$ or $\kappa = \omega_2$. The consistency strengths of these are exactly the same for the bounded subcomplete and the bounded proper forcing axioms (a reflecting cardinal for $\kappa = \omega_1$, and a +1-reflecting, or strongly unfoldable cardinal for $\kappa = \omega_2$). The weak proper forcing axiom, wPFA, of [4] takes a characterization of the proper forcing axiom that guarantees the existence of certain elementary embeddings (see Fact 3.8) and weakens this to the existence of such an embedding in some forcing extension, that is, a generic embedding. It was shown that the consistency strength of wPFA is a remarkable cardinal, and it turned out that this is the case for the corresponding weak subcomplete forcing axiom, wSCFA, as well. A remarkable cardinal can be viewed as a "virtual" version of a supercompact cardinal, if one takes the characterization of supercompactness given in [22] and replaces the embeddings in the characterization by generic ones. Thus, the virtual version of PFA has the consistency strength of a virtual supercompact cardinal, and the same is true of the virtual version of SCFA.

The weak subcomplete forcing axiom is not a bounded forcing axiom, and I realized that there is a hierarchy of weak forcing axioms leading up to it, and since the same can be done with other classes of forcings, I did not limit the investigation to this class. At stages ω_1 and ω_2, the weak hierarchy and the usual hierarchy coincide, but then they diverge, and it turns out that there is a hierarchy of partially remarkable cardinals that precisely capture the levels of the weak bounded forcing axioms beyond ω_2.

The paper is organized as follows. In section 2, I give a little bit of background on how Jensen obtained the failure of $\Box_\kappa$ for every uncountable cardinal κ from the assumption of SCFA, and I show that his arguments actually show an amount of stationary reflection that implies failures of weak square principles almost as strong as those known to be implied by Martin's Maximum. Most of these results come from combining known connections between stationary reflection and the failure of weak square, established by work of Cummings, Foreman and Magidor. There is a difference at ω_1, because SCFA is consistent with CH and even $\diamondsuit$, and it is unclear whether SCFA implies the failure of weak square at cardinals of cofinality ω. Theorem 2.11 summarizes the situation. There are two potential routes to clarifying the situation at cofinality ω. One might try to show directly that SCFA implies that for singular λ, there is no good scale of length λ^+, using a variant of Namba forcing consisting of trees that are stationarily splitting (as was done in the context of Martin's Maximum in [7]). One would have to find such a variant that is provably subcomplete, which I leave for a future project. The other route would be by proving a principle of stationary reflection strong enough to derive the desired failure of weak square, such as a principle known as $\mathsf{Refl}^*([\lambda^+]^\omega)$. This principle is known to follow from the "plus" versions of forcing axioms, even from

$\mathsf{MA}^+(\sigma\text{-closed})$, and it follows from Martin's Maximum, but at this point, I cannot see how to get it from SCFA. The known proofs using Martin's Maximum use the saturation of the nonstationary ideal, which is not available under SCFA. In search of strong reflection principles, though, I prove that SCFA implies a form of simultaneous stationary reflection, called OSR_{ω_2}, that Larson showed to be a consequence of Martin's Maximum, see Theorem 2.25. I also answer a question of Tsaprounis in this section.

In section 3, I study the hierarchy of bounded (subcomplete) forcing axioms. I show that BSCFA, at the bottom, is equiconsistent with a reflecting cardinal (Theorem 3.6), and that the next level, $\mathsf{BSCFA}(\leq\omega_2)$, is equiconsistent with a +1-reflecting, or strongly unfoldable cardinal (Theorem 3.11). I also explore the effects of the bounded forcing axioms on the failure of square, via stationary reflection, see Theorem 3.13.

Section 4 introduces the weak subcomplete forcing axiom wSCFA, and lifts the argument from the previous section to show that wSCFA is equiconsistent with a remarkable cardinal. This is Theorem 4.5. Subsequently, I introduce the hierarchy of weak bounded forcing axioms, explore their relationships to the bounded forcing axioms, introduce the remarkably reflecting cardinals, and show that they measure the consistency strengths of the axioms in this hierarchy. The main result in this section is Theorem 4.14.

2. The subcomplete forcing axiom

Definition 2.1. Let Γ be a class of forcings and κ a cardinal. Then Martin's Axiom for Γ, or the Forcing Axiom for Γ, denoted $\mathsf{FA}(\Gamma)$, or $\mathsf{MA}(\Gamma)$, says that for any forcing $\mathbb{P}$ in Γ and any collection $\mathcal{A}$ of ω_1 many maximal antichains in $\mathbb{P}$, there is an $\mathcal{A}$-generic filter F, that is, a filter in $\mathbb{P}$ that intersects each member of $\mathcal{A}$. If Γ is the class of all subcomplete forcings, then I write SCFA for $\mathsf{FA}(\Gamma)$. If Γ is the class of c.c.c., semiproper, proper, or stationary set preserving forcings, then $\mathsf{FA}(\Gamma)$ is known as MA, SPFA, PFA and MM, respectively.

Even though I will not use the definition of subcompleteness until later, I will give it now, for the reader's orientation. For a good introduction to this concept, I refer to [19]. The definition of subcompleteness is due to Jensen.

Definition 2.2. A transitive set N (usually a model of ZFC^-) is *full* if there is an ordinal γ such that $L_\gamma(N) \models \mathsf{ZFC}^-$ and N is regular in $L_\gamma(N)$, meaning that if $x \in N$, $f \in L_\gamma(N)$ and $f : x \longrightarrow N$, then $\mathrm{ran}(f) \in N$.

Definition 2.3. Let $\mathbb{P}$ be a poset, $\delta(\mathbb{P})$ the minimal cardinality of a dense subset of $\mathbb{P}$. and θ a cardinal. Then θ verifies the subcompleteness of $\mathbb{P}$ if $\mathbb{P} \in H_\theta$, and if for any ZFC^- model $N = L_\tau^A$ with $\theta < \tau$ and $H_\theta \subseteq N$, any $\sigma : \bar{N} \prec N$ such that $\bar{N}$ is countable, transitive and full and such that $\mathbb{P}, \theta \in \mathrm{ran}(\sigma)$, any $\bar{G} \subseteq \bar{\mathbb{P}}$ which is $\bar{\mathbb{P}}$-generic over $\bar{N}$, and any $s \in \mathrm{ran}(\sigma)$, the following holds. Letting $\sigma(\bar{s}, \bar{\theta}, \bar{\mathbb{P}}) = s, \theta, \mathbb{P}$, there is a condition $p \in \mathbb{P}$ such that whenever $G \subseteq \mathbb{P}$ is $\mathbb{P}$-generic over V with $p \in G$, there is in $\mathrm{V}[G]$ a σ' such that

(1) $\sigma' : \bar{N} \prec N$,
(2) $\sigma'(\bar{s}, \bar{\theta}, \bar{\mathbb{P}}) = s, \theta, \mathbb{P}$,
(3) $(\sigma')``\bar{G} \subseteq G$,
(4) $\mathrm{Hull}^N(\delta(\mathbb{P}) \cup \mathrm{ran}(\sigma')) = \mathrm{Hull}^N(\delta(\mathbb{P}) \cup \mathrm{ran}(\sigma))$.

$\mathbb{P}$ is subcomplete if all sufficiently large cardinals verify the subcompleteness of $\mathbb{P}$.

There is a typo in the definition of subcompleteness as stated in [19, p. 114] ($C^N_\delta(X)$ should be defined as the hull of $X \cup \delta$ in N, not of $X \cup \{\delta\}$; this is fixed in the above definition, as in [17, §3, p. 2ff.]).

Many arguments in the present paper revolve around the combinatorial principle $\square_\kappa$ introduced by Jensen in his seminal work [16]. Many variations of this principle were subsequently devised, let me recall the ones needed in here.

Definition 2.4. Let κ be a cardinal. A $\square_\kappa$-sequence is a sequence

$$\langle C_\alpha \mid \kappa < \alpha < \kappa^+,\ \alpha \text{ limit}\rangle$$

such that each C_α is club in α, $\mathrm{otp}(C_\alpha) \le \kappa$ and for each β that is a limit point of C_α, $C_\beta = C_\alpha \cap \beta$. $\square_\kappa$ is the statement that there is a $\square_\kappa$-sequence.

If λ is also a cardinal, then a $\square_{\kappa,\lambda}$-sequence is a sequence

$$\langle \mathcal{C}_\alpha \mid \kappa < \alpha < \kappa^+,\ \alpha \text{ limit}\rangle$$

such that each $\mathcal{C}_\alpha$ has size at most λ, and each $C \in \mathcal{C}_\alpha$ is club in α, has order-type at most κ, and satisfies the coherency condition that if β is a limit point of C, then $C \cap \beta \in \mathcal{C}_\beta$. Again, $\square_{\kappa,\lambda}$ is the assertion that there is a $\square_{\kappa,\lambda}$-sequence. $\square_{\kappa,\kappa}$ is known as weak square, denoted by $\square^*_\kappa$. $\square_{\kappa,<\lambda}$ is defined like $\square_{\kappa,\lambda}$, except that each $\mathcal{C}_\alpha$ is required to have size less than λ.

In [17], Jensen showed that SCFA implies the failure of $\square_\kappa$, for every cardinal κ. He derives this conclusion from the fact SCFA implies Friedman's property.

Definition 2.5. Let $\kappa > \omega_1$ be a regular cardinal. Friedman's property at κ, FP_κ, states that if S is a stationary subset of κ consisting of ordinals of countable cofinality, then there is a closed set $C \subseteq S$ of order type ω_1.

Friedman's property was known to be a consequence of MM ([8]). It obviously implies an instance of stationary reflection: every stationary subset of $\kappa \cap \mathsf{cof}(\omega)$ reflects to an ordinal of cofinality ω_1. This stationary reflection, in turn, implies a failure of a weak square principle. The following is a special case of [6, Theorem 2].

Lemma 2.6. *Assume FP_κ, and suppose $\kappa = \kappa^\omega$. Then $\square_{\kappa,\omega}$ fails.*

Proof. Suppose otherwise. Let $\langle \mathcal{C}_\alpha \mid \alpha < \kappa^+,\ \alpha \text{ limit}\rangle$ be a $\square_{\kappa,\omega}$-sequence. Let

$$S_0 = \{\alpha < \kappa^+ \mid \kappa < \alpha \text{ and } \mathrm{cf}(\alpha) = \omega\}$$

For $\alpha \in S_0$, let $F(\alpha) = \{\mathrm{otp}(C) \mid C \in \mathcal{C}_\alpha\}$. Since every $C \in \mathcal{C}_\alpha$ has order type at most κ, there are only $\kappa^\omega = \kappa$ many possible values for $F(\alpha)$. So there is a stationary set $S_1 \subseteq S_0$ on which F is constant, say the constant set of order types is O.

Now, using Friedman's property, let $C \subseteq S_1$ be closed, with $\mathrm{otp}(C) = \omega_1$. Let $\gamma = \sup C$. Let $E \in \mathcal{C}_\gamma$, and let $D = C \cap E$. Since $\mathrm{cf}(\gamma) = \omega_1$, D is club in γ. Now, for every $\alpha < \gamma$ that's a limit point of D, it follows that it is also a limit point of E, and so, $E \cap \alpha \in \mathcal{C}_\alpha$. But further, since $\alpha \in S_1$, it follows that $\mathrm{otp}(E \cap \alpha) \in O$. This is a contradiction, since there are ω_1 many possibilities for α, and they give different values for $\mathrm{otp}(E \cap \alpha)$, while there are only ω many ordinals in O. $\square$

Since SCFA implies that $\kappa^{\omega_1} = \kappa$ for regular $\kappa > \omega_1$, this lemma shows that SCFA implies the failure of $\square_{\kappa,\omega}$ for every such κ. But in fact, Jensen showed a stronger version of Friedman's property to be a consequence of SCFA. I will write

$\mathsf{cof}(\omega)$ for the class of all ordinals of countable cofinality. In the following theorem, a function is *normal* if it is strictly increasing and continuous.

Theorem 2.7 ([17, §3, p. 13, Lemma 9])**.** *Assume* SCFA*. Let $\tau > \omega_1$ be a regular cardinal. Then the following principle holds, let's call it the Strong Friedman Poperty at τ (SFP_τ):*

Let $\langle A_i \mid i < \omega_1\rangle$ be a sequence of stationary subsets of $\tau \cap \mathsf{cof}(\omega)$. Let $\langle D_i \mid i < \omega_1\rangle$ be a partition of ω_1 into stationary sets. Then there is a normal function $f : \omega_1 \longrightarrow \tau$ such that for every $i < \omega_1$, $f{\restriction}D_i : D_i \longrightarrow A_i$.

Observation 2.8. *Let $\tau > \omega_1$ be a regular cardinal. Then SFP_τ implies the following version of simultaneous stationary reflection: if $\langle A_i \mid i < \omega_1\rangle$ is a sequence of stationary subsets of $\tau \cap \mathsf{cof}(\omega)$, then the set of $\alpha < \tau$ such that $\mathrm{cf}(\alpha) = \omega_1$ and for all $i < \omega_1$, $A_i \cap \alpha$ is stationary in α, is stationary in τ.*

Proof. Let τ and $\vec{A}$ be as stated. Let $C \subseteq \tau$ be club. By renumbering the $\vec{A}$ sequence and adding an additional stationary subset at the beginning, one may assume that $A_0 = C \cap \mathsf{cof}(\omega)$. Let $\langle D_i \mid i < \omega_1\rangle$ be a partition of ω_1 into stationary sets. By Theorem 2.7, let $f : \omega_1 \longrightarrow \tau$ be a normal function such that for all $i < \omega_1$, $f{\restriction}D_i : D_i \longrightarrow A_i$. Let $\alpha = \sup f``\omega_1$. Then $\mathrm{cf}(\alpha) = \omega_1$, and for each $i < \omega_1$, $A_i$ reflects to $\alpha$: if $D \subseteq \alpha$ is club, then $\bar{D} = f^{-1}``D$ is club in ω_1, hence there is a $\delta \in D_i \cap \bar{D}$, which means that $f(\delta) \in A_i \cap D$. In particular, for $i = 0$, it follows that α is a limit point of A_0, and hence of C, so $\alpha \in C$. So the set of points of cofinality ω_1 to which $\vec{A}$ reflects simultaneously is stationary in τ. □

It turns out that simultaneous stationary reflection has strong consequences in terms of the failure of weak square principles, and the arguments used draw on PCF theory. Fortunately, many results from [7], stated there in the context of MM, are general enough to carry over to the context of SCFA.

Lemma 2.9 ([7], Lemma 2.1)**.** *If κ is singular and $\square_{\kappa,\mu}$ holds for some $\mu < \kappa$, then every stationary subset of κ^+ has a collection of $\mathrm{cf}(\kappa)$ many stationary subsets which do no reflect simultaneously at any point of uncountable cofinality.*

This lemma, together with Observation 2.8 and Theorem 2.7, shows that if SCFA holds and κ is singular with $\mathrm{cf}(\kappa) \le \omega_1$, then $\square_{\kappa,\mu}$ fails for every $\mu < \kappa$.

Lemma 2.10 ([7], Lemma 2.2)**.** *If κ is an uncountable cardinal and $\square_{\kappa,\mu}$ holds for some $\mu < \mathrm{cf}(\kappa)$, then every stationary subset of κ^+ has a stationary subset which does reflect at any point of uncountable cofinality.*

This lemma shows that SCFA implies that for every uncountable cardinal κ and every $\mu < \mathrm{cf}(\kappa)$, $\square_{\kappa,\mu}$ fails. Note that under CH, $\square^*_{\omega_1}$ holds, because CH implies the existence of a special ω_2-Aronszajn tree, and the existence of a special κ^+-Aronszajn tree is equivalent to $\square^*_\kappa$ (to my knowledge, these facts are due to Jensen; proofs can be found in [23, Thm. 3.1, 3.2]). So, to summarize, we have the following information about the extent of $\square$ principles under SCFA.

Theorem 2.11. *Assume* SCFA*, and let λ be an uncountable cardinal.*

1. *If $\mathrm{cf}(\lambda) \le \omega_1$, then $\square_{\lambda,\mu}$ fails, for every $\mu < \lambda$.*
2. *If $\mathrm{cf}(\lambda) \ge \omega_2$, then $\square_{\lambda,\mu}$ fails for every $\mu < \mathrm{cf}(\lambda)$.*
3. *If* CH *holds, then $\square^*_{\omega_1}$ holds.*

So [7, Theorem 1.2] draws a stronger conclusion about the failure of weak square in the ω-cofinal case from MM, namely, MM implies the failure of $\square^*_\kappa$ for all cardinals κ of countable cofinality, while we so far only know that SCFA implies the failure of $\square_{\kappa,\lambda}$ for all $\lambda < \kappa$ in this situation. Another difference to the MM situation is that MM implies that $\square^*_{\omega_1}$ fails. This is because MM, and even PFA, implies that there are no ω_2-Aronszajn trees (this was shown by Baumgartner, see [29, Thm. 7.7]).

The forcing construction of [7] shows that SCFA + CH has no stronger consequences in terms of the failure of weak square principles at cardinals of uncountable cofinality. Starting in a model with a supercompact cardinal κ, one first forces to make it indestructible, using Laver's forcing. Then, one does the preparatory forcing described in section 3 of that paper, to maximize the extent of partial square, resulting in a model where κ is still supercompact, GCH holds at and above κ, and for singular $\lambda > \kappa$, if $\mathrm{cf}(\lambda) \geq \kappa$, then $\square_{\lambda,\mathrm{cf}(\lambda)}$ holds, and if $\mathrm{cf}(\lambda) < \kappa$, then there is a partial square on points of cofinality at least κ. If one forces over this model in the natural way to obtain SCFA, the resulting model will satisfy CH, and even $\diamondsuit$, and $\kappa = \omega_2$. So this model will satisfy $\square^*_{\omega_1}$, and in general, for cardinals λ with $\mathrm{cf}(\lambda) \geq \omega_1$, $\square^*_\lambda$ will hold. For singular λ with $\mathrm{cf}(\lambda) \geq \omega_2$, $\square_{\lambda,\mathrm{cf}(\lambda)}$ will hold. So the consequences of SCFA on the failure of weak square listed in Theorem 2.11 are optimal, except for the case $\mathrm{cf}(\lambda) = \omega$, in which it is unclear whether $\square^*_\lambda$ has to fail or may hold.

Question 2.12. Suppose λ is a cardinal of cofinality ω, and that SCFA holds. Does it follow that $\square^*_\lambda$ fails?

Recall the "+" versions of $\mathsf{MA}(\Gamma)$.

Definition 2.13. Let Γ be a class of forcings. Then the axiom $\mathsf{MA}^+(\Gamma)$ says that for any forcing $\mathbb{P}$ in Γ, any collection $\mathcal{A}$ of ω_1 many maximal antichains in $\mathbb{P}$, and any $\mathbb{P}$-name $\dot{S}$ such that $\Vdash_{\mathbb{P}}$ "$\dot{S}$ is a stationary subset of ω_1", there is a filter F in $\mathbb{P}$ that intersects each member of $\mathcal{A}$ and such that $\dot{S}^F = \{\alpha < \omega_1 \mid \exists p \in F \quad p \Vdash \check{\alpha} \in \dot{S}\}$ is stationary in ω_1. If Γ is the class of all subcomplete forcings, then I write SCFA^+ for $\mathsf{MA}^+(\Gamma)$. If Γ is the class of countably closed, semiproper, proper, or stationary set preserving forcings, then $\mathsf{MA}^+(\Gamma)$ is known as $\mathsf{MA}^+(\sigma\text{-closed})$, SPFA^+, PFA^+ and MM^+, respectively.

The principle $\mathsf{MA}^{++}(\Gamma)$ is defined similarly, but allowing ω_1 many names for stationary subsets of ω_1.

While $\mathsf{MA}(\sigma\text{-closed})$ is a theorem of ZFC, the axiom $\mathsf{MA}^+(\sigma\text{-closed})$ has considerable consistency strength. It is also consistent with CH, as it holds in $\mathrm{V}^{\mathrm{Col}(\omega_1,<\kappa)}$, where κ is supercompact ([8]), and it implies some reflection phenomena that have many consequences.

Definition 2.14 ([6],[8])**.** For an uncountable transitive set X, a stationary set $S \subseteq [X]^\omega$ *reflects to* a set Y if $\overline{\overline{Y}} \subseteq Y$ and $S \cap [Y]^\omega$ is stationary in $[Y]^\omega$.

The principle $\mathsf{Refl}^*(S)$ holds if every stationary $T \subseteq S$ reflects to an uncountable Y such that $\mathrm{cf}(\mathrm{otp}(Y \cap \mathrm{On})) = \omega_1$.

For a regular cardinal λ, $\mathsf{RP}(\lambda)$ says that for every stationary set $S \subseteq [\lambda]^\omega$ and every $X \subseteq \lambda$ of cardinality ω_1, there is a $Y \subseteq \lambda$ of cardinality ω_1 such that $X \subseteq Y$ and S reflects to Y.

The reflection principle RP says that $\mathsf{RP}(\lambda)$ holds for every regular $\lambda > \omega_1$.

In [6], the fact that after collapsing a supercompact cardinal κ to be ω_2 (by forcing with $\mathrm{Col}(\omega_1, {<}\kappa)$), then in the generic extension, $\mathsf{Refl}^*([\lambda]^\omega)$ holds for every regular cardinal λ greater than ω_1, is attributed to the paper [8]. It is shown in that paper that the generic extension satisfies $\mathsf{MA}^+(\sigma\text{-closed})$, and that this forcing axiom, in turn, implies that $\mathsf{RP}(\lambda)$ holds, for every regular cardinal $\lambda > \omega_1$, but nothing is said about the cofinality of the set to which the stationary set reflects. The proof given there can be modified fairly easily, in order to yield $\mathsf{Refl}^*(\lambda)$. Alternatively, one may use the following concept, taken from [15, Ex. 37.23], where it is given no name.

Definition 2.15. Let $\lambda > \omega_1$ be a regular cardinal. Then the principle $\mathsf{Refl}^+(\lambda)$ says: if $S \subseteq [H_\lambda]^\omega$ is stationary, then there is an elementary chain $\langle M_\alpha \mid \alpha < \omega_1 \rangle$ (i.e., an increasing, continuous sequence with $M_\alpha \in M_{\alpha+1}$ for all $\alpha < \omega_1$) of submodels of H_λ such that $\{\alpha < \omega_1 \mid M_\alpha \in S\}$ is stationary in ω_1.

It is shown in [15] that $\mathsf{MA}^+(\sigma\text{-closed})$ implies that $\mathsf{Refl}^+(\lambda)$ holds, for every regular $\lambda > \omega_1$, and it is an easy observation that $\mathsf{Refl}^+(\lambda)$ implies $\mathsf{Refl}^*(\lambda)$.

Observation 2.16. $\mathsf{Refl}^+(\lambda)$ *implies* $\mathsf{Refl}^*([\lambda]^\omega)$.

Proof. Let $T \subseteq [\lambda]^\omega$ be stationary. Then

$$S = \{N \prec \langle H_\lambda, \in, <\rangle \mid \overline{\overline{N}} = \omega \wedge N \cap \lambda \in T\}$$

is stationary in $[H_\lambda]^\omega$. Let $\langle M_\alpha \mid \alpha < \omega_1 \rangle$ be as in $\mathsf{Refl}^+(\lambda)$ for S. This is a continuous sequence of elementary submodels of H_λ such that $M_\alpha \in M_{\alpha+1}$, and it follows inductively that $\alpha \subseteq M_\alpha$, for all $\alpha < \omega_1$. Let $Y = \bigcup_{\alpha<\omega_1} M_\alpha$. Then $\omega_1 \subseteq Y$, and it is clear that $\mathrm{cf}(\mathrm{otp}(Y \cap \mathrm{On})) = \omega_1$.

To see that $T \cap [Y]^\omega$ is stationary, let $g : Y^{<\omega} \longrightarrow Y$. Then clearly, the set of $\alpha < \omega_1$ such that M_α is closed under g is club in ω_1. So there is such an α with the property that $M_\alpha \in S$. This means that $a = M_\alpha \cap \lambda \in T$. Clearly, a is closed under g. □

It was shown in [6] that if $\mathrm{cf}(\lambda) = \omega$ and $\mathsf{Refl}^*([\lambda^+]^\omega)$ holds, then $\square^*_\lambda$ fails.

Corollary 2.17 (Cummings, Foreman, Magidor)**.** MA^+(*σ-closed*) *implies the failure of* $\square^*_\lambda$ *for* $\mathrm{cf}(\lambda) = \omega$*, and it implies that there is no very good scale for* λ*.*

This connection answers [30, Question 4.12], which asks whether the principle $\mathsf{UR}_{\sigma\text{-closed}}$ (which says that for any cardinal κ and any σ-closed forcing extension $\mathrm{V}[G]$, there is a further forcing extension by a σ-closed forcing, $\mathrm{V}[G][H]$, such that there is in $\mathrm{V}[G][H]$ an elementary embedding $j : \langle H_\kappa, \in\rangle \prec \langle H_\lambda, \in\rangle^{\mathrm{V}[G][H]}$) implies the failure of $\square^*_\lambda$ for all cardinals of cofinality ω (it was shown in that paper that it fails for all strong limits λ of cofinality ω). The answer to the question is yes, because by [30, Cor. 2.6], $\mathsf{UR}_{\sigma\text{-closed}}$ implies $\mathsf{MA}^+(\sigma\text{-closed})$, which implies the desired failure of weak square, by the previous corollary.

On another note, it was shown in [9] that if one collapses an indestructibly weakly compact cardinal to be ω_2, the extension also satisfies $\mathsf{MA}^+(\sigma\text{-closed})$. So we get the following fact (which is usually stated with “supercompact” in place of “indestructibly weakly compact”).

Corollary 2.18. *If* κ *is indestructibly weakly compact and* G *is generic for* $\mathrm{Col}(\omega_1, {<}\kappa)$*, then in* $\mathrm{V}[G]$*,* $\square^*_\lambda$ *fails whenever* $\mathrm{cf}(\lambda) = \omega$*.*

Putting together Corollary 2.17, Theorem 2.11 and the obvious fact that SCFA^+ implies $\mathsf{MA}^+(\sigma\text{-closed})$ results in the following.

Corollary 2.19. *Assume* SCFA^+. *Let* λ *be a cardinal.*

(1) *If* $\mathrm{cf}(\lambda) = \omega$, *then* $\square^*_\lambda$ *fails.*
(2) *If* $\mathrm{cf}(\lambda) = \omega_1$, *then* $\square_{\lambda,\mu}$ *fails, for every* $\mu < \lambda$.
(3) *If* $\mathrm{cf}(\lambda) \geq \omega_2$, *then* $\square_{\lambda,\mu}$ *fails, for every* $\mu < \mathrm{cf}(\lambda)$.

And if CH *holds, then* $\square^*_{\omega_1}$ *holds.*

The question is whether SCFA alone implies this. In particular, does it imply the principle $\mathsf{Refl}^*([\lambda^+]^\omega)$, when $\mathrm{cf}(\lambda) = \omega$? MM implies the strong reflection principle SRP (see [15, Def. 37.20]), which, in turn, implies $\mathsf{Refl}^*([\lambda^+]^\omega)$. SRP also implies that the nonstationary ideal on ω_1 is ω_2-saturated, which makes the following simple observation relevant.

Observation 2.20. *Neither* SCFA *nor* $\mathsf{MA}^+(\sigma\textit{-closed})$ *imply that the nonstationary ideal on* ω_1 *is* 2^{ω_1}*-saturated.*

Proof. This is because both SCFA and $\mathsf{MA}^+(\sigma\text{-closed})$ are compatible with $\diamondsuit$. But of course, if $\langle S_\alpha \mid \alpha < \omega_1\rangle$ is a $\diamondsuit$-sequence, then for every set $A \subseteq \omega_1$, the set $S_A = \{\alpha < \omega_1 \mid A \cap \alpha = S_\alpha\}$ is stationary, and so, the sequence $\langle S_A \mid A \subseteq \omega_1\rangle$ is an almost disjoint sequence of stationary subsets of ω_1 (almost disjoint meaning that for $A \neq B$, $S_A \cap S_B$ is bounded in ω_1). $\square$

The situation is interesting now: $\mathsf{MA}^+(\sigma\text{-closed})$ implies $\mathsf{Refl}^*([\lambda]^\omega)$ for every regular $\lambda > \omega_1$, and MM implies this principle too, it even implies $\mathsf{MA}^+(\sigma\text{-closed})$, but all known proofs of this (even just of the regular reflection principle RP) filter through the fact that MM implies that the nonstationary ideal on ω_1 is ω_2-saturated. These kinds of arguments cannot work when starting from the assumption that SCFA holds.

Question 2.21. Does SCFA imply that $\mathsf{Refl}^*([\lambda]^\omega)$ holds, for every regular $\lambda > \omega_1$? Or even just $\mathsf{RP}(\lambda)$? Does it imply $\mathsf{MA}^+(\sigma\text{-closed})$?

In general, the extent of stationary reflection under SCFA seems crucial in the context of square principles. Let's consider the stationary reflection principle OSR_{ω_2} from [21].

Definition 2.22. The principle OSR_{ω_2} says that whenever $\langle T_\alpha \mid \alpha < \omega_2\rangle$ is a sequence of stationary subsets of ω_2, each consisting of ordinals of cofinality ω, then there is a $\gamma < \omega_2$ with $\mathrm{cf}(\gamma) = \omega_1$ such that for all $\alpha < \gamma$, $T_\alpha \cap \gamma$ is stationary in γ.

If OSR_{ω_2} holds, then it follows as in the proof of Observation 2.8 that the set of γ as in the definition is stationary in ω_2. I aim to show that SCFA implies OSR_{ω_2}. To this end, I will show that the forcing used by Larson in the context of MM is not only stationary set preserving, but subcomplete. The forcing in question is the following. It is an explicit rendering of the one used by Larson in [21].

Definition 2.23. Let κ be regular, and let $\vec{S} = \langle S_\alpha \mid \alpha < \omega_1\rangle$ be a partition of ω_1 into stationary sets, and let $\vec{T} = \langle T_\alpha \mid \alpha < \kappa\rangle$ be a sequence of stationary subsets of κ, each T_α consisting of ordinals of cofinality ω. The forcing $\mathbb{P}_{\vec{S},\vec{T}}$ consists of the pairs $\langle p, q\rangle$ such that

(1) p is a function with $\mathrm{dom}(p) \subseteq \omega_1$, $\mathrm{ran}(p) \subseteq \kappa$ and $\overline{\overline{p}} < \omega_1$,

(2) $q : \gamma + 1 \longrightarrow \omega_1$ is normal, for some $\gamma < \omega_1$,
(3) $\sup(\mathrm{ran}(q)) \subseteq \mathrm{dom}(p)$,
(4) for all $\xi \in \mathrm{dom}(q)$, if α is such that $q(\xi) \in S_\alpha$, then $\alpha \in \mathrm{dom}(p)$ and $\sup p``q(\xi) \in T_{p(\alpha)}$.

The ordering is by reverse inclusion in each component.

Lemma 2.24. *The forcing $\mathbb{P} = \mathbb{P}_{\vec{S},\vec{T}}$ is subcomplete.*

Proof. Clearly, $\delta(\mathbb{P}) \geq \kappa$, where $\delta(\mathbb{P})$ is the smallest size a dense subset of $\mathbb{P}$ can have, or else $\mathbb{P}$ would be κ-c.c., but $\mathbb{P}$ collapses κ to ω_1. Let θ be sufficiently large, $N = L^A_\tau$ a ZFC^- model with $\tau > \theta$ and $\mathbb{P} \in H_\theta \subseteq N$. Let $\sigma : \bar{N} \prec N$ be an elementary embedding with $\theta, \mathbb{P}, \vec{S}, \vec{T} \in \mathrm{ran}(\sigma)$, $\bar{N}$ transitive, countable and full. Let $s \in \mathrm{ran}(\sigma)$ be fixed. Let $\sigma(\bar{s}, \bar{\theta}, \bar{\kappa}, \bar{\mathbb{P}}, \bar{\vec{S}}, \bar{\vec{T}}) = s, \theta, \kappa, \mathbb{P}, \vec{S}, \vec{T}$. Let $\bar{G}$ be $\bar{\mathbb{P}}$-generic over $\bar{N}$. Let $\Omega = \omega_1^{\bar{N}} = \mathrm{crit}(\sigma)$.

Let $\bar{P}$ be the union of the first coordinates occurring in $\bar{G}$, and $\bar{Q}$ the union of the second coordinates. So

$$\bar{P} : \Omega \longrightarrow \bar{\kappa} \text{ is onto, and } \bar{Q} : \Omega \longrightarrow \Omega \text{ is a normal cofinal function.}$$

Let

$$D = \{\tau < \kappa \mid \tau = \kappa \cap \mathrm{Hull}^N(\tau \cup \mathrm{ran}(\sigma))\}$$

Then D is club in κ, and the proof of [19, Lemma 6.3] shows that for every $\kappa' \in D$ with $\mathrm{cf}(\kappa') = \omega$, there is a σ' (in V) such that

(1) $\sigma' : \bar{N} \prec N$,
(2) $\sigma'(\bar{s}, \bar{\theta}, \bar{\kappa}, \bar{\mathbb{P}}, \bar{\vec{S}}, \bar{\vec{T}}) = s, \theta, \kappa, \mathbb{P}, \vec{S}, \vec{T}$,
(3) $\mathrm{Hull}^N(\kappa \cup \mathrm{ran}(\sigma)) = \mathrm{Hull}^N(\kappa \cup \mathrm{ran}(\sigma'))$,
(4) $\sup \sigma'``\bar{\kappa} = \kappa'$.

Note that this works for any specified $\bar{s}$, s. In other words, there is such a σ' that agrees with σ on any finite number of specified points.

Let α_0 be such that $\Omega \in S_{\alpha_0}$. I will find an embedding σ' as above and a master condition $\langle P, Q\rangle \in \mathbb{P}$, in the sense that $\langle P, Q\rangle$ extends every condition in $\sigma'``\bar{G}$. Clearly, if $G$ is $\mathbb{P}$-generic over V with $\langle P, Q\rangle \in G$, then $\sigma'``\bar{G} \subseteq G$. This will show that $\mathbb{P}$ is subcomplete. I will consider two cases separately, but in both cases, I will define $Q : \Omega + 1 \longrightarrow \omega_1$ by setting $Q{\restriction}\Omega = \bar{Q}$ and $Q(\Omega) = \Omega$.

Case 1: $\alpha_0 < \Omega$.

Let

$$\bar{\delta} = \bar{P}(\alpha_0),\ \ \delta = \sigma(\bar{\delta})$$

Let $\kappa' \in D \cap T_\delta$, and let σ' be as above, with $\sigma'(\bar{\delta}) = \delta$ and $\sup \sigma'``\bar{\kappa} = \kappa' \in T_\delta$. Define $P : \Omega \longrightarrow \kappa$ by

$$P(\xi) = \sigma'(P(\xi))$$

for $\xi < \Omega$.

Then $\langle P, Q\rangle \in \mathbb{P}_{\vec{S},\vec{T}}$. Let's go through the requirements of Definition 2.23: 1.,2. are clear. 3. is true because $\sup \mathrm{ran}(Q) = \Omega = \mathrm{dom}(P)$. For 4., let $\xi \in \mathrm{dom}(Q) = \Omega + 1$. If $\xi < \Omega$, then let $Q(\xi) \in S_\alpha$. Then $\bar{Q}(\xi) \in \bar{S}_\alpha$. So $\alpha \in \mathrm{dom}(\bar{P})$ and $\sup \bar{P}``(\bar{Q}(\xi)) \in \bar{T}_{\bar{P}(\alpha)}$. Let $\gamma = (\bar{Q}(\xi) + 1) \cup (\alpha + 1)$, and set $\bar{p} = \bar{P}{\restriction}\gamma$. Then $\bar{p} \in \bar{N}$, and in $\bar{N}$, it is the case that $\sup \bar{p}``\bar{Q}(\xi) \in \bar{T}_{\bar{p}(\alpha)}$. So by elementarity of σ', $\sup \sigma'(\bar{p})``Q(\xi) \in T_{\sigma'(\bar{p})(\alpha)}$, because $\bar{Q}(\xi) = Q(\xi) < \Omega$ and $P(\zeta) = \sigma'(\bar{P})(\zeta)$ for $\zeta < \Omega$. But $\sigma'(\bar{p}) = P{\restriction}\gamma$, and so, $\sup P``Q(\xi) \in T_{P(\alpha)}$. If $\xi = \Omega$, then we have that $Q(\xi) \in S_{\alpha_0}$ and, by assumption, $\alpha_0 < \Omega$, so $\alpha_0 \in \mathrm{dom}(P)$, and

$\sup P``Q(\xi) = \sup P``\Omega = \sup \sigma'``\bar{P}``\Omega = \sup \sigma'``\bar{\kappa} = \kappa' \in T_\delta = T_{\sigma'(\bar{P}(\alpha_0))}$ by construction, and $\sigma'(\bar{P}(\alpha_0)) = P(\alpha_0)$, as wished. It is then clear that $\langle P, Q\rangle$ is a master condition for $\sigma'``\bar{G}$, because if $\langle \bar{p}, \bar{q}\rangle \in \bar{G}$, then $\sigma'(\bar{p}) \subseteq P$ and $\sigma'(\bar{q}) \subseteq Q$.
Case 2: $\alpha_0 \geq \Omega$.

In this case, pick $\kappa' \in D \cap T_0$ and find a σ' as above with $\sup \sigma'``\bar{\kappa} = \kappa' \in T_0$. Define $Q : \Omega + 1 \longrightarrow \Omega + 1$ as above, and define $P : \alpha_0 + 1 \longrightarrow \kappa$ by

$$P(\xi) = \begin{cases} \sigma'(\bar{P}(\xi)) & \text{if } \xi < \Omega, \\ 0 & \text{if } \Omega \leq \xi \leq \alpha_0 \end{cases}$$

for $\xi \leq \alpha_0$. It is then easily checked that $\langle P, Q\rangle$ is in $\mathbb{P}$: points 1.-3. of Definition 2.23 are satisfied as in Case 1, and for condition 4., let $\xi \in \mathrm{dom}(Q) = \Omega+1$. If $\xi < \Omega$, the situation is as in Case 1, and if $\xi = \Omega$, then $Q(\xi) = Q(\Omega) = \Omega \in S_{\alpha_0}$. By definition of P, $\alpha_0 \in \mathrm{dom}(P)$ and $\sup P``(Q(\xi)) = \sup P``\Omega = \sigma'``\bar{\kappa} = \kappa' \in T_0 = T_{P(\alpha_0)}$. So $\langle P, Q\rangle \in \mathbb{P}$, and clearly, $\langle P, Q\rangle$ extends every member of $\sigma'``\bar{G}$. □

To draw the desired conclusion, I will use the characterization of SCFA given by Fact 3.8.

Theorem 2.25. SCFA *implies* OSR_{ω_2}. *In fact,* SCFA($\leq\omega_2$) *suffices - see Def. 3.1 and Fact 3.8.*

Proof. Let $\vec{T} = \langle T_\alpha \mid \alpha < \omega_2\rangle$ be a sequence of stationary subsets of ω_2, each consisting of ordinals of cofinality ω. Let $\vec{S} = \langle S_\alpha \mid \alpha < \omega_1\rangle$ be a partition on ω_1 into disjoint stationary sets, and let $\mathbb{P} = \mathbb{P}_{\vec{S},\vec{T}}$. Let G be generic for $\mathbb{P}$. Let $M \prec H_{\omega_3}$ with $\omega_2 \subseteq M$, so that M has size ω_2, with $\vec{S}, \vec{T} \in M$. Let M also be equipped with constant symbols for the countable ordinals. Let P be the union of the first components of conditions in G, and Q the union of the second components. Then in V$[G]$, the following Σ_1-statement about M holds: "there is a club $C \subseteq \omega_1^M$ and a function $g : \omega_1 \longrightarrow \omega_2^M$ such that g is onto, and such that for every $\zeta \in C$, if α is such that $\zeta \in S_\alpha$, then $\sup g``\zeta \in T_{g(\alpha)}$." This is witnessed by $g = P$ and $C = \mathrm{ran}(Q)$.

So, by SCFA, there is in V a model $\bar{M}$ such that the same Σ_1 statement is true of $\bar{M}$, and there is an elementary embedding $j : \bar{M} \prec M$. Let $\bar{C}, \bar{g}$ witness that the statement holds for $\bar{M}$. Let $\bar{\vec{S}} = j^{-1}(\vec{S})$, $\bar{\vec{T}} = j^{-1}(\vec{T})$. Let $\gamma = \omega_2^{\bar{M}}$. Note that $\omega_2^{\bar{M}}$ is the critical point of j and $\omega_1^{\bar{M}} = \omega_1$, so that for $\alpha < \omega_1$, $\bar{S}_\alpha = S_\alpha$, and for $\alpha < \gamma$, $\bar{T}_\alpha = T_\alpha \cap \gamma$. Let $e : \omega_1 \longrightarrow \bar{C}$ be the monotone enumeration of $\bar{C}$, and define $h : \omega_1 \longrightarrow \gamma$ by $h(\xi) = \sup \bar{g}``e(\xi)$. Clearly, $h$ witnesses that the cofinality of $\gamma$ is $\omega_1$. It follows now that for every $\alpha < \gamma$, $T_\alpha \cap \gamma$ is stationary in $\gamma$. To see this, let $d \subseteq \gamma$ be club, and let $\bar{g}(\bar{\alpha}) = \alpha$. Let $\bar{d} = h^{-1}``d$. Then $\bar{d}$ is club in ω_1, so let $\zeta \in \bar{d} \cap \bar{C} \cap S_{\bar{\alpha}}$. By the properties of $\bar{g}$ and $\bar{C}$, it follows that $h(\zeta) = \sup \bar{g}``\zeta \in \bar{T}_{\bar{g}(\alpha)} = \bar{T}_\alpha = T_\alpha \cap \gamma$, and of course $h(\zeta) \in d$ since $\zeta \in \bar{d}$. □

3. A hierarchy of bounded subcomplete forcing axioms

In this section, I will consider bounded versions of SCFA and determine their consistency strength. The study of bounded forcing axioms has proven fruitful in many contexts. These axioms were introduced in [11] as follows. There is no uniform terminology in the literature, and my notation, while a little verbose, is hopefully hard to misunderstand.

Definition 3.1. Let Γ be a class of forcings, and let κ, λ be cardinals. Then $\mathsf{BFA}(\Gamma, \le\kappa, \le\lambda)$ is the statement that if $\mathbb{P}$ is a forcing in Γ, $\mathbb{B}$ is its complete Boolean algebra, and $\mathcal{A}$ is a collection of at most κ many maximal antichains in $\mathbb{B}$, each of which has size at most λ, then there is a $\mathcal{A}$-generic filter in $\mathbb{B}$, that is, a filter that intersects each antichain in $\mathcal{A}$. If Γ is the class of proper, stationary set preserving or subcomplete forcings, I write BPFA, BMM, BSCFA (respectively) for $\mathsf{BFA}(\Gamma, \le\omega_1, \le\omega_1)$. In general, for a cardinal λ, I will write $\mathsf{BFA}(\Gamma, \le\lambda)$ for $\mathsf{BFA}(\Gamma, \le\omega_1, \le\lambda)$. $\mathsf{BSCFA}(\le\lambda)$, $\mathsf{BPFA}(\le\lambda)$, etc., then have the obvious meaning.

What makes bounded forcing axioms so attractive is the fact that they can be equivalently expressed as a statement about generic absoluteness.

Theorem 3.2 ([3, Thm. 5])**.** *Let κ be a cardinal of uncountable cofinality, and let $\mathbb{P}$ be a poset. Then $\mathsf{BFA}(\{\mathbb{P}\}, \le\kappa, \le\kappa)$ is equivalent to $\Sigma_1(H_{\kappa^+})$-absoluteness for $\mathbb{P}$. The latter means that whenever $\varphi(x)$ is a Σ_1-formula and $a \in H_{\kappa^+}$, then $\mathrm{V} \models \varphi(a)$ iff for every $\mathbb{P}$-generic g, $\mathrm{V}[g] \models \varphi(a)$.*

3.1. **Bounded subcomplete forcing axioms and reflecting cardinals.** The focus of [11] is of course the bounded forcing axiom for the class of proper forcings, BPFA, and the main result of that paper is that BPFA is equiconsistent with the existence of a reflecting cardinal, defined as follows.

Definition 3.3 ([11, Def. 2.2])**.** A regular cardinal κ is *reflecting* if for every $a \in H_\kappa$, and every formula $\varphi(x)$, the following holds: if there is a regular cardinal $\theta \ge \kappa$ such that $H_\theta \models \varphi(a)$, then there is a cardinal $\bar{\theta} < \kappa$ such that $H_{\bar{\theta}} \models \varphi(a)$.

It is pointed out in [11, Remark 2.3(2)] that "for all θ" in this definition can be replaced by "for unboundedly many θ". This means that if X is an unbounded class of regular cardinals, then a regular cardinal κ is reflecting iff for every formula $\varphi(x)$ and every $a \in H_\kappa$, if there is a $\theta \in X$ such that $H_\theta \models \varphi(a)$, then there is a cardinal $\bar{\theta} < \kappa$ such that $H_{\bar{\theta}} \models \varphi(a)$. This is not hard to see, because one can replace $\varphi(x)$ with the statement "there exists a regular cardinal μ such that $\varphi^{H_\mu}(x)$". The concept also doesn't change if one drops the clause that θ be regular, and if it is regular, one may request that $\bar{\theta}$ also be regular, etc., so it is very robust.

The idea of proof of the following lemma traces back to Todorčević, see [1, Lemma 2.4], where Todorčević's argument to show that BPFA implies that ω_2 is reflecting in L is given (originally, this was shown, by a different argument, in [11]).

Lemma 3.4. BSCFA *implies that ω_2 is reflecting in L.*

Proof. We may assume that $0^\#$ does not exist, as otherwise, every Silver indiscernible is reflecting in L. Let $\kappa = \omega_2$, fix $a \in L_\kappa = (H_\kappa)^L$, a formula $\varphi(x)$, a singular cardinal $\gamma > \kappa$, and let $\theta = \gamma^+ = (\gamma^+)^L$, by covering. By the remark after Definition 3.3, it suffices to show that if $L_\theta \models \varphi(a)$, then there is an L-cardinal $\bar{\theta} < \kappa$ such that $L_{\bar{\theta}} \models \varphi(a)$. I will show that there is $\bar{\theta}$ like this that's regular in L. So let's assume that $L_\theta \models \varphi(a)$.

Let $\langle C_\xi \mid \xi \text{ is a singular ordinal in } L\rangle$ be the canonical global $\square$ sequence for L ([16]). It is Σ_1-definable in L and has the properties that for every L-singular ordinal ξ, the order type of C_ξ is less than ξ, and if ζ is a limit point of C_ξ, then ζ is singular in L and $C_\zeta = C_\xi \cap \zeta$.

Let $B = \{\xi < \theta \mid \kappa < \xi < \theta \text{ and } \mathrm{cf}(\xi) = \omega\}$. Note that by covering, every $\xi \in B$ is singular in L, since a countable cofinal subset of ξ in V can be covered by a set in L of cardinality at most ω_1, so that its order type will be less than κ, and hence less than ξ. So C_ξ is defined for every $\xi \in B$, and since the function $\xi \mapsto \mathrm{otp}(C_\xi)$ is regressive, there is a stationary subset A of B on which this function is constant.

Since A consists of ordinals of cofinality ω and is stationary in a regular cardinal greater than ω_1, the forcing $\mathbb{P}_A$, which adds a normal function $F : \omega_1 \longrightarrow A$ cofinal in θ, is subcomplete – see [19, p. 134ff., Lemma 6.3]. Let G be generic for $\mathbb{P}_A$, and let $F : \omega_1 \longrightarrow A$ be the function corresponding to G.

In $\mathrm{V}[F]$, the statement "there is an ordinal α and a set C such that $L_\alpha \models \varphi(a)$, C is club in α, $\mathrm{otp}(C) = \omega_1$, for every $\xi \in C$, C_ξ is defined, and for all $\xi, \zeta \in C$, $\mathrm{otp}(C_\xi) = \mathrm{otp}(C_\zeta)$" holds, as witnessed by $\alpha = \theta$ and $C = \mathrm{ran}(F)$. This is a Σ_1 statement about the parameters ω_1 and a. So by BSCFA (which is applicable as $\mathbb{P}_A$ is subcomplete), the same statement is true in V. Let $\bar{\theta}, \bar{C}$ witness this. Since $\omega_1, a \in H_{\omega_2}$, such witnesses for a Σ_1 formula can be found in H_{ω_2}, so we may take $\bar{\theta} < \omega_2 = \kappa$. The point is now that $\bar{\theta}$ must be regular in L. The reason is that if $\bar{\theta}$ were singular in L, then $C_{\bar{\theta}}$ would be defined. Note that $\mathrm{cf}(\bar{\theta}) = \omega_1$, since $\mathrm{otp}(\bar{C}) = \omega_1$. So, letting $C'_{\bar{\theta}}$ be the set of limit points of $C_{\bar{\theta}}$, $C'_{\bar{\theta}} \cap \bar{C}$ is club in $\bar{\theta}$. Now take $\xi < \zeta$, both in $C'_{\bar{\theta}} \cap \bar{C}$. Then, since $\xi, \zeta \in \bar{C}$, C_ξ and C_ζ have the same order type, but since both are limit points of $C_{\bar{\theta}}$, $C_\xi = C_{\bar{\theta}} \cap \xi$, which is a proper initial segment of $C_\zeta = C_{\bar{\theta}} \cap \zeta$.

So $\bar{\theta}$ is a regular cardinal in L, $\bar{\theta} < \omega_2$, and $H^L_{\bar{\theta}} = L_{\bar{\theta}} \models \varphi(a)$, showing that ω_2 is reflecting in L. □

Lemma 3.5. *If κ is reflecting, then there is a subcomplete forcing that forces* BSCFA.

Proof. Instead of going through the usual proof, [2, Lemma 2.2] applies. It says that if κ is a reflecting cardinal and Γ is a class of forcings such that

(1) Every $\mathbb{P}$ in Γ preserves ω_1,
(2) If $\mathbb{P} \in \Gamma$ and $\Vdash_{\mathbb{P}}$ "$\dot{\mathbb{Q}} \in \Gamma$", then $\mathbb{P} * \dot{\mathbb{Q}} \in \Gamma$,
(3) Whenever $\langle\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha\rangle \mid \alpha < \kappa\rangle$ is an iteration of posets in V_κ, with some suitable support (RCS being a possibility), and $\mathbb{P}_\kappa$ is the corresponding limit, the following hold:
    (a) $\mathbb{P}_\kappa \in \Gamma$,
    (b) $\mathbb{P}_\kappa / \mathbb{P}_\alpha \in \Gamma^{\mathrm{V}^{\mathbb{P}_\alpha}}$, for all $\alpha < \kappa$,
    (c) if for all $\alpha < \kappa$, $\mathbb{P}_\alpha \in \mathrm{V}_\kappa$, then $\mathbb{P}_\kappa$ is the direct limit and satisfies the κ-c.c.
(4) "$\mathbb{P} \in \Gamma$" is expressible by a Σ_2-formula,

then there is a poset $\mathbb{P} \in \Gamma$ that forces $\mathsf{BFA}(\Gamma)$.

When aiming to use this lemma with Γ being the class of subcomplete forcings, the only point worth looking at closely here is that subcompleteness is a Σ_2 property. Recall that Definition 2.3 says that a forcing, let's say here a complete Boolean algebra $\mathbb{B}$, is subcomplete if all sufficiently large θ verify its subcompleteness. Let's write $\mathrm{ver}(\mathbb{B}, \theta)$ to say that θ verifies the subcompleteness of $\mathbb{B}$. Then, [19, Lemma 2.1] says that if $\theta' > \overline{\overline{H_\theta}}$ (and $\mathbb{B} \in H_\theta$), then the statement "$\mathrm{ver}(\mathbb{B}, \theta)$" is absolute in $H_{\theta'}$, and finally, [19, Lemma 2.4] says that if *there is* a θ with $\mathrm{ver}(\mathbb{B}, \theta)$, then $\mathbb{B}$ is subcomplete. So "for all sufficiently large θ" in the original definition of subcompleteness can be replaced with "there is a θ". Thus,

$$\begin{array}{ll} & \mathbb{B} \text{ is subcomplete} \\ \iff & \exists\theta \quad \mathrm{ver}(\mathbb{B},\theta) \\ \iff & \exists\theta',\theta,H,X \quad (\theta' \text{ is a cardinal} \wedge H = H_{\theta'} \wedge X = H_\theta \in H_{\theta'} \wedge \mathrm{ver}(\mathbb{B},\theta)^H) \end{array}$$

Being a cardinal is Π_1, and "$H = H_{\theta'}$" is Σ_2. For one just has to say that H is transitive (which is Σ_0), and that there is a Z such that for every $x \in H$, there is a transitive $y \in H$ with $x \in y$ and a $\xi < \theta'$ and an $f \in Z$ such that $f : \xi \to y$ is surjective, and for all $\xi < \theta'$ and all functions g with domain ξ, if the range of g is transitive, then it is in X. So, altogether, this last formula can be expressed in a Σ_2 way. □

So, putting Lemma 3.4 and 3.5 together results in:

Theorem 3.6. BSCFA *is equiconsistent with the existence of a reflecting cardinal.*

Note that as a result, BSCFA does not imply any failure of $\square$, since the strength of the failure of $\square_{\omega_1}$ is a Mahlo cardinal (Solovay showed that one can force to a model where $\square_{\omega_1}$ fails from a model with a Mahlo cardinal, and Jensen showed that if $\square_{\omega_1}$ fails, then ω_2 is Mahlo in L), which is strictly stronger than a reflecting cardinal (if κ is Mahlo, then there are stationarily many $\bar{\kappa} < \kappa$ which are reflecting in V_κ, see [2, Fact 2.1]).

Question 3.7. Does BSCFA have any combinatorial consequences? Does it imply $2^\omega \le \omega_2$?

Note that even SCFA, being compatible with MM, doesn't determine the size of 2^ω, and it doesn't determine whether $\diamondsuit$ holds or fails. There could be Souslin trees, but it could also be that every Aronszajn tree is special.

3.2. **Pushing the boundaries:** BSCFA($\le\omega_2$) **and +1-reflecting cardinals.** For any class Γ of forcings, the principles BFA($\le\kappa$) (see Definition 3.1) give closer and closer approximations to MA(Γ), as κ increases; in fact, MA(Γ) is BFA($\Gamma,\le\infty$), or, for all κ, BSCFA($\le\kappa$). The following characterization of these axioms is easily seen to be equivalent to the one given in [5, Thm. 1.3], see also [4].

Fact 3.8. BFA($\{\mathbb{Q}\},\le\kappa$) *is equivalent to the following statement: if $M = \langle |M|, \in, \langle R_i \mid i < \omega_1\rangle\rangle$ is a transitive model for the language of set theory with ω_1 many predicate symbols $\langle \dot{R}_i \mid i < \omega_1\rangle$, of size κ, and $\varphi(x)$ is a Σ_1-formula, such that $\Vdash_{\mathbb{Q}} \varphi(\check{M})$, then there is in* V *a transitive $\bar{M} = \langle |\bar{M}|, \in, \langle \bar{R}_i \mid i < \omega_1\rangle\rangle$ and an elementary embedding $j : \bar{M} \prec M$ such that $\varphi(\bar{M})$ holds.*

Miyamoto has analyzed the strength of these principles for proper forcing and introduced the following large cardinal concept.

Definition 3.9 ([25, Def. 1.1])**.** Let κ be a regular cardinal, α an ordinal, and $\lambda = \kappa^{+\alpha}$. Then κ is H_λ-reflecting, or I will say $+\alpha$-reflecting, iff for every $a \in H_\lambda$ and any formula $\varphi(x)$, the following holds: if there is a cardinal θ such that $H_\theta \models \varphi(a)$, then the set of $N \prec H_\lambda$ such that

(1) N has size less than κ,
(2) $a \in N$,
(3) if $\pi_N : N \longrightarrow H$ is the Mostowski-collapse of N, then there is a cardinal $\bar{\theta} < \kappa$ such that $H_{\bar{\theta}} \models \varphi(\pi_N(a))$

is stationary in $\mathcal{P}_\kappa(H_\lambda)$.

Clearly, being reflecting is the same as being +0-reflecting. The +1-reflecting cardinals are also known as strongly unfoldable cardinals, introduced independently in [31], and treated in [20]. In the context of bounded forcing axioms, it seems to make the most sense to emphasize that they generalize reflecting cardinals, so I will stick to calling them +1-reflecting.

It is shown in [25, Thm. 4.2] that $\mathsf{BPFA}(\le\omega_2)$ is equiconsistent with the existence of a +1-reflecting cardinal. The next goal is to show that the corresponding statement is true of $\mathsf{BSCFA}(\le\omega_2)$ as well.

Lemma 3.10. *The axiom* $\mathsf{BSCFA}(\le\omega_2)$ *implies that* ω_2 *is +1-reflecting in* L.

Proof. Let $\kappa=\omega_2$. As in the proof of [25, Thm. 4.2], we may assume that $0^\#$ does not exist, since every Silver indiscernible is +1-reflecting, and it suffices to show that if $X\in L\cap\mathcal{P}(\kappa)$, $\varphi(x)$ is a formula, γ is a singular cardinal, $\theta=(\gamma)^{+L}$ (so that $\theta=\gamma^{+\mathrm{V}}$) and $L_\theta\models\varphi(X)$, then the set B of ordinals $\alpha<\kappa$ such that there is an L-cardinal $\bar\theta<\kappa$ with $L_{\bar\theta}\models\varphi(X\cap\alpha)$ is stationary in κ.

So let $C\subseteq\kappa$ be club in κ. While the proof of [25, Thm. 4.2] is a reworking of the original argument from [11], I will argue along the lines of the proof of Lemma 3.4, and I will use some notation introduced in that proof.

Let $\langle C_\xi \mid \xi \text{ is a singular ordinal in } L\rangle$ be the canonical global $\square$ sequence for L. As before, we can find a stationary set $A\subseteq\theta\cap\mathsf{cof}(\omega)$ on which the function $\xi\mapsto\mathrm{otp}(C_\xi)$ is constant. Let F be a normal function added by the subcomplete forcing $\mathbb{P}_A$, so that $F:\omega_1\longrightarrow A$ is cofinal in θ. Let $M=\langle H_\kappa,\in,C,X,0,1,\ldots,\xi,\ldots\rangle_{\xi<\omega_1}$. Let g be generic over $\mathrm{V}[F]$ for $\mathrm{Col}(\omega_1,M)$. In $\mathrm{V}[F][g]$, the Σ_1-statement $\Phi(M)$ saying "there is an ordinal $\alpha>\mathrm{On}\cap M$, a set D and a function h such that $L_\alpha\models\varphi(\dot X^M)$, D is club in α, $\mathrm{otp}(D)=\omega_1^M$, h is a surjection from ω_1^M onto the universe of M, and for all $\xi,\zeta\in D$, $\mathrm{otp}(C_\xi)=\mathrm{otp}(C_\zeta)$" holds, as witnessed by θ, $\mathrm{ran}(F)$ and g. So according to the characterization of $\mathsf{BSCFA}(\le\omega_2)$ given by Fact 3.8, there is in V a transitive $\bar M=\langle|\bar M|,\in,\bar C,\bar X,\langle\xi\mid\xi<\omega_1\rangle\rangle$ such that $\Phi(\bar M)$ holds, and an elementary embedding $j:\bar M\prec M$. Note that since M calculates ω_1 correctly and thinks that it is the largest cardinal, it follows that $\mathrm{ran}(j)$ is transitive, and hence that j is the identity. So letting $\bar\kappa=\mathrm{On}^{\bar M}$, it follows that $\bar C=C\cap\bar\kappa$ and $\bar X=X\cap\bar\kappa$. Let $\bar\theta$, $\bar D$ and $\bar h$ witness that $\Phi(\bar M)$ holds. Then $\bar h:\omega_1\longrightarrow|\bar M|$ is onto, so $\bar\kappa<\kappa$. So by elementarity, $\bar C$ is unbounded in $\bar\kappa$, and hence, $\bar\kappa\in C$. Moreover, since $\bar M\in H_{\omega_2}$, $\bar\theta$ may be chosen to be less than ω_2.

It follows as before that $\bar\theta$ is a regular cardinal in L. So, since by $\Phi(\bar M)$, $L_{\bar\theta}\models\varphi(X\cap\bar\kappa)$, it follows that $\bar\kappa\in B\cap C$, showing that B is stationary. □

In [25, Thm. 3.1], it is shown that one can force $\mathsf{BPFA}(\le\omega_{1+\xi})$ over a model with a $+\xi$-reflecting cardinal κ such that $(\kappa^{+\xi})^\kappa=\kappa^{+\xi}$, with a countable support iteration of proper forcings that is κ-c.c. and collapses κ to be ω_2, assuming there is a version of a Laver function suitable for $+\xi$-reflecting cardinals. This proof can be made to work with subcomplete forcings instead of proper forcings, and with revised countable support instead of countable support.

In [25, Thm. 1.5], Miyamoto shows that such a Laver function for a $+\xi$-reflecting cardinal κ such that $(\kappa^{+\xi})^\kappa=\kappa^{+\xi}$ can be added by a κ^+-c.c. forcing that preserves that κ is $+\xi$-reflecting, and that $(\kappa^{+\xi})^\kappa=\kappa^{+\xi}$, assuming there is no inaccessible cardinal above κ. Rather than going through the details of these constructions now, I will later show that +1-reflecting cardinals have a different characterization in terms of partial remarkability (see Lemma 4.9). Lemma 4.13 shows that one can force an axiom called $\mathsf{wBSCFA}(\le\omega_2)$ over a model with a $+1$-reflecting cardinal, but $\mathsf{wBSCFA}(\le\omega_2)$ is equivalent to $\mathsf{BSCFA}(\le\omega_2)$, see Observation 4.7. So, combining this with Lemma 3.10, one arrives at the desired equiconsistency.

Theorem 3.11. $\mathsf{BSCFA}(\le\omega_2)$ *is equiconsistent with the existence of a* $+1$*-reflecting cardinal.*

I do also want to state Miyamoto's result in the context of subcomplete forcing:

Fact 3.12 (Miyamoto)**.** *Suppose that* κ *is* $+\xi$*-reflecting, with* $(\kappa^{+\xi})^\kappa = \kappa^{+\xi}$*, and that there is no inaccessible cardinal above* κ*. Then there is a subcomplete forcing extension in which* $\mathsf{BSCFA}(\omega_{1+\xi})$ *holds.*

Finally, I would like to say something about the effects of the bounded versions of SCFA on the failure of weak square.

Theorem 3.13. *Let* $\tau > \omega_1$ *be regular. Then* $\mathsf{BSCFA}(\le\tau)$ *implies* SFP_τ*, introduced in Theorem 2.7.*

Proof. I use the characterization of $\mathsf{BSCFA}(\le\tau)$ given in Fact 3.8. Let $\vec{A}$ and $\vec{D}$ be given. Let $M = \langle H, \in, \vec{A}, \vec{D}, \vec{\xi}\rangle_{\xi<\omega_1}$, where $\tau \subseteq H \prec H_\tau$ and H has size τ. Let $\mathbb{P}$ be the forcing to add a normal, cofinal function $f : \omega_1 \longrightarrow \tau$ such that for every $i < \omega_1$, $f{\restriction}D_i : D_i \longrightarrow A_i$. Jensen showed that this forcing is subcomplete, see [17, §3]. Let G be $\mathbb{P}$-generic. Then in $\mathrm{V}[G]$, the statement that there is a normal function $f : \omega_1^M \longrightarrow \mathrm{On}^M$, cofinal in On^M, such that for every $i < \omega_1$, $f{\restriction}D_i : D_i \longrightarrow A_i$ holds, and it is a Σ_1-statement about M. Using the characterization given in Fact 3.8 of $\mathsf{BSCFA}(\le\tau)$, let $\bar{M} = \langle|\bar{M}|, \in, \vec{\bar{A}}, \vec{\bar{D}}, \langle\xi \mid \xi < \omega_1\rangle, \bar{\tau}\rangle$, $j \in \mathrm{V}$ be such that $j : \bar{M} \prec M$ is elementary, $\bar{M}$ is transitive, and such that the same statement is true in V of $\bar{M}$. Note that $\omega_1^{\bar{M}} = \omega_1$. Let $\bar{f} : \omega_1 \longrightarrow \mathrm{On}^{\bar{M}}$ be normal and cofinal. Clearly, $j \circ \bar{f}$ is continuous, because j is continuous at limits of $\bar{M}$-cofinality ω, and the range of $\bar{f}$ consists of ω-cofinal points (in $\bar{M}$). And clearly, if $\xi \in D_i$, then $\xi \in \bar{D}_i$ (since $j{\restriction}\omega_1 = \mathrm{id}$), which implies that $\bar{f}(\xi) \in \bar{A}_i$, so $j(\bar{f}(\xi)) \in D_i$. So $j \circ \bar{f}$ is as wished. □

So, remembering Observation 2.8 and Lemmas 2.9 and 2.10, one arrives at the following.

Corollary 3.14. *Let* τ *be a cardinal. Then* $\mathsf{BSCFA}(\le\tau^+)$ *implies the failure of* $\Box_{\tau,\mu}$*, for all* $\mu < \mathrm{cf}(\tau)$*, and if* τ *is singular with* $\mathrm{cf}(\tau) \le \omega_1$*, then it implies the failure of* $\Box_{\tau,\mu}$ *for all* $\mu < \tau$*.*

4. Weak forcing axioms

The weak proper forcing axiom, wPFA, was introduced in [4]. I will add some generality, by considering versions of this axiom for other classes of forcing, and by introducing bounded versions. First, let's deal with the unbounded forms of the axiom.

4.1. **The weak forcing axiom and remarkable cardinals.** The idea is to turn the characterization given in Fact 3.8 into a definition, saying that wPFA holds if it holds for every proper forcing $\mathbb{Q}$, except that the requirement that the cardinality of M be κ is dropped, and the embedding j is only required to exist in some forcing extension of V, not necessarily in V. This move from embeddings existing in V to ones that can be added by forcing is similar to the move from classical large cardinals to virtual large cardinals (see [4]), but in the context of forcing axioms. I want to add some generality here, and consider the version of the axiom for arbitrary classes of forcings, instead of only proper forcings. Later, the class of subcomplete forcings will become focal.

Definition 4.1. Let Γ be a class of forcings. The *weak forcing axiom for* Γ, $\mathsf{wFA}(\Gamma)$, says that whenever $M = \langle |M|, \in, R_0, R_1, \ldots, R_i, \ldots\rangle_{i<\omega_1}$ is a transitive model for a language $\mathcal{L}$ with ω_1 many predicates $\langle \dot{R}_i \mid i < \omega_1\rangle$ and the binary relation symbol $\dot{\in}$, and if $\varphi(x)$ is a Σ_1-formula and $\mathbb{P}$ is a forcing in Γ such that $\mathbb{P}$ forces that $\varphi(\check{M})$ holds, then there is (in V) a transitive model $\bar{M} = \langle |\bar{M}|, \in, \ldots, R_i, \ldots\rangle_{i<\omega_1}$ for $\mathcal{L}$ such that $\varphi(\bar{M})$ holds (in V), and such that in $\mathrm{V}^{\mathrm{Col}(\omega,|\bar{M}|)}$, there is an elementary embedding $j : \bar{M} \prec M$.

If Γ is the class of subcomplete forcings, then $\mathsf{wFA}(\Gamma)$ is denoted wSCFA . Similarly, the axiom for the class of proper forcings is abbreviated by wPFA.

It is shown in [4] that wPFA is equiconsistent with the existence of a remarkable cardinal. Remarkable cardinals were introduced by Schindler in [27], and an alternative characterization of remarkability was given in [28]. As in [4], this characterization can be stated as follows.

Definition 4.2. A regular cardinal κ is remarkable if for every regular $\lambda > \kappa$, there is a regular cardinal $\bar{\lambda} < \kappa$ such that in $\mathrm{V}^{\mathrm{Col}(\omega,H_{\bar{\lambda}})}$, there is an elementary embedding $j : H^{\mathrm{V}}_{\bar{\lambda}} \prec H^{\mathrm{V}}_{\lambda}$ with $j(\mathrm{crit}(j)) = \kappa$.

The next goal is to show that wSCFA is also equiconsistent with a remarkable cardinal. Here is one direction of this equiconsistency.

Lemma 4.3. wSCFA *implies that* ω_2 *is remarkable in* L.

Proof. I again use the idea of proof of Lemmas 3.4 and 3.10, as well as the notation introduced there. We may again assume that $0^\#$ does not exist because otherwise all V-cardinals are remarkable in L and there is nothing to show.

Let $\kappa = \omega_2^{\mathrm{V}}$, let γ be a singular cardinal, and let $\theta = \gamma^+ = (\gamma^+)^L$. By [4, Proposition 2.4.(4)], it suffices to show that there is a $\bar{\theta} < \kappa$ that is regular in L, such that in $L^{\mathrm{Col}(\omega,\bar{\theta})}$, there is a remarkable embedding $j : L_{\bar{\theta}} \prec L_\theta$ (i.e., such that $j(\mathrm{crit}(j)) = \kappa$).

Let $B = \{\xi < \theta \mid \kappa < \xi \text{ and } \mathrm{cf}(\xi) = \omega\}$. Then for all $\xi \in B$, ξ is singular in L (by covering), and so, C_ξ is defined and has order type less than ξ, where $\vec{C}$ is the global $\square$ sequence of L. Let $A \subseteq B$ be stationary such that the function $\xi \mapsto \mathrm{otp}(C_\xi)$ is constant on A. Let $\mathbb{P}_A$ be the subcomplete forcing to shoot a club of order type ω_1 through A, and let F be a $\mathbb{P}_A$-generic normal, cofinal function, $F : \omega_1 \longrightarrow A$.

Let λ be some regular cardinal, greater than all the objects considered so far, and let $M = \langle H_\lambda, \in, \theta, 0, 1, \ldots, \xi, \ldots\rangle_{\xi<\omega_1}$. Let g be $\mathrm{Col}(\omega_1, M)$-generic over $\mathrm{V}[F]$. In $\mathrm{V}[F][g]$, the statement "there is a surjective function $h : \omega_1^M \longrightarrow M$ and a club $C \subseteq \dot{\theta}^M$ of order type ω_1 such that for all $\xi, \zeta \in C$, $\mathrm{otp}(C_\xi) = \mathrm{otp}(C_\xi)$ (and both are defined)" is true, and this is a Σ_1-statement about M. So, by wSCFA, there is in V a transitive model $\bar{M} = \langle \bar{H}, \in, \bar{\theta}, 0, 1, \ldots, \xi, \ldots\rangle_{\xi<\theta}$, a surjective function $\bar{g} : \omega_1^{\bar{M}} \longrightarrow \bar{M}$, and a club $\bar{C} \subseteq \bar{\theta}$ of order type ω_1 such that for all $\xi, \zeta \in \bar{C}$, $\mathrm{otp}(C_\xi) = \mathrm{otp}(C_\zeta)$, and such that in $\mathrm{V}^{\mathrm{Col}(\omega,\bar{M})}$, there is an elementary embedding $\sigma : \bar{M} \prec M$. Clearly, $\omega_1^{\bar{M}} = \omega_1$, and hence, $\bar{M} \in H_{\omega_2}$ and $\bar{\theta} < \omega_2 = \kappa$.

Let $j = \sigma \upharpoonright L_{\bar{\theta}}$, so that $j : L_{\bar{\theta}} \prec L_\theta$ is elementary.

It follows that $\bar{\theta}$ is a regular cardinal in L for otherwise $C_{\bar{\theta}}$ would be defined, and the usual argument shows that $C'_{\bar{\theta}} \cap \bar{C}$ could have at most one element, while it has to be club, as $\bar{\theta}$ has cofinality ω_1.

The rest of the argument is now straightforward, but let's go through the details. We want to show that j is as wished (except that it exists in $\mathrm{V}^{\mathrm{Col}(\omega,|\bar{M}|)}$, rather than in $L^{\mathrm{Col}(\omega,|\bar{M}|)}$). To see that $j(\mathrm{crit}(j)) = \kappa$, first note that since constants for the countable ordinals were included in M and $\bar{M}$, it follows that $j \upharpoonright (\omega_1^{\mathrm{V}})$ is the identity. So $\mathrm{crit}(j) = \mathrm{crit}(\sigma) = \omega_2^{\bar{M}}$, because $\omega_1^{\bar{M}} = \omega_1^M = \omega_1$, and so $j(\mathrm{crit}(j)) = j(\omega_2^{\bar{M}}) = \omega_2^M = \omega_2 = \kappa$.

So, in $\mathrm{V}^{\mathrm{Col}(\omega,\bar{\theta})}$, there is an embedding $j : L_{\bar{\theta}} \prec L_\theta$ with $\bar{\theta} \in \mathrm{Card}^L$ and $j(\mathrm{crit}(j)) = \kappa$. But then, there is such an embedding in $L^{\mathrm{Col}(\omega,\beta)}$ as well, by absoluteness, since there is a tree searching for such an embedding in $L^{\mathrm{Col}(\omega,\beta)}$, and it is ill-founded in $\mathrm{V}^{\mathrm{Col}(\omega,\beta)}$, hence in $L^{\mathrm{Col}(\omega,\beta)}$. □

Lemma 4.4. *Let κ be remarkable. Then* wSCFA *holds in a forcing extension by a subcomplete forcing.*

Proof. The proof of [4, Theorem 6.3] carries over verbatim. Instead of a countable support iteration, one has to use a revised countable support iteration, and instead of proper forcings, one uses subcomplete ones. There is one place where one has to use the fact that subcompleteness is a local property. I leave it to the reader to go through the steps of the argument. I will also give a detailed argument for a more general equiconsistency result, see Lemma 4.13. □

Taken together, the previous two lemmas show the following.

Theorem 4.5. wSCFA *is equiconsistent with the existence of a remarkable cardinal.*

4.2. **A hierarchy of weak bounded forcing axioms.** Here, I will study bounded versions of the weak forcing axiom, defined as follows.

Definition 4.6. Let Γ be a class of forcings, and let κ be an uncountable cardinal. The *weak κ-bounded forcing axiom for Γ,* $\mathsf{wBFA}(\Gamma, \le\kappa)$, says that whenever $M = \langle |M|, \in, \ldots, R_i, \ldots \rangle_{i<\omega_1}$ is a transitive model of size κ for a language $\mathcal{L}$ with ω_1 many predicates $\langle \dot{R}_i \mid i < \omega_1 \rangle$ and the binary relation symbol $\dot{\in}$, and if $\varphi(x)$ is a Σ_1-formula and $\mathbb{P}$ is a forcing in Γ that forces that $\varphi(\check{M})$ holds, then there is (in V) a transitive model $\bar{M} = \langle |\bar{M}|, \in, \langle \bar{R}_i \mid i < \omega_1 \rangle \rangle$ for $\mathcal{L}$ such that $\varphi(\bar{M})$ holds (in V), and such that in $\mathrm{V}^{\mathrm{Col}(\omega,|\bar{M}|)}$, there is an elementary embedding $j : \bar{M} \prec M$.

$\mathsf{wBFA}(\Gamma)$ is $\mathsf{wBFA}(\Gamma, \le\omega_1)$. If Γ is the class of subcomplete forcings, then wBSCFA is $\mathsf{wBFA}(\Gamma)$, and $\mathsf{wBSCFA}(\le\kappa)$ is $\mathsf{wBFA}(\Gamma, \le\kappa)$. Similarly, we abbreviate these axioms for the class of proper forcings by wBPFA and $\mathsf{wBPFA}(\le\kappa)$.

$\mathsf{wBFA}(\Gamma, <\kappa)$ says that $\mathsf{wBFA}(\Gamma, \le \bar{\kappa})$ holds for every $\bar{\kappa} < \kappa$, and $\mathsf{wBSCFA}(<\kappa)$, $\mathsf{wBPFA}(<\kappa)$ have the obvious meaning.

Let's first observe some simple relationships between the bounded weak forcing axioms and the bounded forcing axioms.

Observation 4.7. *Let Γ be a class of forcings.*

(1) *For any cardinal κ,* $\mathsf{BFA}(\Gamma, \le\kappa)$ *implies* $\mathsf{wBFA}(\Gamma, \le\kappa)$.
(2) $\mathsf{wBFA}(\Gamma)$ *is equivalent to* $\mathsf{BFA}(\Gamma)$.
(3) $\mathsf{wBFA}(\Gamma, \le\omega_2)$ *is equivalent to* $\mathsf{BFA}(\Gamma, \le\omega_2)$.

Proof. For 1., this is obvious, since the characterization of $\mathsf{BFA}(\Gamma, \leq\kappa)$ given in Fact 3.8 clearly implies $\mathsf{wBFA}(\Gamma, \leq\kappa)$.

For 2, the direction from right to left follows from 1. For the converse, assume $\mathsf{wBFA}(\Gamma)$. To prove $\mathsf{BFA}(\Gamma)$, I use the characterization of $\mathsf{BFA}(\Gamma)$ given in Theorem 3.2. So let $\varphi(x)$ be a Σ_1-formula, $a \in H_{\omega_2}$, $\mathbb{P} \in \Gamma$, G generic for $\mathbb{P}$ over V, and assume that in $\mathrm{V}[G]$, $\varphi(a)$ holds. In V, let $\mathsf{TC}(\{a\}) \in X \prec H_{\omega_2}$, $\omega_1 \subseteq X$, and let the size of X be ω_1. Let $M = \langle X, \in, a, \vec{\xi}\rangle_{\xi<\omega_1}$. Then there is a Σ_1 statement $\psi(x)$ such that $\psi(M)$ expresses that $\varphi(a)$ holds. So $\psi(M)$ is true in $\mathrm{V}[G]$, and by $\mathsf{wBFA}(\Gamma)$, there is a $\bar{M} = \langle \bar{X}, \in, \bar{a}, \bar{\vec{\xi}}\rangle$ such that $\psi(\bar{M})$ holds (which means that $\varphi(\bar{a})$ holds), and there is an H generic over V for some forcing, such that in $\mathrm{V}[H]$, there is an elementary $j : \bar{M} \prec M$. By choice of $\vec{\xi}$, $j{\restriction}\omega_1 + 1$ is the identity, hence $\bar{a} = a$. So $\varphi(a)$ holds, and we are done.

For 3, the direction from right to left again follows from 1. For the converse, assume $\mathsf{wBFA}(\Gamma, \leq\omega_2)$ holds. To prove $\mathsf{BFA}(\Gamma, \leq\omega_2)$, I again use the characterization given by Fact 3.8. So let $M = \langle |M|, \in, \vec{R}\rangle$ be a transitive model of size at most ω_2, $\varphi(x)$ a Σ_1-formula, $\mathbb{P} \in \Gamma$ a forcing, and G generic for $\mathbb{P}$ over V, such that $\varphi(M)$ holds in $\mathrm{V}[G]$. If M has size ω_1, then we're done by 2., so we may assume that M has size ω_2. Then, let $E \subseteq \omega_2 \times \omega_2$ code M, in the sense that $\langle \omega_2, E\rangle$ is extensional and well-founded, and such that if π_E is the collapsing isomorphism, then $\pi_E : \langle \omega_2, E\rangle \to \langle |M|, \in\rangle$. Let $\vec{R}'$ be the pullbacks of the $\vec{R}$, so that $\pi_E : \tilde{M} = \langle \omega_2, E, \vec{R}'\rangle \longrightarrow M$ is an isomorphism.

Let $X \prec H_{\omega_3}$ have size ω_2, with $\tilde{M} \in X$ and $\omega_2 \subseteq X$. Clearly, X is transitive. Let $M' = \langle X, \in, \tilde{M}, \vec{\xi}\rangle_{\xi<\omega_1}$. Let $\varphi'(x)$ be the canonical Σ_1-formula such that $\varphi'(M')$ expresses that $\varphi(M)$ holds. So $\varphi'(x)$ says that x is a model in the language of M', and that if N is the structure that arises by collapsing $\langle |x|, \dot{E}^x\rangle$ (where $|x|$ is the universe of x and $\dot{E}^x$ is x's interpretation of $\dot{E}$) and moving the predicates $(\dot{\vec{R}}')^x$, then $\varphi(N)$ holds.

So $\varphi'(M')$ holds in $\mathrm{V}[G]$, and by $\mathsf{wBFA}(\omega_2)$, there is a transitive model $\bar{M}'$ such that $\varphi'(\bar{M}')$ holds and such that in some $\mathrm{V}[H]$, there is an elementary $j : \bar{M}' \prec M'$. Let $\bar{M}' = \langle \bar{X}, \in, \bar{\tilde{M}}, \vec{\xi}\rangle_{\xi<\omega_1}$.

The constants $\vec{\xi}$ ensure that $j{\restriction}(\omega_1 + 1)$ is the identity, so that $j{\restriction}\omega_2^{\bar{M}'}$ is the identity. Let $\bar{M}$ be the transitive structure isomorphic to $\bar{\tilde{M}}$, with the isomorphism $\pi_{\bar{E}} : \bar{\tilde{M}} \longrightarrow \bar{M}$. Since $\varphi'(\bar{M}')$ holds, it follows that $\varphi(\bar{M})$ holds, and it is obvious that $\pi_E \circ \pi_{\bar{E}}^{-1} : \bar{M} \prec M$ is an elementary embedding that exists in V. $\square$

4.3. **Weak bounded forcing axioms and remarkably reflecting cardinals.** I would like to establish a precise correspondence between $\mathsf{wBSCFA}(\leq\lambda)$ and a suitably weakened notion of remarkability.

Definition 4.8. Let κ be an inaccessible cardinal and let $\lambda \geq \kappa$ be a cardinal. κ is *remarkably $\leq\lambda$-reflecting* if the following holds: for any $X \subseteq H_\lambda$ and any formula $\varphi(x)$, if there is a regular cardinal $\theta > \lambda$ such that $\langle H_\theta, \in\rangle \models \varphi(X)$, then there are cardinals $\bar{\kappa} \leq \bar{\lambda} < \bar{\theta} < \kappa$ such that $\bar{\theta}$ is regular, and there is a set $\bar{X} \subseteq H_{\bar{\lambda}}$ such that $\langle H_{\bar{\theta}}, \in\rangle \models \varphi(\bar{X})$, and a generic embedding $j : \langle H_{\bar{\lambda}}, \in, \bar{X}, \bar{\kappa}\rangle \prec \langle H_\lambda, \in, X, \kappa\rangle$ (meaning that j exists in $\mathrm{V}^{\mathrm{Col}(\omega, H_{\bar{\lambda}})}$) such that $j{\restriction}\bar{\kappa} = \mathrm{id}$.

κ is *remarkably $<\lambda$-reflecting* iff it is remarkably $\leq\bar{\lambda}$-reflecting, for every cardinal $\bar{\lambda} < \lambda$ with $\kappa \leq \bar{\lambda}$.

Note that in the notation of the definition, one may of course allow φ to have more than one free variable, and one may use finitely many sets $X_0, \ldots, X_{n-1} \subseteq H_\lambda$ as parameters. If λ is regular, one may insure that $\bar{\lambda}$ is regular by using λ as a parameter in φ, and adding the statement that λ is regular to $\varphi(X)$. Note also that if $\kappa < \lambda$, then it follows that $\bar{\kappa} = \mathrm{crit}(j)$ and $j(\bar{\kappa}) = \kappa$, and if $\kappa = \lambda$, then $\bar{\kappa} = \bar{\lambda}$, and hence $j = \mathrm{id}$. So in the case that $\kappa = \lambda$, we have that $\langle H_{\bar{\kappa}}, \in, \bar{X}\rangle \prec \langle H_\kappa, \in, X\rangle$, and hence $\bar{X} = X \cap H_{\bar{\kappa}}$. The following lemma is not used in the rest of the paper, and is included here to establish some context.

Lemma 4.9. *Let κ be an inaccessible cardinal. The following are equivalent:*

(1) *κ is $+1$-reflecting,*
(2) *κ is remarkably $\leq\kappa$-reflecting.*

Proof. $1 \Longrightarrow 2$: Let $X \subseteq H_\kappa$. Let $\theta > \kappa$ be regular, and suppose that $\langle H_\theta, \in\rangle \models \varphi(X)$. Let $\theta' > \theta$, $X' = H_\kappa$, θ' large enough that $H_\theta \in H_{\theta'}$. Then in $\langle H_{\theta'}, \in\rangle$, the following statement $\psi(X, X')$ holds: "there are regular cardinals $\mu < \nu$ such that $\langle H_\nu, \in\rangle \models \varphi(X)$ and $X' = H_\mu$". This is witnessed by $\mu = \kappa$ and $\nu = \theta$. Since κ is $+1$-reflecting, we can let $Y \prec \langle H_{\kappa^+}, \in, X, X'\rangle$ have size less than κ, $Y \cap \kappa \in \kappa$, $\pi : Y \longrightarrow H$ be the Mostowski-collapse, and $\bar{\theta}' < \kappa$ be a cardinal such that $\langle H_{\bar{\theta}'}, \in\rangle \models \psi(\bar{X}, \bar{X}')$, where $\bar{X}, \bar{X}' = \pi(X, X')$. Let $\bar{\kappa}, \bar{\theta}$ witness that $\psi(\bar{X}, \bar{X}')$ holds in $\langle H_{\bar{\theta}'}, \in\rangle$. Then $\bar{X}' = (H_{\bar{\kappa}})^{H'_{\bar{\theta}}} = H_{\bar{\kappa}}$ and $\langle H_{\bar{\theta}}, \in\rangle \models \varphi(\bar{X})$, where $\bar{\theta}$ is regular in $H_{\theta'}$, and hence in V. Since $Y \cap \kappa \in \kappa$, it follows that $\pi{\restriction}H_{\bar{\kappa}} = \mathrm{id}$ and $\bar{X} = X \cap H_{\bar{\kappa}}$. So we get that $\langle H_{\bar{\kappa}}, \in, \bar{X}\rangle \prec \langle H_\kappa, \in, X\rangle$.

$2 \Longrightarrow 1$: Let $a \in H_{\kappa^+}$, let $\varphi(x)$ be a formula and let θ be a cardinal such that $\langle H_\theta, \in\rangle \models \varphi(a)$. Let $B \subseteq H_{\kappa^+}$. We have to find a $Y \prec \langle H_{\kappa^+}, \in, a, B\rangle$ of size less than κ such that $Y \cap \kappa \in \kappa$, and such that if $\pi : Y \longrightarrow H$ is the Mostowski collapse of Y, then there is a cardinal $\bar{\theta} < \kappa$ such that $\langle H_{\bar{\theta}}, \in\rangle \models \varphi(\pi(a))$.

Let $M = \langle Z, \in, a, B \cap Z\rangle \prec \langle H_{\kappa^+}, \in, a, B\rangle$ such that Z has size κ and $\kappa \subseteq Z$. Note that Z is transitive. Let $f : \kappa \longrightarrow Z$ be a bijection, and let $E = \{\langle\alpha, \beta\rangle \mid f(\alpha) \in f(\beta)\}$, $\alpha_0 = f^{-1}(a)$, $B' = f^{-1}``B$, and $f' = f \cap (\kappa \times \kappa)$. Let $N = \langle H_\kappa, \in, E, B', \alpha_0, f'\rangle$.

Let $\theta' > \theta$ be a regular cardinal with $H_\theta \in H_{\theta'}$. Let $\varphi'(\gamma, e, \xi)$ be the statement saying: $\langle\gamma, e\rangle$ is extensional and well-founded, $\xi \in \gamma$, and there is a cardinal ν such that if σ is the Mostowski collapse of $\langle\gamma, e\rangle$, then $\langle H_\nu, \in\rangle \models \varphi(f(\xi))$. Then $\langle H_{\theta'}, \in\rangle \models \varphi'(\kappa, E, \alpha_0)$. Since κ is remarkably $\leq\kappa$-reflecting, there is a regular $\bar{\theta}' < \kappa$ and a cardinal $\bar{\kappa} < \bar{\theta}'$ such that, letting $\bar{E} = E \cap H_{\bar{\kappa}}$, $\bar{B}' = B \cap H_{\bar{\kappa}}$, $\bar{f}' = f' \cap H_{\bar{\kappa}}$, we have that $\langle H_{\bar{\theta}'}, \in\rangle \models \varphi'(\bar{\kappa}, E \cap H_{\bar{\kappa}}, \alpha_0)$ and such that $\bar{N} = \langle H_{\bar{\kappa}}, \in, \bar{E}, \bar{B}', \alpha_0, \bar{f}'\rangle \prec N$.

Let $\bar{f} : \langle\bar{\kappa}, \bar{E}\rangle \longrightarrow \langle\bar{Z}, \in\rangle$ be the Mostowski collapse, where $\bar{Z}$ is transitive. Let $\bar{B} = \bar{f}``\bar{B}'$, $\bar{a} = \bar{f}(\alpha_0)$, $\bar{M} = \langle\bar{Z}, \in, \bar{a}, \bar{B}\rangle$, and consider the map $j = f \circ \bar{f}^{-1} : \bar{Z} \longrightarrow Z$. It follows that $j : \bar{M} \prec M$ is elementary, because $\bar{M} \models \psi(\vec{a})$ iff $\bar{N} \models$"$\langle\bar{\kappa}, \bar{E}, \bar{B}'\rangle \models \psi(\bar{f}^{-1}(\vec{a}))$" iff $N \models$"$\langle\kappa, E, B'\rangle \models \psi(\bar{f}^{-1}(\vec{a}))$" iff $M \models \psi(f(\bar{f}^{-1}(\vec{a})))$.

Let $Y = \mathrm{ran}(j) = f``\bar{\kappa} \prec M \prec \langle H_{\kappa^+}, \in, a, B\rangle$. Note that $a = f(\alpha_0) \in Y$, and that $Y \cap \kappa = \bar{\kappa}$. This is because if $\gamma \in Y \cap \kappa$, then, letting $\gamma = f(\delta)$, with $\delta < \bar{\kappa}$, it follows that γ is definable from δ in N, since f' is available as a predicate. But since $\bar{N} \prec N$, γ is an ordinal in the domain of $\bar{N}$, hence less than $\bar{\kappa}$. Vice versa, if $\gamma = f(\delta) < \bar{\kappa}$, then δ is definable from γ in N, using the predicate f', hence the same is true in $\bar{N}$, since γ is in the domain of $\bar{N}$, so $\delta < \bar{\kappa}$, so $\gamma \in f``\bar{\kappa} = Y$.

Now let $\bar{\theta}$ witness that $\langle H_{\bar{\theta}'}, \in\rangle \models \varphi'(\bar{\kappa}, E \cap H_{\bar{\kappa}}, \alpha_0)$, i.e., let $\bar{\theta}$ be a cardinal less than $\bar{\theta}'$ such that $\langle H_{\bar{\theta}}, \in\rangle \models \varphi(\bar{f}(\alpha_0))$. Note that $\bar{f}(\alpha_0) = j^{-1}(a)$. We're done, since j^{-1} is the Mostowski-collapse of Y. □

It is possible to make sense of the concept of κ being remarkably $\leq\lambda$-reflecting also for $\lambda < \kappa$, in which case one could let $\bar{\kappa} = \bar{\lambda} = \kappa = \lambda$ and $\bar{X} = X$ in the definition. The result is that κ is reflecting iff κ is remarkably $<\kappa$-reflecting.

Lemma 4.10. *Suppose $\alpha < \kappa$ and assume that κ has the following properties:*

(1) *κ is reflecting,*
(2) *for every $A \subseteq \kappa$, there is a cardinal $\bar{\lambda} < \kappa$, an $\bar{A} \subseteq H_{\bar{\lambda}}$, and a generic embedding $j : \langle H_{\bar{\lambda}}, \in, \bar{A}\rangle \prec \langle H_{\kappa^{+\alpha+1}}, \in, A\rangle$ with $j(\mathrm{crit}(j)) = \kappa$.*

Note: The assumptions of the lemma are of course satisfied if κ is remarkably $\leq\kappa^{+\alpha+1}$-reflecting. Then the set of $\bar{\kappa} < \kappa$ such that $\bar{\kappa}$ is remarkably $\leq\bar{\kappa}^{+\alpha}$-reflecting is stationary in κ.

Proof. Let $C \subseteq \kappa$ be club. Let $\lambda = \kappa^{+\alpha+1}$, $\bar{\lambda} < \kappa$, $j : \langle H_{\bar{\lambda}}, \in, \bar{C}\rangle \prec \langle H_\lambda, \in, C\rangle$ a generic embedding with $\bar{\kappa} = \mathrm{crit}(j)$ and $j(\bar{\kappa}) = \kappa$. It follows that $\bar{\kappa} \in C$. So we're done if we can show that $\bar{\kappa}$ is remarkably $\bar{\kappa}^{+\alpha}$-reflecting. Note that $\bar{\lambda} = \bar{\kappa}^{+\alpha+1}$, because in H_λ, α is definable as the order type of the cardinals greater than κ, so $\alpha \in \mathrm{ran}(j)$, and since $\alpha < \kappa$, $j^{-1}(\alpha) = \alpha$. Now let $X \subseteq H_{\bar{\kappa}^{+\alpha}}$, let $\varphi(x)$ be a formula, and suppose there is a regular cardinal θ such that $H_\theta \models \varphi(X)$. Since κ is reflecting, we may pick $\theta < \kappa$. Note that $X \in H_{\bar{\lambda}}$. Hence, the following statement is true in H_λ: there is an $\bar{X}$, a regular $\lambda' < j(\bar{\kappa})$ and a generic embedding $j' : \langle H_{\lambda'}, \in, \bar{X}\rangle \prec j(\langle H_{\bar{\kappa}^{+\alpha}}, \in, X\rangle)$ with $j'(\mathrm{crit}(j')) = j(\bar{\kappa})$ and a $\theta' < j(\bar{\kappa})$ such that $\langle H_{\theta'}, \in\rangle \models \varphi(\bar{X})$. This is witnessed by $\bar{X} = X$, $\lambda' = \bar{\kappa}^{+\alpha}$, $j' = j{\restriction}H_{\bar{\kappa}^{+\alpha}}$ and $\theta' = \theta$. So, by elementarity, the corresponding statement is true in $H_{\bar{\lambda}}$, with the parameters moved by j^{-1}: there is an $\bar{X}$, a regular $\lambda' < \bar{\kappa}$ and a generic embedding $j' : \langle H_{\lambda'}, \in, \bar{X}\rangle \prec \langle H_{\bar{\kappa}^{+\alpha}}, \in, X\rangle$ with $j'(\mathrm{crit}(j')) = \bar{\kappa}$, and a regular $\theta' < \bar{\kappa}$ such that $\langle H_{\theta'}, \in\rangle \models \varphi(\bar{X})$. Thus, $\bar{\kappa}$ is remarkably $\leq\bar{\kappa}^{+\alpha}$-reflecting. □

The next goal is to establish that the consistency strengths of the levels of the bounded weak subcomplete forcing axiom hierarchy are precisely the steps in the hierarchy of the remarkably reflecting cardinals. Here is one direction of this correspondence.

Lemma 4.11. *Suppose that $\lambda \geq \omega_2$ is a cardinal such that* $\mathsf{wBSCFA}(\leq\lambda)$ *holds. Then ω_2 is remarkably $\leq\lambda$-reflecting in L.*

Proof. As before, we may assume that $0^\#$ does not exist. Fix a formula $\varphi(x)$, a set $X \subseteq L_\lambda$ and a regular cardinal $\theta > \lambda$ such that $\langle L_\theta, \in\rangle \models \varphi(X)$ holds. Let $M \prec \langle H_\lambda, \in, X, \vec{\xi}\rangle_{\xi<\omega_1}$ have size λ, with $\lambda \subseteq M$. Note that since $X \subseteq L_\lambda$, $\dot{X}^M = X$. Let $A \subseteq \theta$ be a stationary set consisting of ω-cofinal ordinals, on which $\xi \mapsto \mathrm{otp}(C_\xi)$ is constant, where $\vec{C}$ is the global $\Box$ sequence of L. Let F be generic for $\mathbb{P}_A$ over V. Let g be $\mathrm{Col}(\omega_1, M)^{\mathrm{V}[F]}$-generic over $\mathrm{V}[F]$. In $\mathrm{V}[F][g]$, the following statement $\Phi(M)$ holds: there are an ordinal θ', a set C and a function g' such that $\theta' > \mathrm{On}^M$, C is club in θ', $\mathrm{otp}(C) = \omega_1^M$, for every $\xi \in C$, C_ξ is defined, for all $\xi, \xi' \in C$, $\mathrm{otp}(C_\xi) = \mathrm{otp}(C_{\xi'})$, $L_{\theta'} \models \varphi(\dot{X}^M)$, $L_{\theta'} \models$"On^M is a cardinal" and $g' : \omega_1 \longrightarrow M$ is a surjection. The truth of $\Phi(M)$ in $\mathrm{V}[F][g]$ is witnessed by $\theta' = \theta$, $C = \mathrm{ran}(F)$ and $g' = g$.

By wBSCFA($\leq\lambda$), let $\bar{M} = \langle H, \in, \bar{X}, \vec{\xi}\rangle_{\xi<\omega_1}$ be transitive such that $\Phi(\bar{M})$ holds and such that there is a generic elementary embedding $j' : \bar{M} \prec M$. As before, $\omega_1^{\bar{M}} = \omega_1$, and by $\Phi(\bar{M})$, $\bar{M} \in H_{\omega_2}$. Let $\bar{\lambda} = \mathrm{On}^{\bar{M}} < \kappa$. Let $\bar{\theta}, \bar{C}, \bar{g}$ witness that $\Phi(\bar{M})$ holds. It follows by familiar arguments that $\bar{\theta}$ is a regular cardinal in L, so that in particular, $L_{\bar{\theta}} = H^L_{\bar{\theta}}$. Moreover, by $\Phi(\bar{M})$, $L_{\bar{\theta}} \models \varphi(\bar{X})$, and $\bar{\lambda}$ is a cardinal in $L_{\bar{\theta}}$ and hence in L. Letting $j = j'{\restriction}L_{\bar{\lambda}}$, it follows that $j : \langle L_{\bar{\lambda}}, \in, \bar{X}\rangle \prec \langle L_\lambda, \in, X\rangle$ is elementary, and that $j(\mathrm{crit}(j)) = \kappa$, since $\mathrm{crit}(j) = \omega_2^{\bar{M}}$. Such an embedding must then exist in $L^{\mathrm{Col}(\omega,\bar{\theta})}$, because a tree searching for such an embedding exists in this model, as $\bar{X}$ and X are in L. This tree is ill-founded, as witnessed by j. □

Working towards the converse, when forcing wBSCFA($\leq\lambda$) over a model with a remarkably $\leq\lambda$-reflecting cardinal, I will want to use an appropriate version of a Menas function, as defined in the following lemma.

Lemma 4.12. *If κ is remarkably $\leq \lambda$-reflecting ($\lambda \geq \kappa$), then the fast function forcing (due to Woodin, and exposited in* [12]*) is κ-c.c., preserves cardinals and the continuum function, preserves that κ is remarkably $\leq\lambda$-reflecting, and adds a "remarkably reflecting Menas function" f, meaning that f is a partial function from κ to κ such that the following holds in $\mathrm{V}[f]$: for every $X \subseteq H_\lambda^{\mathrm{V}[f]}$ and every formula $\varphi(x)$, if there is a regular $\theta > \lambda$ such that $H_\theta^{\mathrm{V}[f]} \models \varphi(X)$, then there are $\bar{\kappa} \leq \bar{\lambda} < \bar{\theta} < \kappa$ and an $\bar{X} \subseteq H_{\bar{\lambda}}^{\mathrm{V}[f]}$ such that*

(∗) $\bar{\theta}$ is regular, $H_{\bar{\theta}}^{\mathrm{V}[f]} \models \varphi(\bar{X})$, $f(\bar{\kappa}) = \bar{\theta}$, and there is a generic embedding $j : \langle H_{\bar{\lambda}}^{\mathrm{V}[f]}, \in, \bar{X}, \bar{\kappa}\rangle \prec \langle H_\lambda^{\mathrm{V}[f]}, \in, X, \kappa\rangle$ with $j{\restriction}\bar{\kappa} = \mathrm{id}$.

Note: All that is needed in the application is that $f(\bar{\kappa}) \geq \bar{\lambda}$.

Proof. The fast function forcing at κ, $\mathbb{F}_\kappa$, consists of partial functions p from a subset of κ of size less than κ to κ such that each $\gamma \in \mathrm{dom}(p)$ is an inaccessible cardinal, is closed under p, and is such that $\overline{\overline{p{\restriction}\gamma}} < \gamma$. The ordering is reverse inclusion. It is shown in [12] that $\mathbb{F}_\kappa$ is κ-c.c., preserves κ as an inaccessible cardinal and doesn't change the continuum function. To prove the conclusion, let $\dot{f}$ be the canonical $\mathbb{F}_\kappa$-name for the generic fast function, i.e., the union of the conditions in the generic filter. Suppose the conclusion fails. Then there is a condition p, an $\dot{X}$, a regular cardinal θ and a formula $\varphi(x)$ such that p forces that $\dot{X} \subseteq H_{\check{\lambda}}^{\mathrm{V}[\dot{f}]}$, that $\check{\theta}$ is regular and that $H_{\check{\theta}}^{\mathrm{V}[\dot{f}]} \models \varphi(\dot{X})$, but also that there are no $\bar{\kappa} \leq \bar{\lambda} < \bar{\theta}$, $\bar{X} \subseteq H_{\bar{\lambda}}^{\mathrm{V}[f]}$ such that (∗) holds. Since $\mathbb{F}_\kappa \subseteq H_\kappa$, $\dot{X}$ may be chosen so that $\dot{X} \subseteq H_\lambda$. Then the statement $\Phi(\mathbb{F}_\kappa, p, \dot{X}, \kappa)$ expressing that p forces with respect to $\mathbb{F}_\kappa$ that $\varphi(\dot{X})$ holds and that κ is inaccessible, is true in H_θ, and it involves only parameters which are subsets of H_λ. Thus, by the fact that κ is remarkably $\leq\lambda$-reflecting, there are $\bar{\kappa} \leq \bar{\lambda} < \bar{\theta} < \kappa$, $\bar{\mathbb{F}}$, $\bar{p}$, $\dot{\bar{X}}$ and a generic j such that $H_{\bar{\theta}} \models \Phi(\bar{\mathbb{F}}, \bar{p}, \dot{\bar{X}}, \bar{\kappa})$ and $j : \langle H_{\bar{\lambda}}, \in, \bar{\mathbb{F}}, \bar{p}, \dot{\bar{X}}, \bar{\kappa}\rangle \prec \langle H_\lambda, \in, \mathbb{F}_\kappa, p, \dot{X}, \kappa\rangle$, where $j{\restriction}\bar{\kappa} = \mathrm{id}$. Let τ be a $\mathrm{Col}(\omega, H_{\bar{\lambda}})$-name for j such that $\mathbb{1}$ forces that τ behaves this way.

Since $\tau^h{\restriction}\bar{\kappa} = \mathrm{id}$ in $\mathrm{V}[h]$, whenever h is $\mathrm{Col}(\omega, H_{\bar{\lambda}})$-generic, it follows that $\bar{p} = p$, and clearly, $\bar{\mathbb{F}} = \mathbb{F}_{\bar{\kappa}}$. Note that $\bar{\kappa}$ is inaccessible, and hence, $p^* = p \cup \langle\bar{\kappa}, \bar{\theta}\rangle \in \mathbb{F}_\kappa$ is a condition extending p. Now let $G \ni p^*$ be $\mathbb{F}_\kappa$-generic over V, and let h be $\mathrm{Col}(\omega, H_{\bar{\lambda}})$-generic over $\mathrm{V}[G]$. Let $f = \bigcup G$, so $f(\bar{\kappa}) = \bar{\theta}$. It follows that $f{\restriction}\bar{\kappa}$ is

$\mathbb{F}_{\bar\kappa}$-generic, and clearly, $j=\tau^h$ lifts to

$$j':\langle H_{\bar\lambda}[f\upharpoonright\bar\kappa],\in,\dot{\bar{X}}^{f\upharpoonright\bar\kappa},\bar\kappa\rangle\prec\langle H_\lambda[f],\in,\dot{\bar{X}},\kappa\rangle$$

and of course, $j'\upharpoonright\bar\kappa=\mathrm{id}$. One might be worried here about the definability of the forcing relation inside $H_{\bar\lambda}/H_\lambda$ if $\lambda/\bar\lambda$ happen to be singular, but instead of pondering this, one may just add the forcing relation for H_λ as a predicate Y, in addition to X, and add to Φ the statement that Y is the forcing relation wrt. $\mathbb{F}_\kappa$ over H_λ. One then gets, in addition, a $\bar Y$ coding the forcing relation wrt. $\bar{\mathbb{F}}$ over $H_{\bar\lambda}$, so j moves the forcing relation correctly, and hence, it lifts to j' as described.

Let $\bar\theta^*$ be the next inaccessible cardinal greater than $\bar\theta$. It follows then that the part of $\mathbb{F}_\kappa$ below p^* is isomorphic to $\mathbb{F}_{\bar\kappa}\times\mathbb{F}_{\bar\theta^*,\kappa}$, where $\mathbb{F}_{\bar\theta^*,\kappa}$ consists of those conditions in $\mathbb{F}_\kappa$ whose domain is contained in $[\bar\theta^*,\kappa)$ (see [12]). The forcing $\mathbb{F}_{\bar\theta^*,\kappa}$ is $<\bar\theta^*$-closed, thus, $H_{\bar\theta}=H_{\bar\theta}^{\mathrm{V}[f\upharpoonright[\bar\kappa,\kappa)]}$, and since $\mathbb{P}_{\bar\kappa}\in H_{\bar\theta}$, $H_{\bar\theta}[f\upharpoonright\bar\kappa]=H_{\bar\theta}^{\mathrm{V}[f]}$. Similarly, $H_\theta[f]=H_\theta^{\mathrm{V}[f]}$, $H_{\bar\lambda}[f\upharpoonright\bar\kappa]=H_{\bar\lambda}^{\mathrm{V}[f]}$ and $H_\lambda[f]=H_\lambda^{\mathrm{V}[f]}$. So we have achieved exactly the situation that p forced not to occur, a contradiction. □

As expected, here is the converse to Lemma 4.11. The proof is very different from the argument that one can force over a model with a remarkable cardinal to obtain wPFA given in [4]. The issue is that the forcings used to make Σ_1 facts about transitive models of size λ true may be very large, and hence, the remarkably $\le\lambda$-reflecting cardinal cannot be used to anticipate them. Only the transitive models can be anticipated.

Lemma 4.13. *Let κ be an inaccessible cardinal. There is a κ-c.c. forcing that is subcomplete, and is such that for every $\lambda\ge\kappa$, if κ is remarkably $\le\lambda$-reflecting, then* wBSCFA($\le\lambda$) *holds in the extension.*

Proof. The first step is to force to add a remarkably reflecting Menas function f, using the fast function forcing from Lemma 4.12, which is κ-c.c. Note that the fast function forcing is much more than countably closed, and in particular, it is subcomplete. Note also that it depends only on κ, not on λ.

Similarly, the second forcing will only depend on f, so it also will be independent of λ, and will produce a model of wBSCFA($\le\lambda$) whenever κ is remarkably $\le\lambda$-reflecting. As a result, if κ is fully remarkable, then the iteration I am about to describe will force wSCFA.

We may work in a universe where κ is $\le\lambda$-remarkably reflecting and f is a remarkably reflecting Menas function. The forcing $\mathbb{P}_\kappa$ will be the result of a length κ iteration, $\langle\langle\mathbb{P}_\alpha\mid\alpha\le\kappa\rangle,\langle\dot{\mathbb{Q}}_\alpha\mid\alpha<\kappa\rangle\rangle$, with revised countable support, which is defined by recursion. For $\alpha<\kappa$, $\mathbb{P}_\alpha$ will be in V_κ, and by standard facts on revised countable support iterations, $\mathbb{P}_\kappa$ will be κ-c.c.

Assume that $\mathbb{P}_\alpha$ has been defined. Then, for every Σ_1-formula $\varphi(x)$ and every $\mathbb{P}_\alpha$-name $\dot N'\subseteq H_{f(\alpha)}$, if, in $\mathrm{V}^{\mathbb{P}_\alpha}$, $\dot N'$ codes a transitive model M for a language of size ω_1, extending the language of set theory, and if there is a subcomplete forcing $\dot R$ in $\mathrm{V}_\kappa^{\mathbb{P}_\alpha}$ that forces $\varphi(M)$, then let $\rho_{\varphi,\dot N'}$ be the minimal ρ such that there is such a forcing whose subcompleteness is verified by ρ.[1] Let $\dot{\mathbb{Q}}_\alpha$ be a $\mathbb{P}_\alpha$-name for the lottery

[1]See Definition 2.3 for the meaning of verifying subcompleteness. Recall also that if ρ verifies the subcompleteness of $\mathbb{R}$, then $\mathbb{R}\in H_\rho$.

sum[2] of all forcings $\dot{\mathbb{R}}$ that are subcomplete and whose subcompleteness is verified by an ordinal less than or equal to $\sup\{\rho_{\varphi,\dot{N}'} \mid \varphi \text{ is a } \Sigma_1\text{-formula}, \ \rho_\varphi < \kappa, \ \dot{N}' \subseteq H_{f(\alpha)}\}$, followed by the collapse $\mathrm{Col}(\omega_1, \overline{\overline{\mathbb{P}_\alpha}})$. Otherwise, $\dot{\mathbb{Q}}_\alpha$ is just a $\mathbb{P}_\alpha$-name for $\mathrm{Col}(\omega_1, \overline{\overline{\mathbb{P}_\alpha}})$.

Letting G be $\mathbb{P}_\kappa$-generic over V, I claim that $\mathsf{wBSCFA}(\leq\lambda)$ holds in $\mathrm{V}[G]$. To see this, work in $\mathrm{V}[G]$, and let $M' \subseteq \lambda$ be such that M' codes a transitive model M of size at most λ for a language of size ω_1 extending the language of set theory, let $\varphi(x)$ be a Σ_1-formula and $\mathbb{R}$ be subcomplete, such that $\Vdash_{\mathbb{R}} \varphi(M)$.

In V, let $\dot{M}'$, $\dot{\mathbb{R}}$ be $\mathbb{P}_\kappa$-names for M', $\mathbb{R}$, and let $p \in G$ and ρ be such that the following statement $\Phi(\mathbb{P}_\kappa, p, \dot{M}', \dot{\mathbb{R}}, \rho)$ is true in some H_θ, where θ is regular: "p forces with respect to $\mathbb{P}_\kappa$ that $\dot{M}'$ codes a transitive model M, $\dot{\mathbb{R}}$ is subcomplete, as verified by the regular cardinal ρ, $\mathcal{P}(H_{\check{\rho}})$ exists, and $\dot{\mathbb{R}}$ forces $\varphi(\check{M})$." In particular, in $\langle H_\theta, \in\rangle$, the statement $\Phi'(\mathbb{P}_\kappa, p, \dot{M}')$, expressing that there is an $\dot{\mathbb{S}}$ and a ρ' such that $\Phi(\mathbb{P}_\kappa, p, \dot{M}', \dot{\mathbb{S}}, \rho')$ holds, is true.

Let's apply the fact that κ is remarkably $\leq\lambda$-reflecting and that f is an appropriate Menas function to H_θ and the formula Φ'. Note that all the parameters occurring in Φ' are subsets of H_λ. So let $\alpha < \bar{\lambda} < \bar{\theta} < \kappa$ be cardinals, $\bar{\theta}$ regular, $\bar{\mathbb{P}}, \bar{p}, \dot{\bar{M}}' \subseteq H_{\bar{\lambda}}$, $j = \tau^h$ a generic elementary embedding, where h is $\mathrm{Col}(\omega, H_{\bar{\lambda}})$-generic, and

$$j : \langle H_{\bar{\lambda}}, \in, \bar{\mathbb{P}}, \bar{p}, \dot{\bar{M}}', \alpha\rangle \prec \langle H_\lambda, \in, \mathbb{P}_\kappa, p, \dot{M}', \kappa\rangle$$

with $j{\restriction}\alpha = \mathrm{id}$ and $f(\alpha) = \bar{\theta}$ and such that $\langle H_{\bar{\theta}}, \in\rangle \models \Phi'(\bar{\mathbb{P}}, \bar{p}, \dot{\bar{M}}')$. It follows that $\bar{p} = p$ and $\bar{\mathbb{P}} = \mathbb{P}_\alpha$.

Since $\langle H_{\bar{\theta}}, \in\rangle \models \Phi'(\mathbb{P}_\alpha, p, \dot{\bar{M}}')$, there are $\dot{\mathbb{S}}, \bar{\rho} \in H_{\bar{\theta}}$ such that

$$\langle H_{\bar{\theta}}, \in\rangle \models \Phi(\mathbb{P}_\alpha, p, \dot{\bar{M}}', \dot{\mathbb{S}}, \bar{\rho})$$

Letting $\mathbb{S} = \dot{\mathbb{S}}^{G\restriction\alpha}$ and $\bar{M}' = (\dot{\bar{M}}')^{G\restriction\alpha}$, $H_{\bar{\theta}}[G{\restriction}\alpha]$ thinks that the subcompleteness of $\mathbb{S}$ is verified by $\bar{\rho}$, $\Vdash_{\mathbb{S}} \varphi(\check{\bar{M}})$ (where $\bar{M}$ is the model coded by $\bar{M}'$), and $\mathcal{P}(H_{\bar{\rho}})^{\mathrm{V}[G\restriction\alpha]} \in H_{\bar{\theta}}[G{\restriction}\alpha]$. As a result, $\mathbb{S}$ is really subcomplete in $\mathrm{V}[G{\restriction}\alpha]$, by [19, p. 115, Cor. 2.3]. Hence, $\dot{\mathbb{Q}}_\alpha = \dot{R} * \mathrm{Col}(\omega_1, \gamma)$, where γ is the size of $\mathbb{P}_\alpha$, and $\dot{R}$ is a lottery sum of subcomplete forcings, at least one of which will make $\varphi(\bar{M})$ true, since $\mathbb{S} \in \mathrm{V}_\kappa[G{\restriction}\alpha]$ and $\dot{\bar{M}}' \subseteq H_{f(\alpha)}$. But once true, $\varphi(\bar{M})$ will persist to further forcing extensions, since φ is Σ_1. So p can be extended to a condition $p' \in \mathbb{P}_{\alpha+1}$ that will force $\varphi(\bar{M})$ to be true in $\mathrm{V}^{\mathbb{P}_\kappa}$. So the set of such conditions is dense below p, so by genericity of G, let's assume that $p' \in G$. Let h be $\mathrm{Col}(\omega, H^{\mathrm{V}}_{\bar{\lambda}})$-generic over $\mathrm{V}[G]$. Then $j = \tau^h$ lifts to an embedding

$$j' : \langle H_{\bar{\lambda}}[G{\restriction}\alpha], \in, \bar{M}'\rangle \prec \langle H_\lambda[G], \in, M'\rangle$$

where $\bar{M}' = \dot{\bar{M}}'^{\,G\restriction\alpha}$, and this embedding j' induces an elementary embedding

$$k : \bar{M} \prec M$$

This shows that $\mathsf{wBSCFA}(\leq\lambda)$ holds in $\mathrm{V}[G]$. □

So, to summarize, we have the following theorem.

[2]The lottery sum of a set of partial orders is just the partial order obtained by taking the disjoint union of the partial orders in the set, and adding a common weakening to all the conditions in the union. Forcing with this lottery sum amounts to generically choosing one of the posets and then forcing with it. The terminology comes from [12]. It was shown in [24] that a lottery sum of subcomplete forcings is subcomplete.

Theorem 4.14. *Let λ be a cardinal.*

(1) *If $\lambda \geq \omega_2$ and* $\mathsf{wBSCFA}(\leq\lambda)$ *holds, then ω_2 is remarkably $\leq\lambda$-reflecting in L.*
(2) *If $\lambda \geq \omega_2$ and* $\mathsf{wBSCFA}(<\lambda)$ *holds, then ω_2 is remarkably $<\lambda$-reflecting in L.*
(3) *If κ is remarkably $\leq\lambda$-reflecting in L, where $\kappa \leq \lambda$, then* $\mathsf{wBSCFA}(\leq\lambda)$ *holds in a κ-c.c. subcomplete forcing extension of L.*
(4) *If κ is remarkably $<\lambda$-reflecting in L, where $\lambda > \kappa$, then* $\mathsf{wBSCFA}(<\lambda)$ *holds in a κ-c.c. subcomplete forcing extension of L.*

Proof. Items 1. and 3. are Lemmas 4.11 and 4.13. 2. follows immediately from Lemma 4.11, and 4. follows by the remark in the beginning of the proof of Lemma 4.13. □

Note that if one defines that κ is remarkably $<\kappa$-reflecting in the way indicated after Definition 4.8, so that it is equivalent to κ being reflecting, then 2. and 4. hold for $\lambda = \kappa$ as well. With the obvious meaning, these points also hold for $\lambda = \infty$. So, since $\mathsf{wBSCFA}(\leq\kappa)$ is equivalent to $\mathsf{BSCFA}(\leq\kappa)$ for $\kappa \leq \omega_2$ (see Observation 4.7), with hindsight, this theorem thus subsumes Theorems 3.6, 3.11 and 4.5.

Since it fits into this context, even though the focus of the present paper is on forcing axioms that are compatible with CH, I would like to point out that in the case of the bounded weak proper forcing axioms, there is a similar one-to-one correspondence between the level of the bounded forcing axiom and the degree of remarkability that ω_2 enjoys in L.

Lemma 4.15. *Let $\lambda \geq \omega_2$ be a cardinal such that* $\mathsf{wBPFA}(\leq\lambda)$ *holds. Then ω_2 is remarkably $\leq\lambda$-reflecting in L.*

Proof. We may assume that $0^\#$ doesn't exist, as otherwise, ω_2 is remarkable in L, and more. Fix a formula $\varphi(x)$, a set $X \subseteq L_\lambda$ and a regular cardinal $\theta > \lambda$ such that $\langle L_\theta, \in\rangle \models \varphi(X)$ holds. Let $M \prec \langle H_\lambda, \in, X, \vec{\xi}\rangle_{\xi<\omega_1}$ have size λ, with $\lambda \subseteq M$. Note that since $X \subseteq L_\lambda$, $\dot{X}^M = X$.

By the usual argument due to Todorčević, there is a proper forcing $\mathbb{P}$ such that if $\mathrm{V}[g]$ is a forcing extension by $\mathbb{P}$, then in $\mathrm{V}[g]$, M has size ω_1 and there is a club $C \subseteq \theta > \mathrm{On}^M$ of order type ω_1 and a function $F : C \longrightarrow \omega$ such that for all $\xi \in C$, $C_\xi^{L_\theta}$ is defined, and such that if $\xi < \zeta$ are both in C and ξ is a limit point of $C_\zeta^{L_\theta}$, then $F(\xi) \neq F(\zeta)$, where $\vec{C}$ is the global $\square$ sequence of L. By $\mathsf{wBPFA}(\leq\lambda)$, there is in V a transitive model $\bar{M} = \langle H, \in, \bar{X}, \vec{\xi}\rangle_{\xi<\omega_1}$ that has size ω_1, and such that, letting $\beta = \mathrm{On}^{\bar{M}}$, there is an ordinal $\bar{\theta} > \beta$ such that in $L_{\bar{\theta}}$, β is a cardinal, and there is a club $\bar{C} \subseteq \bar{\theta}$ of order type ω_1, consisting of ordinals ξ for which $C_\xi^{L_{\bar{\theta}}}$ is defined, and there is a function $\bar{F} : \bar{C} \longrightarrow \omega$ as above, with $\bar{\theta}$ in place of θ and $\bar{C}$ in place of C. It follows then that $\bar{\theta}$ is regular in L, because if it were singular, then $C_{\bar{\theta}}$ would be defined, and it would follow that if $\xi < \zeta$ both are limit points of $C_{\bar{\theta}}$ and members of $\bar{C}$, then $\bar{F}(\xi) \neq \bar{F}(\zeta)$, since $C_{\bar{\theta}} \cap \zeta = C_\zeta = C_\zeta^{L_\beta}$, and so, ξ is a limit point of $C_\zeta^{L_\beta}$. But since $\mathrm{cf}(\bar{\theta}) = \omega_1$, there are ω_1 many members of $\bar{C}$ that are limit points of $C_{\bar{\theta}}$, a contradiction. The rest of the argument is as before. □

The proof of Lemma 4.13 goes through for proper forcing in place of subcomplete forcing almost without changes, and so, Theorem 4.14 holds for proper forcing in place of subcomplete forcing as well, and similarly for semiproper forcing.

4.4. **How the weak bounded hierarchy fits in the bounded hierarchy.** Keeping in mind Observation 4.7.3, which implies that $\mathsf{wBSCFA}(\leq\omega_2)$ is equivalent to $\mathsf{BSCFA}(\leq\omega_2)$, it is an obvious question how wBSCFA relates to $\mathsf{BSCFA}(\leq\omega_3)$. Again, the situation is similar to the proper forcing case, see [4].

Lemma 4.16. (1) *If* wSCFA *is consistent, then so is* $\mathsf{wSCFA} + \forall\kappa \geq \omega_2 \quad \Box_\kappa$.
(2) wSCFA *does not imply* $\mathsf{BSCFA}(\leq\omega_3)$.

Proof. To see 1, note that if one forces wSCFA over L, using a remarkable κ, so that κ becomes ω_2 in the extension, then for any L-cardinal $\lambda \geq \kappa$, the $\Box_\lambda$ sequence of L will be a $\Box_\lambda$ sequence in the extension. 2. follows from 1. together with Corollary 3.14, which shows that $\mathsf{BSCFA}(\leq\omega_3)$ implies the failure of $\Box_{\omega_2}$. □

Since wSCFA does not imply $\mathsf{BSCFA}(\leq\omega_3)$, the question arises how the consistency strengths of these axioms relate.

Lemma 4.17. *The consistency strength of* $\mathsf{BSCFA}(\leq\omega_3)$ *is strictly higher than that of* wSCFA*.*

Proof. By Corollary 3.14, $\mathsf{BSCFA}(\leq\omega_3)$ implies the failure of $\Box_{\omega_2}$. It was shown by Jensen that this, in turn, implies that ω_3 is Mahlo in L (see [16, p. 286]). By Observation 4.7, $\mathsf{BSCFA}(\leq\omega_3)$ implies $\mathsf{wBSCFA}(\leq\omega_3)$. By Lemma 4.11, $\mathsf{wBSCFA}(\leq\omega_3)$ implies that $\kappa = \omega_2$ is remarkably $\leq\omega_3$-reflecting in L. It follows that L_{ω_3} is a model of $\mathsf{ZFC}+$ "κ is remarkable", and by Theorem 4.5, that's the consistency strength of wSCFA. So $\mathsf{BSCFA}(\leq\omega_3)$ is strictly stronger.

In fact, it is known that if $\tau \geq \omega_2$ is a regular cardinal such that for some stationary $S \subseteq \tau$, any two stationary subsets of S reflect simultaneously at some $\gamma < \tau$ of uncountable cofinality, then $\Box(\tau)$ fails - a proof of this fact can be found in [14]. In particular, SFP_τ implies the failure of $\Box(\tau)$, and so $\mathsf{BSCFA}(\leq\omega_3)$ implies the failure of both $\Box(\omega_2)$ and $\Box(\omega_3)$, and in particular of $\Box_{\omega_2}$. It is also known by [8] or [17, §3, Corollary 9.1] that SFP_{ω_2} implies that $\omega_2^{\omega_1} = \omega_2$, and hence that $2^\omega \leq \omega_2$. This constellation has very high consistency strength, and implies that the axiom of determinacy holds in $L(\mathbb{R})$ - see [26]. This is certainly much stronger than the consistency strength of wPFA, a remarkable cardinal. □

The corresponding result holds for proper forcing.

Lemma 4.18. $\mathsf{BPFA}(\leq\omega_3)$ *has consistency strength strictly higher than* wPFA*.*

Proof. $\mathsf{BPFA}(\leq\omega_1, \leq\omega_3)$ implies the failure of $\Box(\omega_3)$ and of $\Box(\omega_2)$. It also implies that $2^\omega = \omega_2$. Again, this implies $\mathsf{AD}^{L(\mathbb{R})}$, and is thus much stronger than the consistency strength of wPFA, a remarkable cardinal. □

So, combining Theorem 4.14, Lemma 4.10 and Observation 4.7.3 results in the following.

Lemma 4.19. *Writing* $<$ *for "has lower consistency strength than", the following inequalities hold:*

$$\mathsf{BSCFA} < \mathsf{BSCFA}(\leq\omega_2) < \mathsf{wSCFA}(\leq\omega_3) < \mathsf{wSCFA}(\leq\omega_4) < \ldots < \mathsf{wSCFA} < \mathsf{BSCFA}(\leq\omega_3)$$

and

$$\mathsf{BPFA} < \mathsf{BPFA}(\leq\omega_2) < \mathsf{wPFA}(\leq\omega_3) < \mathsf{wSCFA}(\leq\omega_4) < \ldots < \mathsf{wPFA} < \mathsf{BPFA}(\leq\omega_3).$$

It would be interesting to explore consequences of the weak forcing axioms, as well as strengthenings of these axioms and their corresponding remarkably reflecting cardinals in which some restrictions are imposed on the forcings allowed to add the requested generic elementary embeddings, such as having to preserve ω_1, having to preserve stationary subsets of ω_1, or having to belong to the same class of forcings used to define the particular forcing axioms, etc.

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