Homework set 4

Forcing in Set Theory, Fall 2024
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Problem 1 (10 points):
A partial order $\mathbb{P}$ has property (K) (the Knaster property) if:

(K) *Every uncountable subset of* $\mathbb{P}$ *has an uncountable subset consisting of pairwise compatible conditions.*

Show:

1. Property (K) implies the ccc.
2. Let I be any set, and let J be countable. Show that $\mathrm{Fn}(I, J)$ has Property (K).
3. If $\mathbb{P}$ and $\mathbb{Q}$ have property (K) then so has $\mathbb{P} \times \mathbb{Q}$.
4. More generally, if $\langle \mathbb{P}_i \mid i \in I \rangle$ is a sequence of partial orders with property (K), each of which has a maximal element, then the product $\prod_{i \in I} \mathbb{P}_i$ with finite support has property (K).

Note: Part 4 can be used to show part 2. The product $\prod_{i \in I} \mathbb{P}_i$ with finite support is defined as follows. Conditions are sequences $\langle p_i \mid i \in I \rangle$, so that for every $i \in I$, $p_i \in \mathbb{P}_i$, and such that the set $\{i \in I \mid p_i \neq \mathbb{1}_{\mathbb{P}_i}\}$ (the support) is finite. The ordering is defined as expected: $\vec{p} \leq \vec{q}$ iff for all $i \in I$, $p_i \leq_{\mathbb{P}_i} q_i$.

Problem 2 (5 points):
Let κ be a regular cardinal. Recall that a set $C \subseteq \kappa$ is club (closed and unbounded) in κ if it is unbounded in κ, that is, for every $\alpha < \kappa$ there is a $\beta \in C$ with $\alpha < \beta$, and if it is closed in κ, that is, whenever $\alpha < \kappa$ is a limit point of C, then $\alpha \in C$. Here, α is a limit point of C if $C \cap \alpha$ is unbounded in α.

Now, if $f : \kappa \longrightarrow \kappa$, then let's say that $\alpha < \kappa$ is a closure point of f if for all $\beta < \alpha$, $f(\beta) < \alpha$.

Show that if κ is an uncountable regular cardinal and $f : \kappa \longrightarrow \kappa$, then the set of closure points of f forms a club subset of κ.

Problem 3 (10 points):
Let M be a countable and transitive with ZFC^M, and let $\mathbb{P} \in M$ be a partial order satisfying the κ-c.c. in M, where κ is an uncountable regular cardinal in M. Let G be $(M, \mathbb{P})$-generic, and let $C \in M[G]$ be club in κ. Show that there is a $\bar{C} \in M$, club in κ, so that $\bar{C} \subseteq C$.

Calling a set $S \subseteq \kappa$ stationary iff it intersects every club subset of κ, conclude that $\mathbb{P}$ preserves stationary subsets of κ, that is, if $S \subseteq \kappa$ is stationary in M, then it is stationary in $M[G]$.