Homework set 7

Forcing in Set Theory, Fall 2024
Dr. Gunter Fuchs — Submit by 11/26/24

Let M be as usual.

Problem 1 (15 points):
In M, let S be a stationary subset of ω_1, and let $\mathbb{P} = \mathbb{P}_S$ be the poset defined in M as follows.

Conditions are subsets of S which are closed. That is, $p \in \mathbb{P}$ if $p \subseteq S$ and whenever α is a limit point of p, then $\alpha \in p$. In particular, p must be bounded in ω_1.

The ordering is by end extension. That is, $p \le q$ if $q = p \cap \alpha$ for some α ($\alpha = \sup(\{\xi + 1 \mid \xi \in q\}$, the least ordinal greater than all elements of a, works). Show:

(1) In M, $c.c.(\mathbb{P}) = (2^{\aleph_0})^+$.

(2) If G is $(M, \mathbb{P})$-generic, then $\bigcup G$ is a club subset of S.

(3) $\mathbb{P}$ is separative.

(4) In M, $\mathbb{P}$ is ω_1-distributive. So forcing with $\mathbb{P}$ adds no new ω-sequences of ordinals, and hence preserves $(\aleph_1)^M$.

(5) Solovay has shown that S can be partitioned into ω_1 stationary subsets of ω_1 (in M). Use this to conclude that for an appropriate choice of a stationary subset T of ω_1 (in M), forcing with $\mathbb{P}_T$ over M destroys the stationarity of some subset of ω_1.

Hint for part (4): Let $\vec{D} = \langle D_n \mid n < \omega \rangle$ be a sequence of dense open subsets of $\mathbb{P}$, and let $q \in \mathbb{P}$ be a condition below which we want to find an element of the intersection of the D_n's. Let $S, \mathbb{P}, \vec{D}, q \in \mathrm{V}_\theta$, θ regular, so that V_θ is sufficiently correct in V. Find a countable X such that $\langle X, \in \rangle \prec \langle \mathrm{V}_\theta, \in \rangle$ and $\alpha := X \cap \omega_1 \in S$. Let $\pi : H \longrightarrow X$ be the inverse of the Mostowski collapse, H transitive. Then $\pi{\restriction}\alpha = \mathrm{id}{\restriction}\alpha$ and $\pi(\alpha) = \omega_1$. Let $\bar{S}, \bar{\mathbb{P}}, \bar{q}$ be the preimages of $S, \mathbb{P}, q$ under π. Then $q = \bar{q}$ and $\bar{S} = S \cap \alpha$. Let G be $(H, \bar{\mathbb{P}})$-generic with $q \in G$. Then $q' = (\bigcup G) \cup \{\alpha\}$ is as wished.

Problem 2 (10 points):
Let κ be an uncountable cardinal in M, let $\mathbb{P} \in M$, and let G be $(M, \mathbb{P})$-generic, where $\mathbb{P}$ has cardinality less than κ in M. Write A^* for $p[T]^{M[G]}$, for $A \in M$ which is κ-u.B., as witnessed by a κ-absolutely complementing pair of trees (T, U). Show the following points.

(1) Let $A, B \in M$ be κ-u.B. in M. Then $A \subseteq B$ iff $A^* \subseteq B^*$.

(2) Let A be Borel in M. Then A^* is Borel in $M[G]$. [You can prove this by induction on the Borel complexity of A.]