Def: Let $i: \mathbb{P} \to \mathbb{Q}$. Define $i_*: V^{\mathbb{P}} \to V^{\mathbb{Q}}$ by:

$$i_*(\sigma) = \{\langle i_*(\tau), i(p)\rangle \mid \langle \tau, p\rangle \in \sigma\}.$$

Often, I'll write i for i_*.

Lemma: In M, let $i: \mathbb{P} \to \mathbb{Q}$ be complete.

(a) If H is $(M, \mathbb{Q})$-generic then letting $G = i^{-1}[H]$ we have for $\tau \in M^{\mathbb{P}}$: $\tau^G = i_*(\tau)^H$.

(b) If $\varphi(\vec{x})$ is absolute for transitive ZFC models then

$$p \Vdash_{\mathbb{P}} \varphi(\vec{\tau}) \iff i(p) \Vdash_{\mathbb{Q}} \varphi(i_*(\vec{\tau}))$$

(c) If i is dense then (b) holds for arbitrary formulas.

Proof: (a) Induction on τ.

[$\subseteq$] Let $a \in \tau^G$. Let $\langle \sigma, p\rangle \in \tau$ be s.t. $p \in G$, $a = \sigma^G$. Then $\langle i_*(\sigma), i(p)\rangle \in i_*(\tau)$, $i(p) \in H$, so $i_*(\sigma)^H \in i_*(\tau)^H$, and inductively, $i_*(\sigma)^H = \sigma^G = a$.

[$\supseteq$] Let $a \in i_*(\tau)^H$. Let $\langle i_*(\sigma), i(p)\rangle \in i_*(\tau)$ s.t. $i(p) \in H$, $a = i_*(\sigma)^H$, $\langle \sigma, p\rangle \in \tau$. Then $p \in G$, so $\sigma^G \in \tau^G$. Inductively, $\sigma^G = i_*(\sigma)^H = a$.

(b) $\boxed{\Rightarrow}$ Suppose $p \Vdash_{\mathbb{P}} \varphi(\vec{\tau})$. To show that $i(p) \Vdash_{\mathbb{Q}} \varphi(i_*(\vec{\tau}))$, let H be $(M, \mathbb{Q})$-generic with $i(p) \in H$ - we have to show that $M[H] \models \varphi(i_*(\vec{\tau})^H)$. Let $G = i^{-1}[H]$. ~~By~~ Then $p \in G$, and since $p \Vdash_{\mathbb{P}} \varphi(\vec{\tau})$, $M[G] \models \varphi(\vec{\tau}^G)$. By (a), $\vec{\tau}^G = i_*(\vec{\tau})^H$, so $M[G] \models \varphi(i_*(\vec{\tau})^H)$. So by absoluteness, $M[H] \models \varphi(i_*(\vec{\tau})^H)$.

[Note: if i is dense, then $M[G] = M[H]$, so this goes thru for any φ.]

$\boxed{\Leftarrow}$ Assume $i(p) \Vdash_{\mathbb{Q}} \varphi(i_*(\vec{\tau}))$ yet $p \not\Vdash_{\mathbb{P}} \varphi(\vec{\tau})$. Let $q \leq p$ be s.t. $q \Vdash_{\mathbb{P}} \neg\varphi(\vec{\tau})$. Since $\neg\varphi$ is absolute, the forward direction applies, giving $i(q) \Vdash_{\mathbb{Q}} \neg\varphi(i_*(\vec{\tau}))$. But $i(q) \leq i(p) \Vdash_{\mathbb{Q}} \varphi(i_*(\vec{\tau}))$. $\lightning$

(c): Included in (b). $\boxtimes$

Def: By an automorphism of $\mathbb{P}$, we mean an automorphism of the structure $\langle|\mathbb{P}|, \leq_{\mathbb{P}}, \mathbb{1}_{\mathbb{P}}\rangle$.

$\mathbb{P}$ is weakly homogeneous if for any $p, q \in \mathbb{P}$ there is an automorphism i of $\mathbb{P}$ s.t. $i(p) \parallel q$.

A name $\tau \in V^{\mathbb{P}}$ is symmetric if $i_*(\tau) = \tau$ for every automorphism i of $\mathbb{P}$.

Note: $\check{x}$ is symmetric.

Lemma: Let $\mathbb{P} \in M$ be weakly homogeneous in M, $\varphi(\vec{x})$ a formula, and $\vec{\tau} \in M^{\mathbb{P}}$ symmetric names (in M). Then $\mathbb{1} \Vert^{M}_{\mathbb{P}} \varphi(\vec{\tau})$

[i.e. either $\mathbb{1} \Vdash^{M}_{\mathbb{P}} \varphi(\vec{\tau})$ or $\mathbb{1}_{\mathbb{P}} \Vdash^{M}_{\mathbb{P}} \neg\varphi(\vec{\tau})$]

Proof: Working in M, if this fails, there are $p, q \in \mathbb{P}$ s.t. $p \Vdash \varphi(\vec{\tau})$ and $q \Vdash \neg\varphi(\vec{\tau})$. Let $i \in \mathrm{Aut}(\mathbb{P})$ be s.t. $i(p) \parallel q$. Then $i_*(\vec{\tau}) = \vec{\tau}$, so $i(p) \Vdash_{\mathbb{P}} \varphi(\vec{\tau})$ by the previous lemma. Since $q \Vdash \neg\varphi(\vec{\tau})$, a common extension of $q, i(p)$ would force both $\varphi(\vec{\tau})$ and its negation. $\lightning$ $\boxtimes$

<u>Cor.</u>: Let $\mathbb{P}$ be weakly homogeneous in M. If $a \in M[G]$, $a \subseteq M$, is of the form

$$a = \{b \mid M[G] \models \varphi(b, c_0, \ldots, c_{n-1})\}$$

where $c_i = \tau_i^G$ for a symmetric τ_i, then $a \in M$.

<u>Pf.</u>: $a = \{b \mid M \models \mathbb{1} \Vdash_{\mathbb{P}} \varphi(\check{b}, \vec{\tau})\}$. $\boxtimes$

Products

Def: Given posets $\mathbb{P}, \mathbb{Q}$ we equip $\mathbb{P}\times\mathbb{Q}$ with the coordinate-wise partial ordering and set $\mathbb{1}_{\mathbb{P}\times\mathbb{Q}} = \langle \mathbb{1}_{\mathbb{P}}, \mathbb{1}_{\mathbb{Q}}\rangle$.

Let $i_0 : \mathbb{P} \longrightarrow \mathbb{P}\times\mathbb{Q}$ be defined by $i_0(p) = \langle p, \mathbb{1}_{\mathbb{Q}}\rangle$ and $i_1 : \mathbb{Q} \longrightarrow \mathbb{P}\times\mathbb{Q}$ by $i_1(q) = \langle \mathbb{1}_{\mathbb{P}}, q\rangle$.

Note: i_0, i_1 are complete embeddings.

Pf: Given $\langle p,q\rangle \in \mathbb{P}\times\mathbb{Q}$, $p\in\mathbb{P}$ is a reduction because f.a. $p'\leq p$ $i_0(p') = \langle p', \mathbb{1}_{\mathbb{Q}}\rangle \,||\, \langle p, \mathbb{1}_{\mathbb{Q}}\rangle$. (similar for i_1).

Lemma: Let G be $(M, \mathbb{P}_0\times\mathbb{P}_1)$-generic. Then $G_0 = i_0^{-1}[G]$ is $(M,\mathbb{P}_0)$-generic and $G_1 = i_1^{-1}[G]$ is $(M,\mathbb{P}_1)$-generic, and $G = G_0\times G_1$.

Proof: Genericity of G_0, G_1 follows from completeness of i_0, i_1.

$G \subseteq G_0\times G_1$: Given $\langle p,q\rangle \in G$, $\langle p, \mathbb{1}\rangle \in G$ so $p\in G_0$. Also, $\langle \mathbb{1}, q\rangle \in G$, so $q\in G_1$.

$G_0\times G_1 \subseteq G$: Given $\langle p,q\rangle \in G_0\times G_1$, $\langle p,\mathbb{1}\rangle, \langle \mathbb{1}, q\rangle \in G$. Let $\langle r,s\rangle \in G$, $\langle r,s\rangle \leq \langle p,\mathbb{1}\rangle, \langle \mathbb{1},q\rangle$. Then $\langle r,s\rangle \leq \langle p,q\rangle \Rightarrow \langle p,q\rangle\in G$. ⊠

Thm: Let $\mathbb{P}_0, \mathbb{P}_1$ be posets in M, $G_0 \subseteq \mathbb{P}_0$, $G_1 \subseteq \mathbb{P}_1$. TFAE:

(a) $G_0 \times G_1$ is $(M, \mathbb{P}_0 \times \mathbb{P}_1)$-generic.

(b) G_0 is $(M, \mathbb{P}_0)$-gen. & G_1 is $(M[G_0], \mathbb{P}_1)$-gen.

(c) G_1 is $(M, \mathbb{P}_1)$-gen. & G_0 is $(M[G_1], \mathbb{P}_0)$-gen.

In either case, $M[G_0 \times G_1] = M[G_0][G_1] = M[G_1][G_0]$.

Proof: Since $\mathbb{P}_0 \times \mathbb{P}_1 \cong \mathbb{P}_1 \times \mathbb{P}_0$ it suffices to prove (a)$\Leftrightarrow$(b).

(a)$\Rightarrow$(b): By the previous lemma, $G_0 = i_0^{-1}[G]$, and G_0 is $(M, \mathbb{P}_0)$-generic.

Claim: G_1 is $(M[G_0], \mathbb{P}_1)$-generic.

Proof: Let $D \in M[G_0]$ be dense in $\mathbb{P}_1$. Let $D = \dot{D}^{G_0}$, $\dot{D} \in M^{\mathbb{P}_0}$. Let $p \in G_0$, $p \Vdash^{M}_{\mathbb{P}}$ "$\dot{D}$ is dense in $\check{\mathbb{P}}_1$".

We want to define from $\dot{D}$ a ~~dense~~ subset of $\mathbb{P}_0 \times \mathbb{P}_1$ lying in M that's dense below an element of $G_0 \times G_1$ so that we can use $(M, \mathbb{P}_0 \times \mathbb{P}_1)$-genericity of $G_0 \times G_1$.

Let $D' = \{\langle r,s\rangle \in \mathbb{P}_0 \times \mathbb{P}_1 \mid r \Vdash^M_{\mathbb{P}_0} \check{s} \in \dot{D}\}$.

<u>Claim</u>: D' is dense below $\langle p, \mathbb{1}\rangle$.

<u>Proof</u>: Let $\langle u,v\rangle \leq \langle p,\mathbb{1}\rangle$, and let H be $(M, \mathbb{P}_0\times\mathbb{P}_1)$-generic, with $\langle u,v\rangle \in H$. Then $H = H_0 \times H_1$ where H_i is $(M,\mathbb{P}_i)$-gen. and $u \in H_0$, so since $u \leq p$, $\dot{D}^{H_0}$ is dense in $\mathbb{P}_1$. So let $s \leq v$, $s \in \dot{D}^{H_0}$. And let $r \leq u$, $r \in H_0$ be s.t. $r \Vdash^M_{\mathbb{P}_0} \check{s} \in \dot{D}$. Then $\langle r,s\rangle \leq \langle u,v\rangle$, $\langle r,s\rangle \in D'$. $\boxtimes$

So let $\langle r,s\rangle \in (G_0\times G_1) \cap D'$ [note that $\langle p,\mathbb{1}\rangle \in G_0 \times G_1$ and $D' \in M$). Since $r \Vdash \check{s} \in \dot{D}$ and $r \in G_0$, $s \in \dot{D}^{G_0} = D$. Since $s \in G_1$, $D \cap G_1 \neq \emptyset$.

(b)$\Rightarrow$(a): $G_0 \times G_1$ is upward closed and nonempty. If $\langle p_0,p_1\rangle, \langle q_0,q_1\rangle \in G_0\times G_1$ then there is $r_0 \leq p_0, q_0$, $r_0 \in G_0$ and $r_1 \leq p_1, q_1$, $r_1 \in G_1$. Then $\langle r_0, r_1\rangle \in G_0 \times G_1$, $\langle r_0,r_1\rangle \leq \langle p_0,p_1\rangle, \langle q_0,q_1\rangle$, so $G_0\times G_1$ is a filter. To see it is $(M, \mathbb{P}_0\times\mathbb{P}_1)$-generic, let $D \subseteq \mathbb{P}_0\times\mathbb{P}_1$, $D \in M$ be dense.

<u>Note</u>: for every $q \in \mathbb{P}_1$, the set

$$D^q = \{p' \in \mathbb{P}_0 \mid \exists q' \leq q \ \langle p', q' \rangle \in D\}$$

is dense in $\mathbb{P}_1$ because given $p \in \mathbb{P}_0$, we can find $\langle p', q' \rangle \leq \langle p, q \rangle$, $\langle p', q' \rangle \in D$. Then $p' \leq p$ & $p' \in D^q$ as witnessed by q'.

Now let $D_1 = \{q \in \mathbb{P}_1 \mid \exists p \in G_0 \ \langle p, q \rangle \in D\}$.

Then $D_1 \in M[G_0]$, and we claim that D_1 is dense in $\mathbb{P}_1$:

Let $q \in \mathbb{P}_1$. Since D^q is dense in $\mathbb{P}$ and $D^q \in M$, we can pick $p' \in G_0 \cap D^q$. Let $q' \leq q$ witness this. Then $\langle p', q' \rangle \in D$ ~~and~~ and $p' \in G_0$, so $q' \in D_1$. Since $q' \leq q$, D is dense in $\mathbb{P}_1$.

So let ~~q_1~~ q ~~ξ~~ $\in D_1 \cap G_1$, and let $p \in G_0$ witness this. Then $\langle p, q \rangle \in D \cap (G_0 \times G_1)$. $\square$ (b)$\Rightarrow$(a)

$M[G_0 \times G_1] = M[G_0][G_1]$:

$\subseteq$: $G_0, G_1 \in M[G_0][G_1]$, so $G_0 \times G_1 \in M[G_0][G_1]$, so $M[G_0 \times G_1] \subseteq M[G_0][G_1]$

$\supseteq$: $G_0 \in M[G_0 \times G_1]$ so $M[G_0] \subseteq M[G_0 \times G_1]$, and $G_1 \in M[G_0 \times G_1]$ so $M[G_0][G_1] \subseteq M[G_0 \times G_1]$. $\boxtimes$

<u>Lemma</u>: Let $\mathbb{P}_0, \mathbb{P}_1 \in M$, and let $G_0 \times G_1$ be $(M, \mathbb{P}_0 \times \mathbb{P}_1)$-generic over M. Then $M[G_0] \cap M[G_1] = M$.

<u>Proof</u>: Suppose not. Let a be $\in$-minimal in $M[G] \cap M[G_0] \setminus M$. So $a \subseteq M$. Let $\alpha \in On^M$ be large enough that $a \subseteq S = V_\alpha^M$. Let $\dot{a}_0 \in M^{\mathbb{P}_0}$, $\dot{a}_1 \in M^{\mathbb{P}_1}$ be s.t. $a = \dot{a}_0^{G_0} = \dot{a}_1^{G_1}$. Let i_0, i_1 be the canonical complete embeddings $\mathbb{P}_0, \mathbb{P}_1$ into $\mathbb{P}_0 \times \mathbb{P}_1$ ~~projections from $\mathbb{P}_0 \times \mathbb{P}_1$ onto $\mathbb{P}_0, \mathbb{P}_1$~~. Let Γ be the canonical $\mathbb{P}_0 \times \mathbb{P}_1$-name for the generic filter, and let Γ_0, Γ_1 be the canonical $\mathbb{P}_0 \times \mathbb{P}_1$-names for $\check{i}_0^{-1}``\Gamma$, $\check{i}_1^{-1}``\Gamma$.

Write $\mathrm{val}(\tau, \mathbb{Q}, F)$ for the evaluation of the $\mathbb{Q}$-name τ by the filter F. We have in $M[G_0][G_1]$:

$$\mathrm{val}(\dot{a}_0, \mathbb{P}_0, G_0) = \mathrm{val}(\dot{a}_1, \mathbb{P}_1, G_1) \subseteq \check{S}.$$

So let $\langle p_0, p_1 \rangle \in G_0 \times G_1$ be s.t.

$$\langle p_0, p_1 \rangle \Vdash^M_{\mathbb{P}_0 \times \mathbb{P}_1} \text{“}\mathrm{val}((\dot{a}_0)^\vee, \check{\mathbb{P}}_0, \Gamma_0) = \mathrm{val}((\dot{a}_1)^\vee, \check{\mathbb{P}}_1, \Gamma_1) \subseteq \check{S}\text{”}$$

~~Thus~~ So whenever $H_0 \times H_1$ is $(M, \mathbb{P}_0 \times \mathbb{P}_1)$-generic, and $\langle p_0, p_1 \rangle \in H_0 \times H_1$, then $\dot{a}_0^{H_0} = \dot{a}_1^{H_1} \subseteq S$. (*)

~~Now fix $x \in M$.~~ [Strictly speaking, this is true in $M[H_0 \times H_1]$, but it's absolute.]

(1) Let $t \in a$. Then $p_0 \Vdash^{M}_{\mathbb{P}_0} \check{t} \in \dot{a}_0$.

Proof: ~~Since~~ $\cancel{t \in \dot{a}_0^{G_0}}$, ~~there is a~~ $\cancel{q_0 \leq p_0,\ q_0 \in G}$, ~~s.t.~~ $\cancel{q_0 \Vdash^{M}_{\mathbb{P}_0} \check{t} \in \dot{a}_0}$. Suppose p_0 ~~doesn't force this~~ not.

Then there is an $r_0 \leq p_0$ s.t. $r_0 \Vdash^{M}_{\mathbb{P}_0} \check{t} \notin \dot{a}_0$.

Let G_0' be $(M[G_1], \mathbb{P}_0)$-generic with $r_0 \in G_0'$.

Then $t \notin \dot{a}_0^{G_0'}$. But by (*):

$$\dot{a}_0^{G_0'} = \dot{a}_1^{G_1} = a \ni t. \quad \lightning$$

(2) Let $t \in M$, $t \notin a$. Then $p \Vdash^{M}_{\mathbb{P}_0} \check{t} \notin \dot{a}_0$.

Pf: As before. This time, if not, there is $r_0 \leq p$ s.t. $r_0 \Vdash^{M}_{\mathbb{P}_0} \check{t} \in \dot{a}_0$. But if we let G_0' be $(M[G_1], \mathbb{P}_0)$-gen. with $r_0 \in G_0'$, then again by (*), $\dot{a}_0^{G_0'} = \dot{a}_1^{G_1} = a$. $\lightning$

Hence, $a = \{t \in s \mid p \Vdash^{M}_{\mathbb{P}_0} \check{t} \in \dot{a}_0\} \in M$.

(2) Let $t \in M$, $t \notin a$. Then $\neg\, p_0 \Vdash^{M}_{\mathbb{P}_0} \check{t} \in \dot{a}_0$.

Proof: Clearly, if we had $p_0 \Vdash^{M}_{\mathbb{P}_0} \check{t} \in \dot{a}_0$, then $t \in \dot{a}_0^{G_0} = a$. $\lightning$.

So $a = \{t \in s \mid p \Vdash^{M}_{\mathbb{P}_0} \check{t} \in \dot{a}_0\} \in M$. $\boxtimes$