Homework set 2

Forcing in Set Theory, Fall 2024
Dr. Gunter Fuchs — Submit by 10/01/24

Problem 1 (6 points):
Let M be a ctm with $(\mathsf{ZF})^M$, and let $\mathbb{P} \in M$ be a partial order. Show that $\mathbb{P}$ is separative iff for all $p, q \in \mathbb{P}$:

$$(p \Vdash^M_{\mathbb{P}} \check{q} \in \Gamma) \iff (p \leq q);$$

see Homework set 1 for the meaning of separativity. Γ is the canonical name for the generic filter.

Problem 2 (12 points):
Recall that in Homework set 1, fixing a partial order $\mathbb{P}$, we defined a hull operator $C(X) = \perp\perp X$, for $X \subseteq \mathbb{P}$, and we let $\mathbb{B} = \{X \subseteq \mathbb{P} \mid C(X) = X\}$ and $\mathbb{B}^+ = \mathbb{B} \setminus \{\emptyset\}$. We have seen that $i : \mathbb{P} \longrightarrow \mathbb{B}^+$, defined by $i(p) = C(\{p\})$, is a dense embedding. We want to show here that $\mathbb{B}$ is a complete Boolean algebra.

To this end, define for $X, Y \in \mathbb{B}$:

$$X \cdot Y = X \cap Y,\ -X = \perp X \text{ and } X + Y = C(X \cup Y).$$

Further, let $\langle A_j \mid j \in J\rangle$ be a sequence of elements of $\mathbb{B}$. Define

$$\sum_{j\in J} A_j = C(\bigcup_{j\in J} A_j) \text{ and } \prod_{j\in J} A_j = \bigcap_{j\in J} A_j.$$

Show:

1. $\prod_{i\in J} A_i \in \mathbb{B}$. **(2 P.)**
2. $-\prod_{i\in J} A_i = \sum_{i\in I}(-A_i)$. **(2 P.)**
3. $-\sum_{i\in J} A_i = \prod_{i\in J}(-A_i)$ – we even have that for arbitrary $\vec{B} \in {}^J\mathcal{P}(\mathbb{P})$, $\perp\bigcup_{i\in J} B_i = \bigcap_{i\in J}(\perp B_i)$. **(2 P.)**
4. Set $0 := \emptyset \in \mathbb{B}, 1 := \mathbb{P} \in \mathbb{B}$. Show that for $A \in \mathbb{B}$, we have: $A \cdot (-A) = 0$, $A + (-A) = 1$ and $-(-A) = A$. **(2 P.)**
5. For A, B, $C \in \mathbb{B}$, we have:
   (a) $A \cdot (B + C) = (A \cdot B) + (A \cdot C)$. **(2 P.)**
   (b) $A + (B \cdot C) = (A + B) \cdot (A + C)$. **(2 P.)**

It follows from 1-5 that $\langle \mathbb{B}, +, -, 0, 1\rangle$ is a Boolean algebra. Recall that we ordered $\mathbb{B}$ by inclusion. This is the same ordering we get by setting $X \leq Y$ iff $X = X \cdot Y$ iff $Y = X + Y$ (which can always be done in a Boolean algebra). So with this ordering, $\mathbb{B}$ is complete, and hence, we have embedded $\mathbb{P}$ densely into a complete Boolean algebra.

Problem 3 (7 points):
Let $\dot{A}$ be a new unary predicate symbol, and let $M = (M, A)$ be a transitive $\mathsf{ZFC}_{\dot{A}}$-model. Let $A \subseteq M^{\mathbb{P}} \times \mathbb{P}$. If G is $(M, \mathbb{P})$-generic, then let

$$A^G = \{\tau^G \mid \exists p \in G \quad \langle \tau, p\rangle \in A\}.$$

Define a formula "$p \Vdash^*_{\mathbb{P}} \dot{A}(y)$" in the language including $\dot{A}$, with three free variables $(p, \mathbb{P}, y)$ so that we have:

1. If $(M, A, \in) \models (p \Vdash^*_{\mathbb{P}} \dot{A}(\tau))$, G is $(M, \mathbb{P})$-generic and $p \in G$, then:
$$(M[G], A^G) \models \dot{A}(\tau^G).$$
2. If G is $(M, \mathbb{P})$-generic and $(M[G], A^G) \models \dot{A}(\tau^G)$, then there is a $p \in G$ with $(p \Vdash^*_{\mathbb{P}} \dot{A}(\tau))^{(M,A)}$.

Total score: **(25 P.)**