Homework set 5

Forcing in Set Theory, Fall 2024
Dr. Gunter Fuchs — Submit by 11/12/24

Problem 1 (4 points):
Let $\mathbb{P}$ be a partial order for which $cl(\mathbb{P})$, the least ordinal α such that there is a weakly decreasing sequence $\langle p_\xi \mid \xi < \alpha\rangle$ of conditions in $\mathbb{P}$ that does not have a lower bound, exists. Observe that $\mathbb{P}$ is $cl(\mathbb{P})$-closed, that $cl(\mathbb{P})$ is the largest ordinal with this property, and that $cl(\mathbb{P})$ is a regular cardinal.

Problem 2 (16 points):
Let $\mathbb{P}$ be a partial order. Let $c.c.(\mathbb{P})$ be the least cardinal κ such that $\mathbb{P}$ satisfies the κ-c.c.

1. Let $D \subseteq \mathbb{P}$ be dense. Then $c.c.(\mathbb{P}) = c.c.(\mathbb{P}{\restriction}D)$, where $\mathbb{P}{\restriction}D$ is the restriction of the ordering of $\mathbb{P}$ to D. **(2 P.)**
2. For $p \in \mathbb{P}$, let $\mathbb{P}_p = \langle |\mathbb{P}_p|, \leq_{\mathbb{P}} \cap |\mathbb{P}_p| \times |\mathbb{P}_p|\rangle$, where $|\mathbb{P}_p| = \{q \mid q \leq_{\mathbb{P}} p\}$. Say that p is *stable* (in $\mathbb{P}$), if for all $q \leq p$:
$$c.c.(\mathbb{P}_q) = c.c.(\mathbb{P}_p).$$
Show that the set S consisting of all conditions stable in $\mathbb{P}$ is dense and open in $\mathbb{P}$, and that every $p \in S$ is stable in $\mathbb{P}{\restriction}S$. **(3 P.)**
3. Let A and B be antichains in $\mathbb{P}$. Say that A refines B if the following hold:
   (a) for every $a \in A$ there is a $b \in B$ such that $a \leq b$.
   (b) for every $b \in B$ there is an $a \in A$ such that $a \leq b$.

   Show that in this case, $\mathrm{card}(B) \leq \mathrm{card}(A)$.
   Show further that if A and B are maximal antichains in $\mathbb{P}$, then they have a common refinement. **(3 P.)**
4. Show that $c.c.(\mathbb{P})$ is either finite or uncountable. **(4 P.)**
   Hint: Assuming there are arbitrarily large finite antichains in $\mathbb{P}$, you have to show that there is an infinite one. If the set of atoms of $\mathbb{P}$ is dense, then choose a maximal antichain A in $\mathbb{P}$ which consists of atoms. The cardinality of A is then $\geq$ the cardinality of any maximal antichain of $\mathbb{P}$, by part 3. If the atoms of $\mathbb{P}$ are not dense in $\mathbb{P}$, then pick a condition below which there is no atom, and construct directly an infinite antichain below it.
5. Show that, assuming $c.c.(\mathbb{P})$ is infinite, it is regular. **(4 P.)**
   Hint: By part 4, $\lambda := c.c.(\mathbb{P})$ is uncountable, and by parts 1 and 2, one may assume that every condition in $\mathbb{P}$ is stable. Assuming towards a contradiction that λ is singular with cofinality $\kappa < \lambda$, say, fix a maximal antichain A of cardinality $\geq \kappa$. Using part 3, show that $C = \{c.c.(\mathbb{P}_p) \mid p \in A\}$ is unbounded in λ (case 1) or that $\lambda \in C$ (case 2). In each case, you can construct an antichain of cardinality λ refining A.

Problem 3 (5 points):
Let κ be a cardinal, and let $\mathbb{P}$ be a poset in which the set of atoms is not dense. Show that $\mathbb{P}$ is not both κ-closed and κ-c.c.
Hint: Otherwise, working below a condition that has no atom below it, construct an antichain of size κ, using the κ-closure of $\mathbb{P}$ to keep the construction going.