# Homework set 6

Forcing in Set Theory, Fall 2024
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Let $M$ be as usual.

**Problem 1 (5 points):**
In $M$, let $\mathbb{P}$ be a separative poset. Let $G$ be $(M, \mathbb{P})$-generic. Show that $G$ is $M$*-directed* in the sense that whenever $D \subseteq G$ and $D \in M$, then $D$ has a lower bound in $G$, i.e., there is a $p \in G$ such that for all $q \in D$, $p \leq q$.
*Hint:* You can use Problem 1 of Homework Set 2 here.

**Problem 2 (10 points):**
For a cardinal $\kappa$, a partial order $\mathbb{P}$ is $<\kappa$-distributive if the intersection of fewer than $\kappa$ open dense subsets of $\mathbb{P}$ is dense (and of course open) in $\mathbb{P}$.

Now let $M$ be as usual, and in $M$, let $\kappa$ be an infinite cardinal and $\mathbb{P}$ a separative partial order. Show that the following are equivalent:

(1) Whenever $G$ is $(M, \mathbb{P})$-generic, then $M[G] \cap ({}^{<\kappa}M) \subseteq M$.

(2) $M \models$ "$\mathbb{P}$ is $\kappa$-distributive".

*Note:* Problem 1 comes in handy for the direction from (1) to (2). Separativity is not needed for the other direction.

**Problem 3 (10 points):**
Suppose that in $M$, $\kappa$ is an uncountable regular cardinal, $\mathbb{P}$ is $\kappa$-closed and $S \subseteq \kappa$ is stationary, meaning that $S$ has nonempty intersection with every club subset of $\kappa$; cf. Problem 3 of HW set 4.

Now let $G$ be $(M, \mathbb{P})$-generic. Show that $S$ is stationary in $M[G]$.

*Remark:* Problem 2 shows that $\kappa$-distributivity is a weak form of $\kappa$-closure. Problem 3 gives a consequence of $\kappa$-closure that does not follow from $\kappa$-distributivity. Maybe this will be a problem for the next homework set...