Homework set 1

Forcing in Set Theory, Fall 2024
Dr. Gunter Fuchs — Submit by 9/24/24

Fix a countable transitive model M of ZFC.

Problem 1 (5 points):
Let $\mathbb{P} \in M$ be a partial order. Show that the following properties are equivalent for a set $D \subseteq \mathbb{P}$ and a condition $p \in \mathbb{P}$:

1. D is predense below p (i.e., for every $q \leq p$, there is an $r \in D$ such that $r || q$.)
2. Whenever G is $\mathbb{P}$-generic over M and $p \in G$, then $G \cap D \neq \emptyset$.

Problem 2 (10 points):
Let $\mathbb{P} \in M$, and set

$$\text{gen} = \{G \mid G \text{ is } (M, \mathbb{P}) - \text{generic}\},$$

and for $p \in \mathbb{P}$ let

$$\text{gen}_p = \{G \mid p \in G \in \text{gen}\}.$$

1. Say that an *antichain* in $\mathbb{P}$ is a set of pairwise incompatible conditions. Note that any antichain can be extended to a maximal one, and show that if $A \subseteq \mathbb{P}$ is a maximal antichain, then
$$\text{gen} = \dot{\bigcup_{p \in A}} \text{gen}_p$$
(the disjoint union). **(2 P.)**
2. Show that $p \leq q \implies \text{gen}_p \subseteq \text{gen}_q$. **(2 P.)**
3. Show that
$$p \parallel q \iff \text{gen}_p \cap \text{gen}_q \neq \emptyset.$$
(2 P.)
4. Say that $\mathbb{P}$ is *separative* iff for all $p, q \in \mathbb{P}$:
$$p \not\leq q \implies \exists r \leq p \quad r \perp q.$$
Show that $\mathbb{P}$ is separative iff the reverse implication of 2 holds. **(4 P.)**

Problem 3 (10 points):
Let $\mathbb{P}$ be a partial order. For $A \subseteq \mathbb{P}$ and $p \in \mathbb{P}$, write $p \perp A$ to express that $p \perp a$, for every $a \in A$. Set:

$$\perp A = \{p \mid p \perp A\}.$$

Show for $X \subseteq \mathbb{P}$:

1. $\perp X$ is downward closed, i.e., if $p \leq q \in \perp X$, then $p \in \perp X$. **(1 P.)**
2. $X \cap \perp X = \emptyset$. **(1 P.)**
3. $X \subseteq \perp\perp X$. **(1 P.)**
4. $X \subseteq Y \subseteq \mathbb{P} \implies \perp Y \subseteq \perp X$. **(1 P.)**
5. $\perp X = \perp\perp\perp X$. **(1 P.)**
6. The map $C : \mathcal{P}(\mathbb{P}) \longrightarrow \mathcal{P}(\mathbb{P})$ defined by
$$C(X) = \perp\perp X$$
is a hull operator, i.e., $X \subseteq C(X)$, $X \subseteq Y \implies C(X) \subseteq C(Y)$, and $C(C(X)) = C(X)$. **(1 P.)**
7. Let $\mathbb{B}$ be the partial order consisting of all $X \subseteq \mathbb{P}$ with $C(X) = X$, ordered by inclusion, and let $\mathbb{B}^+$ be the restriction of $\mathbb{B}$ to $\{X \in \mathbb{B} \mid X \neq \emptyset\}$. Check: if $X, Y \in \mathbb{B}$, then $X \cap Y \in \mathbb{B}$. In particular, if $X, Y \in \mathbb{B}^+$, then $X \perp^{\mathbb{B}^+} Y \iff X \cap Y = \emptyset$. **(1 P.)**

8. Let $i : \mathbb{P} \longrightarrow \mathbb{B}^+$ be defined by

$$i(p) = C(\{p\}).$$

Show that i is a dense embedding, that is:

(a) $p \leq q \implies i(p) \subseteq i(q)$, **(1 P.)**

(b) $p \perp q \implies i(p) \perp^{\mathbb{B}^+} i(q)$, **(1 P.)**

(c) the range of i is dense in $\mathbb{B}^+$. **(1 P.)**

Total score: **(25 P.)**