Homework set 3

Forcing in Set Theory, Fall 2024
Dr. Gunter Fuchs — Submit by 10/08/24

Let M be countable and transitive, with ZFC^M.

Problem 1 (5 points):
Let $I, J \in M$, $\mathbb{P} = \mathrm{Fn}(I, J)$. Let G be an $(M, \mathbb{P})$-generic filter, and let $f = \bigcup G$. Show that if N is any transitive ZFC model with $f \in N$, then $G \in N$. Conclude that $M[G]$ is the minimal model of ZFC containing M as a subset and f as an element. In this sense, $M[G] = M[f]$.

Problem 2 (10 points):
Let $\mathbb{P}, \mathbb{Q} \in M$ be partial orders, and let $i : \mathbb{P} \longrightarrow \mathbb{Q}$ be a dense embedding (see HW set 1, Problem 3, part 8 for the meaning of this), $i \in M$. Show that if G is $(M, \mathbb{P})$-generic and $H = i[G]^{\geq_{\mathbb{Q}}}$ is the upward closure of $i[G]$, then H is $(M, \mathbb{Q})$-generic. Vice versa, if H is $(M, \mathbb{Q})$-generic, then $i^{-1}[H]$ is $(M, \mathbb{P})$-generic. Conclude that if G is $(M, \mathbb{P})$-generic, then $M[G] = M[i[G]^{\geq_{\mathbb{Q}}}]$, and if H is $(M, \mathbb{Q})$-generic, then $M[H] = M[i^{-1}[H]]$.

Problem 3 (5 points):
Let $\mathbb{P}, \mathbb{Q} \in M$ be partial orders, and let $i : \mathbb{P} \longrightarrow \mathbb{Q}$ be such that $i \in M$ and

1. $p_0 \leq_{\mathbb{P}} p_1 \implies i(p_0) \leq_{\mathbb{Q}} i(p_1)$.
2. whenever $p \in \mathbb{P}$ and $q \leq_{\mathbb{Q}} i(p)$, there is a $p' \leq p$ such that $i(p') \leq q$.

Show that then, forcing with $\mathbb{P}$ also adds a $\mathbb{Q}$-generic filter. That is, if G is $(M, \mathbb{P})$-generic, then there is an $H \in M[G]$ that is $(M, \mathbb{Q})$-generic.

Problem 4 (5 points):
Let κ be a cardinal in M, and let $\mathbb{P} = \mathrm{Fn}(\kappa \times \omega, 2)$. Suppose G is $(M, \mathbb{P})$-generic. Prove the following statements that were sketched in class:

1. $\bigcup G : \kappa \times \omega \longrightarrow 2$ is a total function.
2. For $\alpha < \kappa$, let $g_\alpha : \omega \longrightarrow 2$ be defined by $g_\alpha(n) = (\bigcup G)(\alpha, n)$. Then $g_\alpha \notin M$.
3. Further, for $\alpha < \beta < \kappa$, $g_\alpha \neq g_\beta$.

Total score: **(25 P.)**