Club degrees of rigidity and almost Kurepa trees

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Abstract

A highly rigid Souslin tree T is constructed such that forcing with T turns T into a Kurepa tree. Club versions of previously known degrees of rigidity are introduced, as follows: for a rigidity property P, a tree T is said to have property P on clubs if for every club set C (containing 0), the restriction of T to levels in C has property P. The relationships between these rigidity properties for Souslin trees are investigated, and some open questions are stated.

1 Introduction

A wide range of set-theoretic research has been undertaken concerning automorphisms and isomorphisms of trees. [GS64] analyzes the number of isomorphism types of trees, depending on their cardinality, while [Jec72] focuses on the possible size of the automorphism group of a normal ω_1-tree, and it turns out that if such a tree has no Souslin subtree, then the size of its automorphism group is either finite, $2^{\aleph_0}$ or $2^{\aleph_1}$, while there is much more flexibility for the number of automorphisms of a Souslin tree. In [Jen69], Jensen announced some results concerning the existence of rigid as well as homogeneous Souslin trees; see also [DJ74]. Answering a question of Jech, Todorcevic showed in [Tod80] that it is provable in ZFC that there is a rigid Aronszajn tree (this was also shown by Abraham in [Abr79], independently). Hamkins and the author analyzed rigidity properties of a Souslin tree, having to do with what happens after forcing with the tree, in [FH09]. Motivated by [Jec74] and [AS85], I am here comparing these notions of rigidity to their corresponding "club degrees". For example, a tree T is rigid on clubs if for every club subset of ω_1, the restriction of the tree to levels in the club is rigid. Any notion of rigidity can be strengthened in this way, and I analyze the relationships between them, and between the club and the non-club degrees. This is done in Section 3.

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Before that analysis, I develop a notion of forcing that adds a Souslin tree T with a peculiar property: T is rigid (and even absolutely totally rigid, a notion to be defined later), yet forcing with T turns T into a Kurepa tree, i.e., a normal ω_1-tree with at least $\aleph_2$ branches. Hamkins and the author had previously tried to tackle the question whether such a tree can exist, unsuccessfully. Such a tree basically shows that there is a huge gap between absolute total rigidity and the unique branch property (saying that forcing with the tree adds precisely one branch). It was shown in [FH09] that the unique branch property (UBP) implies total rigidity, but that this implication cannot be reversed, and that absolute total rigidity does not imply the UBP. The present construction shows that an absolutely totally rigid Souslin tree can fail to have the UBP very badly. However, the tree is not rigid on clubs, and it is an interesting open question whether there can be such a tree that is. I would like to mention two somewhat related results. Firstly, Jensen had constructed a Souslin tree with ω_2 automorphisms (see [DJ74]), which is easily seen to have the property that forcing with it turns it into a Kurepa tree – the point of the present construction is that the tree is highly rigid. In a different direction, Jin and Shelah ([JS97]) constructed a model in which there is no Kurepa tree, yet there is an ω-distributive Aronszajn tree such that forcing with that tree adds a Kurepa tree.

Let's fix some notation and terminology. A *tree* T is a well-founded partial order on a set of *nodes* with precisely one minimal element, called the *root*, and such that the predecessors of any given node are linearly ordered by the tree order. Two nodes p and q are *compatible* if they are comparable, i.e., $p \leq_T q$ or $q \leq_T p$, otherwise they are *incompatible*. The *height of a node* p, denoted by $|p|$, is the order-type of the set of its predecessors. The *height of a tree* is the least strict upper bound of the heights of its nodes. If α is an ordinal less than the height of a tree T, then the α-th level of T, denoted by $T(\alpha)$, is the collection of nodes of T which have height α. If T is a tree and X is a set of ordinals, then $T|X$ is the ordered structure resulting from restricting the ordering of T to those nodes whose height belongs to X. This need not necessarily be a tree, but it will be if $0 \in X$. If p is a node in T, then T_p denotes the cone above p, so it is the tree consisting of the nodes in T that are above p, and inheriting the ordering from T. An $\aleph_1$-*tree* is a tree of height $\aleph_1$ each of whose levels is at most countable. A tree T is *normal* if it satisfies the following three conditions: Firstly, every node has to have successors with arbitrarily large heights below the height of the tree (trees with that property are sometimes called *well-pruned*). Secondly, T has *unique limits*: Nodes of limit height are determined by their predecessors (i.e., if p and q are nodes of limit height and they have the same predecessors, then $p = q$). Lastly, every node of T must have incompatible immediate successors. An *antichain* in a tree is a set of pairwise incompatible nodes. Most trees under consideration will be *uniformly splitting*, which means that every node has the same number of immediate successors. A *Souslin Tree* is a normal $\aleph_1$-tree that has no uncountable antichain. A *branch* of a tree is a set of nodes that is downward closed under and linearly ordered by the tree relation. A branch of a tree is *cofinal* in the tree if its order type is equal to the height of the tree (equivalently, if it intersects every level of the tree). A *Kurepa*

Tree is a normal $\aleph_1$-tree that has at least $\aleph_2$ many cofinal branches. Note that a Souslin Tree has no cofinal branch. When forcing with a tree, the order is turned around, so that the stronger conditions are higher up in the tree. This way, two nodes are compatible in the tree order iff they are compatible in the forcing sense. So a Souslin tree will satisfy the countable chain condition as a notion of forcing. It is also easy to see and well-known that it is σ-distributive.

A tree T is *rigid* if it has no non-trivial automorphism. It is *totally rigid* if whenever p and q are distinct nodes of T, it follows that T_p and T_q are non-isomorphic. By regarding a tree as a notion of forcing, one arrives at interesting degrees of rigidity, as introduced in [FH09]: A normal tree T has the *unique branch property* (UBP) if forcing with T adds precisely one new cofinal branch. It is *absolutely rigid* if after forcing with T, T is still rigid (i.e., if T forces that it is rigid). It is *absolutely totally rigid* if T forces that T is totally rigid, and it has the absolute UBP if T forces that T has the UBP.

It was shown in [FH09] that the following diagram exhibits all the ZFC-provable implications between these notions of rigidity.

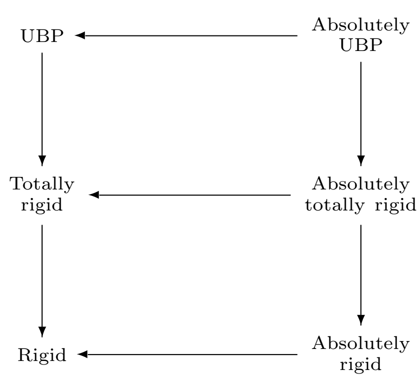

Figure 1: Implication Diagram

When proving the nonimplications in the diagram, we focused on uniformly splitting Souslin trees, mostly in order to exclude trivialities. For example, it is known that it is provable in ZFC that there is a rigid Aronszajn tree, and it is consistent that every Aronszajn tree is special. When forcing with a special, rigid Aronszajn tree, ω_1 is collapsed, and hence such a tree will become countable. Of course, countable trees have continuum many branches, and so this would be an example of a rigid ω_1-tree that is not UBP. By requiring the tree to be Souslin, though, it is insured that forcing with it will preserve ω_1. Similarly, the requirement that the trees be uniformly splitting is there to exclude trivial counterexamples. For example, it would be easy to build an ω_1-tree with the property that for every countable ordinal α and every natural number $n \geq 2$, there is exactly one node p at level α of the tree that has exactly n immediate successors. Such a tree will be rigid, but for a trivial reason.

So, in order to show that the implication from UBP to rigidity cannot be reversed, we constructed a 2-splitting Souslin tree that's rigid but doesn't have the unique branch property. The first part of the present paper addresses the question just how badly a rigid Souslin tree can fail to have the unique branch property. Continuing the idea of looking at the properties a tree may have in forcing extensions obtained by forcing with the tree itself, let's define that a tree is *almost Kurepa* if after forcing with T, T is a Kurepa tree (i.e., $T \Vdash$"$\check{T}$ is a Kurepa tree"). What is shown in the next part of the paper is that it is consistent that there is an absolutely totally rigid Souslin tree that's almost Kurepa. So this shows that a rigid Souslin tree may fail the unique branch property in the strongest imaginable way.

The Souslin almost Kurepa tree constructed in Section 2 has the property that when restricting it to its limit levels, it has $\aleph_2$ many automorphisms. This leads quite naturally to the club degrees of rigidity, defined by saying that for any rigidity property P, a Souslin tree T has property P *on clubs* if for every club subset C of ω_1, the restriction $T|C$ of T to the levels in C has property P. The question arises whether there can be a Souslin almost Kurepa tree that's rigid on clubs. Rigidity on clubs is a very natural notion, in view of the fact that automorphisms of the Boolean algebra of a Souslin tree give rise to club automorphisms of the Souslin tree. So club rigidity of a Souslin tree implies the rigidity of its Boolean algebra, which is a very natural property when viewing Souslin trees as notions of forcing. So Section 3 is devoted to an analysis of the club degrees of rigidity and how they relate to the previously introduced rigidity degrees.

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2 A rigid Souslin almost Kurepa tree

The goal of this section is to produce an (absolutely totally) rigid Souslin almost Kurepa tree by forcing.

Definition 2.1. An ordinal is *appropriate* if it is a limit of limit ordinals or the successor of a limit ordinal.

Suppose t is a normal tree of appropriate height α. Let C be a subset of α which is unbounded in $\sup\alpha$. A function $\sigma : t|C \longrightarrow t|C$ is *totally far from extending to an automorphism of* t if for every non maximal $p \in t$ and every isomorphism $\pi : t_p \longrightarrow t_q$ (where $|p| = |q|$ and $p \neq q$), there is a node $p' \geq p$ s.t. $\sigma(p') \neq \pi(p')$.

Using the terminology introduced above, I am now going to define the forcing notion $\mathbb{S}$, designed to add an absolutely rigid Souslin tree that's almost Kurepa, over a model of CH. Conditions u are of the form $\langle t, \langle \pi_\alpha \mid \alpha \in I \rangle\rangle$, where we write $t = t_u$, $I = I_u$ and $\langle \pi_\alpha \mid \alpha \in I \rangle = \vec{\pi}^u$. We demand:

1. t is a normal, ω-splitting[1] tree of countable height η, where η is appropriate. The set of nodes of t is a countable ordinal. Let's call trees of this form *standard*.

2. I is a countable subset of ω_2.

3. Every π_α ($\alpha \in I$) is an automorphism of $t|\text{Lim}$, where Lim is the set of limit ordinals (less than ω_1).

4. Letting Γ be the group of automorphisms of $t|\text{Lim}$ generated by $\vec{\pi}^u$, every $\sigma \in \Gamma$ is totally far from extending to an automorphism of t.

Condition 4 is there to insure that automorphisms can be sealed. The ordering is the obvious one: $\langle s, \vec{\sigma}\rangle \leq \langle t, \vec{\pi}\rangle$ iff s end-extends t, $\text{dom}(\vec{\sigma}) \supseteq \text{dom}(\vec{\pi})$ and for all $\alpha \in \text{dom}(\vec{\pi})$, $\sigma_\alpha \upharpoonright t = \pi_\alpha$.

Of course, imposing the apparently strong requirement of being totally far from extending to an automorphism on our forcing conditions necessitates additional arguments when constructing extensions of a given condition.

2.1 Closure

Lemma 2.2. *Suppose that t is a standard tree of height α, which is a limit of limits, $I \in [\omega_2]^{\leq\omega}$ and $\vec{\pi} = \langle \pi_i \mid i \in I\rangle$ is a sequence of functions such that the domain of every π_i is $t|\text{Lim}$, and such that for unboundedly many $\beta < \alpha$, $\langle t|\beta, \langle \pi_i \upharpoonright (t|\beta) \mid i \in I\rangle\rangle$ is a forcing condition in $\mathbb{S}$. Then $\langle t, \vec{\pi}\rangle$ is a forcing condition.*

Proof. It is clear that $\langle t, \vec{\pi}\rangle$ satisfies requirements 1.-3. of the definition of the forcing conditions. It remains to be shown that every non-identity automorphism σ in the group of automorphisms Γ of $t|\text{Lim}$ generated by $\vec{\pi}$ is totally far from extending to an automorphism of t. So let p, $q \in t$ be distinct nodes at the same level β of t. Let $\pi : t_p \longrightarrow t_q$ be an isomorphism. Let $\beta' > \beta$, $\beta' < \alpha$ be appropriate such that $\langle t|\beta', \langle \pi_i \upharpoonright (t|\beta') \mid i \in I\rangle\rangle$ is a forcing condition. Then $\sigma \upharpoonright \beta'$ belongs to the group of automorphisms of $(t|\beta')|\text{Lim}$ generated by $\langle \pi_i \upharpoonright (t|\beta') \mid i \in I\rangle$, and $\pi \upharpoonright (t|\beta') : (t|\beta')_p \longrightarrow (t|\beta')_q$ is an isomorphism. So, since $\sigma \upharpoonright \beta'$ is totally far from extending to an automorphism of $t|\beta'$, there is a $p' \geq p$ such that $\sigma(p') \neq \pi(p')$. This shows that σ is totally far from extending to an automorphism of t. □

Lemma 2.3. $\mathbb{S}$ *is countably closed.*

Proof. Let $\langle u_n \mid n < \omega\rangle$ be a strictly decreasing sequence of conditions in $\mathbb{S}$, $u_n = \langle t_n, \vec{\pi}^n\rangle$. Define $u = \langle t, \vec{\pi}\rangle$ in the natural way by setting $t = \bigcup_{n<\omega} t_n$, $I_u = \bigcup_{n<\omega} \text{dom}(\vec{\pi}^n)$, and for $\alpha \in I_u$, let $\pi_\alpha = \bigcup_{\alpha \in \text{dom}(\vec{\pi}^n)} \pi^n_\alpha$.

Obviously, t is a normal tree of countable height. The height of t will be a limit of limit ordinals, unless the heights of the t_ns are eventually constant

[1] The tree should be uniformly splitting, but it is not important for the construction whether it is ω-splitting or n-splitting, for some n.

and are the successor of a limit ordinal, in which case this will be the height of t. In any case, the height of t will be appropriate. The domain of $\vec{\pi}$ is clearly a countable subset of ω_2, and it is easily seen that $\vec{\pi}$ is a sequence of automorphisms of $t|\mathrm{Lim}$. Let's check that every σ in the group of automorphisms of $t|\mathrm{Lim}$ generated by $\vec{\pi}$ is totally far from extending to an automorphism of t. Let p and q be distinct non-maximal nodes in t on the same level of t. Let $\tau : t_p \longrightarrow t_q$ be an isomorphism. Let $\sigma = (\pi_{i_0})^{j_0} \circ \ldots \circ (\pi_{i_{m-1}})^{j_{m-1}}$, where each $j_k = \pm 1$. Find n large enough so that for all $k < m$, $i_k \in \mathrm{dom}(\vec{\pi}^n)$, and so that $|p|$ is less than the height of t_n. Now, clearly, $\tau\restriction(t_n)_p : (t_n)_p \longrightarrow (t_n)_q$ is an isomorphism, and $\sigma\restriction t_n = (\pi^n_{i_0})^{j_0} \circ \ldots \circ (\pi^n_{i_{m-1}})^{j_{m-1}}$ belongs to the group of automorphisms of t_n generated by $\vec{\pi}^n$, so since u_n is a condition, $\sigma\restriction t_n$ is totally far from extending to an automorphism, which means that there is a $p' \geq p$, $p' \in t_n$, such that $\tau(p') \neq \sigma(p')$. This finishes the proof. $\square$

2.2 Extending conditions

The following lemma says basically that the set of conditions $\langle t, \vec{\pi}\rangle$ such that the $\vec{\pi}$s are locally different (in the sense that given $i, j \in \mathrm{dom}(\vec{\pi})$ and a nonmaximal node p, there is a $q \in t$ above p such that $\pi_i(q) \neq \pi_j(q)$) is dense in $\mathbb{S}$. This is crucial, and will serve to show that images of a branch added by forcing with the tree we are forcing (under the automorphisms of the restriction of the tree to limit levels) are going to turn it into a Kurepa tree. It also shows that $\mathbb{S}$ is nonempty and nonatomic. The following definition is needed in the proof.

Definition 2.4. For nodes p, q of a normal tree T, $p \wedge q$ is the unique maximal node which is below both p and q in the ordering of T. Call $p \wedge q$ the *meet* of p and q.

Note that the meet always exists in a normal tree by the uniqueness of limits.

Lemma 2.5. *Let $\bar{t}$ and t be standard trees of countable height such that t end-extends $\bar{t}$, $\bar{t}$ has height α, which is the successor of a limit ordinal, and t has height $\alpha+\omega+1$. Let I be a countable subset of ω_2 and $\langle\bar{\pi}_i \mid i \in I\rangle$ be a sequence of automorphisms of $\bar{t}|\mathrm{Lim}$. Then there is a sequence $\vec{\pi} = \langle\pi_i \mid i \in I\rangle$ such that*

1. *$\langle t, \vec{\pi}\rangle \in \mathbb{S}$.*
2. *For all $i \in I$, $\bar{\pi}_i \subseteq \pi_i$.*
3. *If $\{i, j\} \in [I]^2$, and p is a non-maximal node in t, then there is a node q in t above p (at the top level of t) such that $\pi_i(q) \neq \pi_j(q)$.*

Proof. The proof uses a pseudo forcing argument: Let $\mathbb{Q}$ be the poset consisting of conditions of the form $\langle\sigma_i \mid i \in I\rangle$, such that

1. For every $i \in I$, σ_i is an injective partial function from $t|\mathrm{Lim}$ to $t|\mathrm{Lim}$ that's order-preserving and that extends $\bar{\pi}_i$.

2. The set $\bigcup_{i\in I}\{i\}\times(\mathrm{dom}(\sigma_i)\setminus\mathrm{dom}(\bar{\pi}_i))$ is finite.[2]

The ordering is reverse inclusion on each component.

I will specify a certain countable collection of dense subsets of $\mathbb{Q}$ such that a filter H which is $\mathbb{Q}$-generic with respect to this collection will determine a set $\vec{\pi}$ of automorphisms of $t|\mathrm{Lim}$ with the desired properties. More precisely, given a filter H in $\mathbb{Q}$, let, for $i\in I$,

$$\pi_i=\bigcup\{\sigma_i\mid\vec{\sigma}\in H\}.$$

The strategy is to specify the countably many dense subsets of $\mathbb{Q}$ in such a way that any filter that's generic with respect to that collection of dense sets will give rise to a sequence $\vec{\pi}$ with all the properties mentioned in the statement of the lemma.

Let $\bar{I}^*$ be the set of pairs of finite sequences $\langle\vec{\alpha},\vec{j}\rangle$ s.t. $|\vec{\alpha}|=|\vec{j}|$, $\vec{\alpha}\in{}^{|\vec{\alpha}|}I$ and $\vec{j}\in{}^{|\vec{j}|}\{-1,1\}$. The elements of $\bar{I}^*$ can be thought of as names for members of the group generated by $\vec{\pi}$, the set of automorphisms of $t|\mathrm{Lim}$ that's going to be determined by the $\mathbb{Q}$-pseudo-generic filter under construction. Namely, if $\vec{\sigma}$ is a condition in $\mathbb{Q}$, and $\langle\vec{\alpha},\vec{j}\rangle\in\bar{I}^*$ is of length n, set:

$$\vec{\sigma}_{\langle\vec{\alpha},\vec{j}\rangle}=\sigma_{\alpha_{n-1}}^{j_{n-1}}\circ\cdots\circ\sigma_{\alpha_0}^{j_0}.$$

Let's let $\vec{\sigma}_\emptyset=\mathrm{id}$ be the identity automorphism. Say that a sequence $\langle\vec{\alpha},\vec{j}\rangle\in\bar{I}^*$ is *canceled* if there is no $k+1<|\vec{j}|$ such that $\alpha_k=\alpha_{k+1}$ and $j_k=-j_{k+1}$. Let I^* be the collection of elements of $\bar{I}^*$ that are canceled and that are not the empty sequence. So these are the names for potentially nontrivial automorphisms.

For each name $\langle\vec{\alpha},\vec{j}\rangle\in I^*$ and each non-maximal node p of t, we want H to intersect the set

$$D_{\langle\vec{\alpha},\vec{j}\rangle,p}=\{\vec{\sigma}\in\mathbb{Q}\mid\exists q_1,q_2>p\quad(|q_1\wedge q_2|\neq|\vec{\sigma}_{\langle\vec{\alpha},\vec{j}\rangle}(q_1)\wedge\vec{\sigma}_{\langle\vec{\alpha},\vec{j}\rangle}(q_2)|)\}.$$

To clarify, the meets are taken in t here, not in $t|\mathrm{Lim}$.

If we succeed in finding a filter H intersecting all of these sets (and all that's needed in order to guarantee this is a verification that these sets are dense), then clearly, the group Γ generated by the sequence of automorphisms $\vec{\pi}$ which is determined by H will have the property that for every nontrivial $\sigma\in\Gamma$ and every non-maximal node $p\in t$, there are nodes q_1 and q_2 above p (q_1 and q_2 can be chosen to be maximal in t) such that $|q_1\wedge q_2|\neq|\sigma(q_1)\wedge\sigma(q_2)|$. This of course implies that every such σ is totally far from extending to an automorphism of t: Suppose π is an isomorphism between t_p and t_q, where $p,q\in t$ are distinct nonmaximal nodes located at the same level of t. So $\pi(p)=q$. If $\sigma(p)\neq q$, then there is nothing to show, so suppose $\sigma(p)=q$. Let $q_1,q_2\geq p$ be nodes in t such that $|q_1\wedge q_2|\neq|\sigma(q_1)\wedge\sigma(q_2)|$. Since $q_1,q_2\geq p$, it follows that $q_1\wedge q_2\geq p$, and moreover, $\pi(q_1)\wedge\pi(q_2)=\pi(q_1\wedge q_2)$, so $|q_1\wedge q_2|=|\pi(q_1)\wedge\pi(q_2)|$. So

[2] Note that $\bigcup_{i\in I}\{i\}\times(\mathrm{dom}(\sigma_i)\setminus\mathrm{dom}(\bar{\pi}_i))=\bigcup_{i\in I}\{i\}\times(\mathrm{dom}(\sigma_i)\setminus\bar{t})$. So the additional nodes in the domain of σ_i, if there are any, are located at the top level of t.

$\sigma(q_1) \neq \pi(q_1)$ or $\sigma(q_2) \neq \pi(q_2)$. So meeting these sets will insure that condition 1 of the lemma is satisfied.

Before proving the required density, I need a simple, yet technical observation:

Observation 2.6. *Let $u = \vec{\tau} \in \mathbb{Q}$ and let $\langle\langle \alpha_k, j_k\rangle \mid k < 2\rangle \in I^*$. Let $q_1 \neq q_2$ be nodes at the top level of t which don't belong to the domain of $(\tau_{\alpha_0})^{j_0}$. Then there are $r_1 \neq r_2$, also at the top level of t, such that if we let $v = \langle \tau'_i \mid i \in I\rangle$ be defined by*

$$\tau'_i = \begin{cases} \tau_i & \text{if } i \neq \alpha_0 \\ \tau_i \cup \{\langle q_1, r_1\rangle, \langle q_2, r_2\rangle\} & \text{if } i = \alpha_0 \text{ and } j_0 = 1 \\ \tau_i \cup \{\langle r_1, q_1\rangle, \langle r_2, q_2\rangle\} & \text{if } i = \alpha_0 \text{ and } j_0 = -1 \end{cases}$$

then $v \in \mathbb{Q}$ and r_1, r_2 are not in the domain of $(\tau'_{\alpha_1})^{j_1}$. But to clarify, q_1 and q_2 are in the domain of $(\tau'_{\alpha_0})^{j_0}$ and $r_l = (\tau'_{\alpha_0})^{j_0}(q_l)$, for $l = 1, 2$.

Proof of Observation. By symmetry, we may assume that $j_0 = 1$.

Let x_l be the predecessor of q_l in t which is located at the top level of $\bar{t}$ ($l \in \{1, 2\}$), and let $y_l = \tau_{\alpha_0}(x_l)$.

If $\alpha_1 \neq \alpha_0$, then we can just pick distinct nodes r_1, r_2 at the top level of t which are above y_1, y_2, respectively, which are not in the domain of $(\tau_{\alpha_1})^{j_1}$, and which are not in the range of τ_{α_0}. Note that every node at the top level of $\bar{t}$ has infinitely many successors at the top level of t, by normality. Yet only finitely many nodes at the top level of t belong to the domain or the range of τ_i, for any $i \in I$. Defining $\vec{\tau}'$ as in the claim, $\tau'_{\alpha_1} = \tau_{\alpha_1}$, and so, r_1 and r_2 are not in the domain of $(\tau'_{\alpha_1})^{j_1}$.

So assume now that $\alpha_1 = \alpha_0 = \alpha$. Since $\langle\langle \alpha_k, j_k\rangle \mid k < 2\rangle$ is canceled, it follows that $j_1 = j_0$, so by assumption, $j_1 = j_0 = 1$. In that case, pick nodes $r_1 \neq r_2$ at the top level of t which are above y_1, y_2, respectively, which are neither in the domain nor in the range of τ_α, and which are different from q_1 and q_2. Since after defining v as in the claim, the domain of τ'_α will be $\mathrm{dom}(\tau_\alpha) \cup \{q_1, q_2\}$, we have that $r_1, r_2 \notin \mathrm{dom}(\tau'_\alpha) = \mathrm{dom}((\tau'_{\alpha_1})^{j_1})$, as wished.

In both cases, r_1 and r_2 had to be chosen outside of the range of τ_{α_0}, so that τ'_{α_0} will be injective. $\square_{\text{Obs.}}$

Now let's show that $D_{\langle\vec{\alpha},\vec{j}\rangle,p}$ is dense in $\mathbb{Q}$. Let $|\langle\vec{\alpha},\vec{j}\rangle| = n + 1$.

Let $u = \vec{\tau} \in \mathbb{Q}$ be given. Pick distinct nodes q_1 and q_2 above p on the top level of t, so that q_1, q_2 are not in the domain of $\tau^{j_0}_{\alpha_0}$. In addition, they may be chosen so that the meet of q_1 and q_2 (in t) is at least at the maximal level of $\bar{t}$. By applying the previous observation n times, find an extension $v = \vec{\tau}'$ of u such that, letting $\sigma := v_{\langle\vec{\alpha}\restriction n,\vec{j}\restriction n\rangle}$, $q'_l = \sigma(q_l)$ is defined but not in the domain of $\rho := {\tau'}^{j_n}_{\alpha_n}$.

Since the meet of q_1 and q_2 (in t) is at least at the maximal level of $\bar{t}$, so is the meet of q'_1 and q'_2, because σ is order preserving on $t|\mathrm{Lim}$. Let p' be the t-predecessor of q'_1 and q'_2 at the top level of $\bar{t}$. So p' and p are comparable in t, and it follows that $q'_1, q'_2 \geq \sigma(p')$.

Now q_1' and q_2' are not in the domain of ρ. So we can pick extensions $z_1 \neq z_2$ of $\rho(\sigma(p'))$ at the top level of t which are not in the range of ρ (only finitely many are), such that the meet of z_1 and z_2 is at a level of t different from the level of the meet of q_1 and q_2. Define the desired condition to behave just like v, but in addition, specify that its α_n^{th} function (eventually inverted, depending on j_n) maps q_1' to z_1 and q_2' to z_2. This condition extends the one we started with and is in the set which we wanted to prove dense.

Insuring requirement 3 of the lemma corresponds to meeting the following dense subsets of $\mathbb{Q}$:

$$D_{i,j,p} = \{\vec{\sigma} \in \mathbb{Q} \mid \exists q \geq p \quad \sigma_i(q) \neq \sigma_j(q)\},$$

for $\{i,j\} \in [I]^2$ and any nonmaximal node p of t. That each of these sets is dense is obvious. The point is again that every nonmaximal node of t has infinitely many maximal nodes above it, while only finitely many maximal nodes of t are in the domain of σ_i or σ_j.

To insure that our pseudo-generic filter H will generate total functions, let, for each maximal node $p \in t$ and each $i \in I$,

$$D_{i,p} = \{\vec{\sigma} \in \mathbb{Q} \mid p \in \mathrm{dom}(\sigma_i)\}.$$

It is again trivial that $D_{i,p}$ is dense.

Finally, to insure that H will give rise to surjective functions, define

$$D'_{i,p} = \{\vec{\sigma} \in \mathbb{Q} \mid p \in \mathrm{ran}(\sigma_i)\},$$

for maximal nodes $p \in t$ and $i \in I$. Clearly, $D'_{i,p}$ is dense.

To conclude the proof, let H be a filter in $\mathbb{Q}$ which intersects all the dense sets of the form $D_{\langle\vec{\alpha},\vec{j}\rangle,p}$, $D_{i,j,p}$, $D_{i,p}$ and $D'_{i,p}$. Obviously, this is a countable collection of dense subsets of $\mathbb{Q}$, so such an H exists. The dense sets have been chosen so as to guarantee that the sequence $\langle \pi_i \mid i \in I\rangle$ defined as in the beginning of the proof with respect to H has the desired properties. □

2.3 $\mathbb{S}$ preserves cardinals

Lemma 2.7. *Assuming* CH*,* $\mathbb{S}$ *is* $<\aleph_2$*-c.c.*

Proof. Towards a contradiction, assume that $\langle u_\alpha \mid \alpha < \omega_2\rangle$ is an enumeration of an antichain in $\mathbb{S}$ of size $\aleph_2$. Let $u_\alpha = \langle t_\alpha, \vec{\pi}^\alpha\rangle$. Let $t_\alpha = \langle o_\alpha, <_\alpha\rangle$. By definition of $\mathbb{S}$, $o_\alpha < \omega_1$ is a countable ordinal, so by restricting to a subcollection of conditions in the antichain, we may assume that o_α is the same ordinal θ, for all $\alpha < \omega_2$. Each $<_\alpha$ is then a subset of $\theta \times \theta$. There are $2^\theta \leq \aleph_1$ such subsets by CH, so we may assume that $<_\alpha$ is the same for all $\alpha < \omega_2$. So for all $\alpha < \omega_2$, t_α is the same tree t. Let $I_\alpha = \mathrm{dom}(\vec{\pi}^\alpha)$. Each I_α is a countable subset of ω_2, and there are $\aleph_2$ many. By CH, $\omega_1^{<\omega_1} = \omega_1$, so that the Δ-Lemma applies, giving a Δ-system $D = \{I_\alpha \mid \alpha \in X\}$, where X has size $\aleph_2$. Let r be the root of the system. For every $\alpha \in X$, $\vec{\pi}^\alpha \restriction r$ belongs to ${}^r({}^{t|\mathrm{Lim}}t|\mathrm{Lim})$. Since r is countable, and so is $t|\mathrm{Lim}$, it again follows from CH that there is a subset Y of X of size $\aleph_2$ such that for all $\alpha, \beta \in Y$, $\vec{\pi}^\alpha \restriction r = \vec{\pi}^\beta \restriction r$.

Now pick two conditions u_α and u_β with $\alpha, \beta \in Y$, $\alpha \neq \beta$. Let $\vec{\bar{\pi}} = \vec{\pi}^\alpha \cup \vec{\pi}^\beta$, and set $u = \langle t, \vec{\bar{\pi}} \rangle$. u need not be a condition in $\mathbb{S}$, since the requirement on the group of automorphisms of $t|\mathrm{Lim}$ generated by $\vec{\bar{\pi}}$ will not necessarily be satisfied. But I claim that nevertheless, u_α and u_β are compatible. To see this, note that the tree t can be assumed to have a maximal level (by extending it and its automorphism sequence by one level if necessary), and then, t' may be defined to be some arbitrary normal, ω-splitting tree whose nodes form an ordinal, end-extending t, and having height $|t| + \omega + 1$. Now Lemma 2.5 can be invoked to produce a condition v with first coordinate t' whose automorphism sequence will extend $\vec{\bar{\pi}}$, and thus will extend both u_α and u_β. This contradicts the assumption that $\{u_\gamma \mid \gamma < \omega_2\}$ is an antichain. □

Corollary 2.8. *Assuming* CH*,* $\mathbb{S}$ *preserves cardinals.*

Proof. We have already seen that $\mathbb{S}$ is countably closed, so putting this together with the previous lemma, it follows that $\mathbb{S}$ preserves cardinals. □

2.4 Properties of the generic tree

From now on, assume CH, let G be $\mathbb{S}$-generic, and let

$$T = T_G = \bigcup \{t \mid \exists \vec{\pi} \quad \langle t, \vec{\pi} \rangle \in G\}$$

be the tree added by $\mathbb{S}$. The focus will now be on the tree T as a forcing notion rather than on the properties of $\mathbb{S}$. The aim is to see that T is a Souslin tree in $\mathrm{V}[G]$ (so that it is ccc and countably distributive), rigid (as a tree) and finally, almost Kurepa.

Lemma 2.9. *T is a Souslin tree in* $\mathrm{V}[G]$.

Proof. Let $u = \langle t, \vec{\pi} \rangle \in \mathbb{S}$ force that $\dot{A}$ is a maximal antichain in the generic tree added by $\mathbb{S}$. Let $\dot{f}$ be a name for the function that maps each countable ordinal α (which is also a node of T) to the smallest ordinal in $\dot{A}^G$ that is compatible with α. More precisely, let u force that $\dot{f}$ satisfies that definition. Now, we bootstrap as follows: Construct a decreasing sequence of conditions $\langle u_n \mid n < \omega \rangle$ in $\mathbb{S}$. Let $u_0 = u$. Given $u_n = \langle t_n, \vec{\pi}^n \rangle$, find a condition $u_{n+1} = \langle t_{n+1}, \vec{\pi}^{n+1} \rangle \leq u_n$ whose tree is higher than the tree part of u_n, such that u_{n+1} decides $\dot{f}$ on t_n (i.e., such that there is a function $f_n : t_n \longrightarrow \omega_1$ such that $u_{n+1} \Vdash \dot{f} \restriction \check{t_n} = \check{f_n}$), and such that $\mathrm{ran}(f_n) \subseteq t_{n+1}$. Let $u_\omega = \langle t_\omega, \vec{\pi}^\omega \rangle$, where $t_\omega = \bigcup_{n<\omega} t_n$, $\mathrm{dom}(\vec{\pi}^\omega) = \bigcup_{n<\omega} \mathrm{dom}(\vec{\pi}^n)$, and for $i \in \mathrm{dom}(\vec{\pi}^\omega)$, $\pi^\omega_i = \bigcup_{i \in \mathrm{dom}(\vec{\pi}^n)} \pi^n_i$. Lemma 2.3 shows that u_ω is a condition in $\mathbb{S}$ extending u_n, for every n. Let $f = \bigcup_{n<\omega} f_n$ and set $a = \mathrm{ran}(f)$. It is obvious from the construction that u_ω forces that $\check{a} = \dot{A} \cap \check{t_\omega}$ is a maximal antichain in $\check{t_\omega}$.

The next step is the paradigmatic sealing argument. To seal a, we construct a stronger condition in such a way that every new node in the tree part of the condition will be above a condition in a. Note that by construction, the height of t_ω is a limit of limits. Let Γ be the group of automorphisms of $t_\omega|\mathrm{Lim}$ generated by $\vec{\pi}^\omega$. Let a' be the maximal antichain in $t_\omega|\mathrm{Lim}$ consisting of nodes p (of limit height) such that

$$\exists q \in a \quad q \leq p \wedge [|q|, |p|) \cap \mathrm{Lim} = \emptyset.$$

Now Γ is a countable group acting on $t_\omega|\mathrm{Lim}$. We would like to construct a set of cofinal branches B of $t_\omega|\mathrm{Lim}$ such that

1. B respects Γ: For each $\sigma \in \Gamma$ and each $b \in B$, $\sigma``b \in B$.
2. B seals a': For each $b \in B$, $b \cap a' \neq \emptyset$.
3. B covers t_ω: For each $p \in t_\omega$, there is a $b \in B$ such that $p \in b$.

It is again convenient to construct B using pseudo forcing. Let $\Gamma = \{\sigma_n \mid n < \omega\}$. Let $\mathbb{Q}$ be the partial order $(t_\omega|\mathrm{Lim})^\omega$, with finite support and the canonical ordering. The aim is to write down a countable collection of dense subsets of $\mathbb{Q}$ such that any filter H that's generic with respect to this collection will give rise to a "generating" set of branches $\bar{B}$ through $t_\omega|\mathrm{Lim}$, so that the set of images of branches in $\bar{B}$ under automorphisms in Γ will satisfy the requirements listed above. Note that since Γ is closed under compositions, requirement 1 will automatically be satisfied. To meet requirement 2, our filter H should meet the following sets, for every $m, n < \omega$:

$$D_{m,n} = \{\langle p_i \mid i < \omega\rangle \in \mathbb{Q} \mid \exists r \in a' \quad \sigma_m(p_n) \geq r\}.$$

$D_{m,n}$ is dense, since given $\langle p_i \mid i < \omega\rangle$, $\sigma_m(p_n)$ (which can be taken to be the root of t_ω if undefined) can be extended to a node $q \in t_\omega|\mathrm{Lim}$ that lies above a node in a', as a' is a maximal antichain in t_ω. Let $\bar{q} = \sigma_m^{-1}(q)$. Then the condition $\langle q_i \mid i < \omega\rangle$ defined by $q_i = p_i$ for $i \neq n$ and $q_n = \bar{q}$ extends $\vec{p}$ and belongs to $D_{m,n}$.

For requirement 3, it suffices to meet the following sets, for $p \in t_\omega$:

$$D_p = \{\langle p_i \mid i < \omega\rangle \in \mathbb{Q} \mid \exists n \quad p \leq p_n\}.$$

As usual, a tiny bit more is needed in order to insure H will add cofinal branches. For every ordinal $\alpha < \mathrm{ht}(t_\omega)$ and every $n < \omega$, let

$$D_n^\alpha = \{\langle p_i \mid i < \omega\rangle \in \mathbb{Q} \mid |p_n| > \alpha\}.$$

Here, $|p_n|$ is the height of p_n in t_ω, not in $t_\omega|\mathrm{Lim}$.

The sets listed above are dense and open, and there are countably many, so there is a filter H in $\mathbb{Q}$ intersecting all of them. For $n < \omega$, let

$$c_n = \{p \in t_\omega|\mathrm{Lim} \mid \exists\langle p_i \mid i < \omega\rangle \in \mathbb{Q} \quad p \leq p_n\}.$$

Let $B = \{\sigma_m``c_n \mid m, n < \omega\}$. By construction, B satisfies the requirements listed above. Now, define $u' = \langle t', \vec{\pi}'\rangle \in \mathbb{S}$ as follows: t' end-extends t_ω by adding one new level (a limit level). To define this new level, pick the next ω many new ordinals. For each of these ordinals, we have to specify the predecessors they should have. Of course, for each of these ordinals α, we'll pick one branch $b \in B$ and say that the set of predecessors of α in t' should be the closure of b under $<_{t_\omega}$. This defines t'. To define $\vec{\pi}'$, let $\mathrm{dom}(\vec{\pi}') = \mathrm{dom}(\vec{\pi}^\omega)$. For $i \in \mathrm{dom}(\vec{\pi}')$ and $p \in t'|\mathrm{Lim}$, define $\pi'_i(p)$ as follows: If $p \in t_\omega$, then $\pi'_i(p) = \pi^\omega_i(p)$. If not, then let $b = \{q \in (t_\omega|\mathrm{Lim}) \mid q <_{t'} p\} \in B$. Since B is closed under Γ, $c := (\pi^\omega_i)``b \in B$. So c has a unique successor at the top level of t', say p' (i.e., $c = \{q \in (t_\omega|\mathrm{Lim}) \mid q <_{t'} p'\}$). Set $\pi'_i(p) = p'$. It is clear that $u' \in \mathbb{S}$. Moreover, u' forces that $\dot{A} = \check{a}$, since u' forces that every node of the generic tree that's not in t_ω is above a node at the top level of t' which, in turn, is above a node in a. In particular, u' forces that $\dot{A}$ is countable.

So what we have shown is that every condition that forces $\dot{A}$ is a maximal antichain in the generic tree has an extension that forces that $\dot{A}$ is countable. So actually, u forces that $\dot{A}$ is countable, which shows that T is a Souslin tree. □

Lemma 2.10. *T is rigid in* $\mathrm{V}[G]$.

Proof. This is where the property "totally far from extending to an automorphism of t" comes into play. Assume $T[G]$ was not rigid. Let $u \in \mathbb{S}$ force that $\dot{f}$ is a nontrivial automorphism of the generic tree. Bootstrap as in the previous proof to find a stronger condition $\langle t, \vec{\pi}\rangle$ which decides $\dot{f}$ on t (to be nontrivial). Let f be the function $\dot{f}$ is decided by $\langle t, \vec{\pi}\rangle$ to be, when restricted to t.

Again, assume the height of t to be a limit of limit ordinals. Let $p_0 \in t$ be such that $f(p_0) \neq p_0$, and let Γ be the group of automorphisms of $t|\mathrm{Lim}$ generated by $\vec{\pi}$. Now we do a pseudo forcing argument to cover t and seal f:

Consider the notion of forcing $\mathbb{P} = t_{p_0} \times t^\omega$, with finite support. View elements of $\mathbb{P}$ as functions $c : \omega \longrightarrow t$, where $c(0) \in t_{p_0}$ and for all but finitely many $i < \omega$, $c(i)$ is the root of t.

The goal is that if we pick H to be $\mathbb{P}$-generic with respect to a carefully chosen countable collection of dense subsets of $\mathbb{P}$, and let $b(i) = \{q \mid \exists g \in H \quad q \leq g(i)\}$, then $b(i)$ is a cofinal branch through t, the closure of the set $B = \{b(i) \mid i < \omega\}$ under Γ covers t, and, most importantly, we want that the closure of B seals f, in the sense that

$$f``b(0) \notin \Gamma(B),$$

where $\Gamma(B)$ stands for the closure of the branches in B under the automorphisms in Γ. Ensuring the latter corresponds to intersecting the following dense subsets of $\mathbb{P}$: Given $\sigma \in \Gamma$ and $i < \omega$, set

$$D_{\sigma,i} := \{c \in \mathbb{P} \mid \sigma(c(i)) \perp f(c(0))\}.$$

It needs to be checked that $D_{\sigma,i}$ is dense. This is obvious in case $i \neq 0$. For then, given a condition c, find an extension $c' \in D_{\sigma,i}$ as follows: First extend its 0^{th} coordinate arbitrarily so as to determine $f(c'(0))$ up to a limit height above that of $\sigma(c(i))$. Then extend the i^{th} coordinate so that $\sigma(c'(i))$ is different from $f(c'(0))$. This is easily achieved as σ is an automorphism of $t|\mathrm{Lim}$.

So assume now that $i = 0$. Let $c \in \mathbb{P}$ be given. We must find an extension c' which is in $D_{\sigma,0}$. Let $p = c(0)$. We may assume that p has limit height. Note that $f \restriction t_p : t_p \longrightarrow t_{f(p)}$ is an isomorphism, and $p \neq f(p)$ are at the same level of t. So by the fact that σ is totally far from extending to an automorphism of t, we can find an extension q of p s.t. $f(q) \neq \sigma(q)$. Hence, $f(q) \perp \sigma(q)$ and the condition obtained from c by strengthening its 0^{th} coordinate to q is in $D_{\sigma,0}$.

Since the sets $D_{\sigma,i}$ are dense, there is a filter intersecting them all, and additionally some more, to insure that the branches determined by the filter are cofinal in t and cover t (see the proof of Lemma 2.9). The rest of the procedure is as in the case of sealing a maximal antichain: We find a strengthening u' of $\langle t, \vec{\pi} \rangle$ by adding a top level to t in such a way that every branch in the closure of B under Γ gets a successor, and the automorphisms $\vec{\pi}$ are extended in the canonical way, as before. Since u' is a strengthening, it should force that $\dot{f}$ is an automorphism of the generic tree, but from the way u' was constructed, it's clear that $f``b(0)$ is not extended, a contradiction. □

Corollary 2.11. *T is totally rigid in* V$[G]$.

Proof. Indeed, the formulation "totally far from extending to an automorphism" allows us to seal any isomorphism $\sigma : T_r \longrightarrow T_s$, where $r \neq s$ are nodes located on the same level of the tree. But if we had any isomorphism $\sigma : T_p \longrightarrow T_q$, then we could find $p' \geq p$ such that $|\sigma(p')| = |p'|$ as follows: Let $g : \omega_1 \setminus |p| \longrightarrow \omega_1 \setminus |q|$ be defined by $g(\gamma) = |\sigma(r)|$, for any (and all) $r \in T_p$ with $|r| = \gamma$. Clearly then, $g(|p| + \alpha) = |q| + \alpha$, for all $\alpha < \omega_1$. So as soon as α is greater than or equal to an indecomposable ordinal above the maximum of $|p|$ and $|q|$, we have $g(|p| + \alpha) = g(\alpha) = |q| + \alpha = \alpha$. So if $r \in T_p$ is such that $|r|$ is indecomposable and greater than both $|p|$ and $|q|$, it follows that $\sigma \restriction T_r : T_r \longrightarrow T_{\sigma(r)}$ is an isomorphism, and $|r| = |\sigma(r)|$. But such isomorphisms don't exist, by the argument establishing Lemma 2.10. □

Lemma 2.12. *T is absolutely rigid in* V$[G]$.

Proof. Note that Lemma 2.10 is a consequence of the present lemma. Instead of proving the most complicated lemma first, I preferred to present the results incrementally, at each step dealing with the new complications.

Suppose T was not absolutely rigid. Then in V$[G]$, there would be a T-name $\dot{\pi}$ for a nontrivial automorphism of T. $\dot{\pi}$ can be viewed as a function π mapping nodes $p \in T$ to partial automorphisms of T: $\pi(p)(q) = r$ if r is the T-maximal node such that $p \Vdash \dot{\pi}(\check{q}) = \check{r}$. Call such a function a potential nontrivial automorphism of T. Now in V, there is an $\mathbb{S}$-name τ for π, so that $\tau_G = \pi$. Fix an $\mathbb{S}$-condition t_0 that forces that τ is a potential nontrivial automorphism of T.

Using the bootstrap construction, it is easy to see that there is an extension $\langle t, \vec{\pi} \rangle$ of t_0 whose height is a limit of limits, such that t decides τ on t, and such that, writing π for what t decides τ to be, there are p_0, $q_0 \in t$ such that $(\pi(p_0))(q_0) \neq q_0$, and such that for every $p, q \in t$, there are $p' \geq p$ and $q' \geq q$ such that $q' \in \mathrm{dom}(\pi(p'))$, and for every $p, q \in t$, there are $p' \geq p$ and $q' \geq q$

such that $q' \in \operatorname{ran}(\pi(p'))$. For details on potential additional automorphisms, see [FH09].

As before, let Γ be the group of automorphisms of $t|\mathrm{Lim}$ generated by $\vec{\pi}$. Using a pseudo forcing argument, I am going to find a set of branches covering t and respecting Γ such that π is sealed, in the sense that it can't be extended to a potential additional automorphism of any tree t' extending t where t' results from t by adding successor to precisely that set of branches. Similarly to previous constructions, let $\mathbb{P} = t^\omega$, with finite support. For $\sigma \in \Gamma$ and $i < \omega$, let

$$D_{\sigma,i} = \{c \in \mathbb{P} \mid \sigma(c(i)) \perp \pi(c(0))(c(1))\}.$$

The idea is that if H is a filter meeting all of these (countably many) sets, and in addition other dense sets ensuring that H will determine cofinal branches that cover t, as before, then, letting b_i be the branch determined by the i-th coordinates of conditions in H, then $\pi(b_0)(b_1)$ is incompatible with the branch determined by $\sigma(b_i)$. Hence, π cannot be extended to a potential additional automorphism of any tree extending t', where the t' results from t by adding successors to the branches in $\Gamma(B)$. So all that needs to be checked is that $D_{\sigma,i}$ is dense.
Case 1: $i > 1$.

This is the easiest case. I leave the details to the reader.
Case 2: $i = 0$.

Let $\bar{c} \in \mathbb{P}$ be given. Let $\bar{p} = \bar{c}(0)$, $\bar{q} = \bar{c}(1)$. Find $p \geq \bar{p}$, so that $\pi(p)$ is defined on at least two nodes of limit height that are above $\bar{q}$. At least one of these nodes, q, is such that $\sigma(p) \perp \pi(p)(q)$. So define $c \in \mathbb{P}$ to be like $\bar{c}$, except that $c(0) = p$ and $c(1) = q$. Then c is an extension of $\bar{c}$ in $D_{\sigma,i}$.
Case 3: $i = 1$.

Again, let $\bar{c} \in \mathbb{P}$ be given, and let $\bar{p} = \bar{c}(0)$, $\bar{q} = \bar{c}(1)$. Note that since I am forcing with $\mathbb{P}$ below p_0 and q_0, it follows that $\bar{p} \geq p_0$ and $\bar{q} \geq q_0$. Let b be a cofinal branch of t that contains $\bar{p}$, so that $\pi' := \bigcup_{r \in b} \pi(r) : t \overset{\sim}{\longleftrightarrow} t$ is an automorphism. Clearly, $\bar{q}' := \pi'(\bar{q}) \neq \bar{q}$, so that $\pi' {\restriction} t_{\bar{q}} : t_{\bar{q}} \overset{\sim}{\longleftrightarrow} t_{\bar{q}'}$. Now since σ is totally far from extending to an automorphism of t, there is a $q \geq \bar{q}$ in t so that $\sigma(q) \neq \pi'(q)$. Now find $p \in b$, $p \geq \bar{p}$, such that $\pi'(q) = (\pi(p))(q)$, and define $c \in \mathbb{P}$ by letting $c(i) = \bar{c}(i)$ if $i > 1$, $c(0) = p$ and $c(1) = q$. Then c is an extension of $\bar{c}$ in $D_{\sigma,i}$. □

Corollary 2.13. *T is absolutely totally rigid in* $\mathrm{V}[G]$.

Proof. This can be shown by a minor variation of the proof establishing Lemma 2.12. □

Lemma 2.14. *T is almost Kurepa in* $\mathrm{V}[G]$*: Forcing with T over* $\mathrm{V}[G]$ *turns T into a Kurepa tree.*

Proof. From G we not only get T, but also a sequence $\langle \pi_i \mid i < \omega_2 \rangle$ of automorphisms of $T|\mathrm{Lim}$, where

$$\pi_i = \bigcup \{\sigma_i \mid \exists u = \langle t, \vec{\sigma} \rangle \in G \quad i \in I_u\}.$$

Recall that $\mathbb{S}$ preserves cardinals, so $\omega_2^{\mathrm{V}} = \omega_2^{\mathrm{V}[G]}$ here.

The point is that in $\mathrm{V}[G]$, if $\{i, j\} \in [\omega_2]^2$, the set

$$D_{i,j} = \{p \in T \mid \pi_i(p) \neq \pi_j(p)\}$$

is dense in T. The reason is that given such i and j, and a node $p \in T$, we can pick a condition $u \in G$ such that $p \in t_u$ and $i,\ j \in I_u$. By (a weak version of) Lemma 2.5, the set of conditions $v \in \mathbb{S}$ that extend u and are such that there is a $q \in t_v$ above p such that $\pi_i^v(q) \neq \pi_j^v(q)$ is dense below u, so there is such a v in G.

So if we force with T over $\mathrm{V}[G]$, letting b be T-generic over $\mathrm{V}[G]$, then letting $b_i = \pi_i``b$ (i.e., the branch generated by this), for $i < \omega_2$, it follows that the b_i's are distinct, because given distinct $i,\ j < \omega_2$, b intersects $D_{i,j}$. Hence this collection of branches witnesses that T is Kurepa in $\mathrm{V}[G][b]$ – recall that T is Souslin in $\mathrm{V}[G]$, so forcing with T over $\mathrm{V}[G]$ preserves cardinals and again, $\omega_2^{\mathrm{V}[G]} = \omega_2^{\mathrm{V}[G][b]}$. □

3 Club Rigidity

The Souslin almost Kurepa tree T constructed in the previous section is rigid and absolutely totally rigid, yet $T|\mathrm{Lim}$ has $\aleph_2$ automorphisms. Obviously, an almost Kurepa tree cannot have the unique branch property. It must have many "potential additional branches", i.e., functions derived from names for additional branches, which are a certain type of homomorphism of the tree, see [FH09]. So it is an obvious question whether there can be a Souslin almost Kurepa tree T such that $T|\mathrm{Lim}$ is rigid. But of course, the construction of the previous section can be modified to produce such a tree. In fact, for any given club C, an obvious modification of the proof can produce a Souslin almost Kurepa tree T such that $T|C$ is totally rigid. Can this work simultaneously for every club subset of ω_1?

Definition 3.1. An $\aleph_1$-tree T is *rigid on clubs* if for every club $C \subseteq \omega_1$ such that $0 \in C$, $T|C$ is rigid.

Of course, demanding that $0 \in C$ is not essential here, and this is only required in order to ensure that the resulting tree will have a unique root.

Question 3.2. Can there be a Souslin almost Kurepa tree T that's rigid on clubs?

The concept of club rigidity is very appealing for many reasons. Firstly, $T|C$ is always a normal $\aleph_1$-tree if T is (the critical condition here is uniqueness at limits, which is maintained because C is closed). And if T is ω-branching, then so is $T|C$. But the main reason why the concept of club rigidity seems important is as follows. The motivation for many of the rigidity degrees introduced in [FH09] was viewing the trees as notions of forcing. But if we are really interested in the trees as notions of forcing, then what really matters should be the rigidity properties of Boolean algebra associated to the tree. The connection here is the known fact that if T is club rigid, then the Boolean algebra of T is rigid (see [Jec03, p. 599, ex. 30.15]).

The idea of considering rigidity properties on clubs opens a whole new array of rigidity degrees. In general, let's say a tree T has a property on clubs if for every club C, $T|C$ has this property. So we can talk about trees being rigid on clubs, totally rigid on clubs, UBP on clubs, et cetera. Recall that T has a property absolutely if the T has the property after forcing with T. It is then unclear, for example, what it should mean that T is absolutely rigid on clubs: Should it mean that after forcing with T, T is rigid on clubs, or should it mean that $T|C$ is absolutely rigid, for any club C? The latter would mean that for any club C, $T|C$ is rigid after forcing with $T|C$. The following lemma shows that the ambiguity is irrelevant, in the context of Souslin trees.

Lemma 3.3. *Let P be a property of a tree such that provably in* ZFC*, if $C \subseteq D$ are club subsets of ω_1, and T is a Souslin tree such that $T|C$ satisfies P, then $T|D$ satisfies P ("P goes up"). Then the following are equivalent for a Souslin tree T:*

1. *T forces that for every club C, $T|C$ satisfies P.*
2. *For every club C, $T|C$ forces that $T|C$ satisfies P.*

The implication 1. $\Longrightarrow$ 2. holds in general, for any property, and any tree T.

Proof. 1. $\Longrightarrow$ 2.: Fix a club C. Since $T|C$ is dense in T, forcing with $T|C$ is equivalent to forcing with T. So by 1., after forcing with $T|C$, $T|C$ satisfies P.

2. $\Longrightarrow$ 1.: Let b be T-generic. Let C be a club subset of ω_1 in $\mathrm{V}[b]$. Since T satisfies the countable chain condition, there is a club $\bar{C} \subseteq C$ such that $\bar{C} \in \mathrm{V}$. Since $b|\bar{C}$ is $T|\bar{C}$-generic, it follows by 2. that in $\mathrm{V}[b|\bar{C}]$, $T|\bar{C}$ satisfies P. But then, since P goes up, and since $\mathrm{V}[b] = \mathrm{V}[b|\bar{C}]$, it follows that $T|C$ satisfies P in $\mathrm{V}[b]$. □

Observation 3.4. *The following properties go up: rigidity, total rigidity, the unique branch property.*

So there is no ambiguity in saying that a Souslin tree is absolutely rigid on clubs, absolutely totally rigid on clubs and absolutely UBP on clubs.

Observation 3.5. *A tree has each of the following properties iff it has the property on clubs:*

1. *UBP*
2. *Souslin off the generic branch*[3]
3. *absolutely UBP*

[3] A Souslin tree T is Souslin off the generic branch if for every V-generic branch b and every node $p \in T \setminus b$, T_p is Souslin in $\mathrm{V}[b]$. This is a strengthening of the unique branch property, introduced in [FH09]. Note that, using this terminology, a Souslin tree is UBP iff it is Aronszajn off the generic branch.

4. absolutely Souslin off the generic branch

This robustness makes these rigidity degrees seem very natural. But observe that all the club degrees have this robustness. For example, if a tree is rigid on clubs, then it is rigid on clubs on clubs - just because the intersection of two clubs is club.

Lemma 3.6. *The following properties are equivalent, for a normal ω_1-tree T:*

1. *T is rigid on clubs.*

2. *T is totally rigid on clubs.*

Proof. 2. $\Longrightarrow$ 1. is trivial.
1. $\Longrightarrow$ 2.: Suppose T is rigid on clubs but not totally rigid on clubs. So let C be club, and let $p, q \in T|C$ be distinct nodes such that there is an isomorphism $\pi : (T|C)_p \overset{\sim}{\longleftrightarrow} (T|C)_q$. Let $p \in T(\alpha)$ and $q \in T(\beta)$, and without limitation of generality, let $\alpha \leq \beta$. It may also be assumed that $p \not\leq_T q$: Otherwise, let $p' \in T(\beta)$ be incompatible with (i.e., different from) q, $p \leq_T p'$. Then $q' := \pi(p') \geq_T \pi(p) = q$. Then p' and q' are incompatible, or else $p' \leq q'$ and also $q \leq q'$, so p' and q would be compatible, a contradiction. So letting $\pi' = \pi\restriction(T|C)_{p'} : (T|C)_{p'} \overset{\sim}{\longleftrightarrow} (T|C)_{q'}$ the desired constellation has been reached, where p' and q' are incompatible. So instead of passing to p', q' and π', we may assume p and q are already incompatible.

For $r \in T$, let $|r|$ be the level at which r occurs in T, i.e., $r \in T(|r|)$. Let $C' = C \setminus |p|$, and let $f : C' \longrightarrow C'$ be such that for every $r \in (T|C)_p$, $|\pi(r)| = f(|r|)$. Now f is continuous, so there is a club $\bar{C} \subseteq C'$ of fixed points of f (in fact, $\bar{C}$ can be chosen to be a tail of C', but that's not important for the argument). Now let $\bar{p} \geq p$, $\bar{p} \in T(\bar{\alpha})$, where $\bar{\alpha} = \min(\bar{C})$. Let $\bar{q} = \pi(\bar{p})$, and let $\bar{\pi} = \pi\restriction(T|\bar{C})_{\bar{p}}$. Then $\bar{\pi} : (T|\bar{C})_{\bar{p}} \overset{\sim}{\longleftrightarrow} (T|\bar{C})_{\bar{q}}$, and $\bar{p} \neq \bar{q}$ occur at the same level of T. Now $\bar{\pi}$ can be extended to an automorphism σ of $T|(\bar{C} \cup \{0\})$ as follows:

$$\sigma(r) = \begin{cases} \bar{\pi}(r) & \text{if } r \in (T|\bar{C})_{\bar{p}} \\ \bar{\pi}^{-1}(r) & \text{if } r \in (T|\bar{C})_{\bar{q}} \\ r & \text{if } r \in (T|(\bar{C} \cup \{0\})) \setminus ((T|\bar{C})_{\bar{p}} \cup (T|\bar{C})_{\bar{q}}). \end{cases}$$

□

Corollary 3.7. *The following properties are equivalent, for a Souslin tree T:*

1. *T is absolutely rigid on clubs.*

2. *T is absolutely totally rigid on clubs.*

Proof. Fix a Souslin tree T. Then for any club C containing 0, $T|C$ is Souslin, and after forcing with $T|C$, $T|C$ is a normal ω_1-tree. So in the forcing extension, $T|C$ is rigid on clubs iff $T|C$ is totally rigid on clubs, by Lemma 3.6. This holds for any club C containing 0, and so, T is absolutely rigid on clubs iff T is absolutely totally rigid on clubs. □

In Observation 3.5, we have seen that the UBP is equivalent to the UBP on clubs, and similarly for the property "Souslin off the generic branch". In the other cases, having the property on clubs is a proper strengthening.

Lemma 3.8. *Let T be a normal ω_1-tree.*

1. *If T is rigid on clubs, then T is totally rigid.*
2. *If T is Souslin and absolutely rigid on clubs, then T is absolutely totally rigid.*

These implications cannot be reversed in ZFC*. So it is consistent that there is a normal ω_1-tree that's totally rigid but not club rigid, and it is consistent that there is a Souslin tree T that's absolutely totally rigid but not absolutely rigid on clubs.*

Proof. First, let's prove the implications. For 1., if T is rigid on clubs, then T is totally rigid on clubs and hence totally rigid. For 2., if T is Souslin and absolutely rigid on clubs, then T is absolutely totally rigid on clubs and hence absolutely totally rigid.

To see that these implications cannot be reversed, we may use the absolutely totally rigid Souslin almost Kurepa tree constructed in the first part of the present paper. That tree is not rigid on clubs, so it shows that neither of the implications proved here can be reversed. □

Taking into account that the distinctions between rigidity and total rigidity, and between absolute rigidity and absolute total rigidity, vanish in the context of club degrees of rigidity, the rigidity diagram is collapsed as follows:

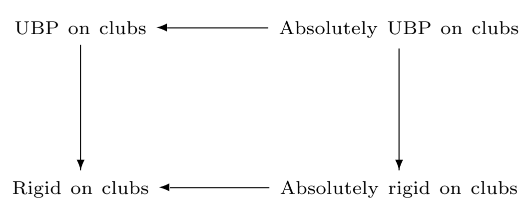

Figure 2: Club Rigidity Implication Diagram

Recall that the UBP is equivalent to the UBP on clubs here.

Lemma 3.9. *The rigidity diagram holds on clubs, for Souslin trees.*

Proof. The diagram can be lifted to the context of club rigidity easily. For example, to show that a Souslin tree T that's absolutely UBP on clubs is also UBP on clubs, it just needs to be shown that for any club C, $T|C$ is UBP. But for that club, $T|C$ is absolutely UBP by assumption. So $T|C$ is a Souslin tree that's absolutely UBP. By the original implication diagram, it follows that $T|C$ is UBP. All the other implications can be shown that way. □

In order to arrive at the result that the club rigidity degree diagram is complete, the following need to be shown:

1. It is consistent that there is a Souslin tree T that is UBP (on clubs) but not absolutely rigid on clubs.

2. It is consistent that there is a Souslin tree U such that U is absolutely rigid on clubs but not UBP (on clubs).

The first of these tasks is easily achieved:

Lemma 3.10. *It is consistent (it actually follows from $\diamondsuit$) that there is a Souslin tree T that is UBP (on clubs) but not absolutely rigid on clubs (not even absolutely rigid).*

Proof. It was shown in [FH09, Theorem 3.13] that the $\diamondsuit$ principle implies the existence of a Souslin tree T that's UBP but not absolutely rigid (in fact, T is absolutely nonrigid in the sense that after forcing with T, T is not rigid). Note that by Observation 3.5, part 1, T is UBP on clubs. But it is not absolutely rigid, and hence not absolutely rigid on clubs. □

I don't know whether the second task is achievable.

Question 3.11. Is it consistent that there is a Souslin tree that is absolutely rigid on clubs but not UBP (on clubs)?

Let's now put together the club degrees and the previous degrees of rigidity into one diagram and assemble what we know about their relationships. In the following diagram, arrows denote implications, double-headed arrows indicate equivalence, and arrows labeled with an "s" stand for strict, that is, non-reversible implications. The question marks label arrows where it is open whether the implication can be reversed.

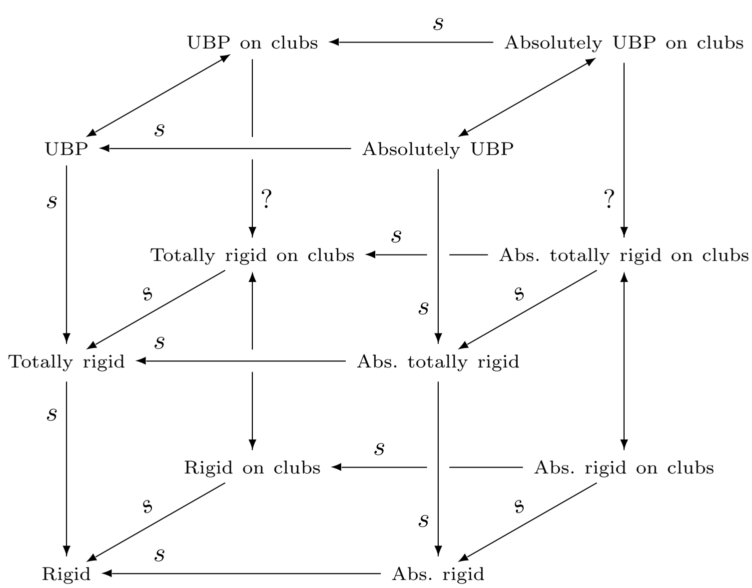

Figure 3: 3D implication diagram

Theorem 3.12. *The implications in the above diagram are* ZFC*-provable in the context of Souslin trees, and the implications marked by an "s" are strict, so not reversible.*

Proof. It is clear that the implications hold. It remains to show that the ones marked by an "s" are strict.

The seven implications between the non-club degrees are strict, as was shown in [FH09].

Furthermore, there is no implication from any nonabsolute degree to an absolute degree, because by Lemma 3.10, the UBP does not imply absolute rigidity. So the strongest nonabsolute degree does not even imply the weakest absolute degree. In particular, all the arrows pointing left are strict.

Concerning implications from non-club degrees to club degrees: the absolute UBP implies any club degree, because it is equivalent to the absolute UBP on clubs. Similarly, the UBP implies all nonabsolute club degrees. Other than that, there are no implications from non-club degrees to club degrees: The strongest non-club degree other than the UBP and the absolute UBP is absolute total rigidity. And absolute total rigidity does not imply even the weakest club degree, namely club rigidity. For the tree constructed in the first part of this paper is a counterexample: it is absolutely totally rigid but not rigid on clubs.

Concerning implications from the rigid level to the totally rigid level: absolute rigidity on clubs implies everything on the totally rigid level, and rigidity on clubs implies the nonabsolute totally rigid degrees. Rigidity on clubs implies no absolute degree on the totally rigid level, as there is no implication from any nonabsolute degree to an absolute one. To see that there is no other implication from a rigid degree to a totally rigid degree, it suffices to see that the strongest rigid degree left (absolute rigidity) does not imply even the weakest totally rigid degree, i.e., total rigidity. This was shown in [FH09].

So far, the only implications that hold result from following the arrows in the diagram.

The only question left is which implications go from the totally rigid level to the UBP level. Note that the UBP level really only consists of two degrees, UBP and absolute UBP. It was shown in [FH09] that absolute total rigidity does not imply the UBP. So there is no implication from absolute total rigidity or total rigidity to any degree on the top UBP level. We know that total rigidity on clubs does not imply the absolute UBP, since there is no implication from a nonabsolute degree to an absolute degree. So the only possible implications that don't arise from following arrows in the diagram are from (total) rigidity on clubs to UBP (on clubs), from absolute (total) rigidity on clubs to UBP (on clubs), and from absolute (total) rigidity on clubs to absolute UBP (on clubs). □

So the question raised by the previous lemma is:

Question 3.13.

1. Does absolute rigidity on clubs imply UBP?

2. Does absolute rigidity on clubs imply the absolute UBP?

3. Does rigidity on clubs imply the UBP?

All three could be answered in the negative by answering Question 3.11 in the positive.

References

[Abr79] Uri Abraham. Construction of a rigid Aronszajn tree. *Proceedings of the American Mathematical Society*, 77(1):136–137, 1979.

[AS85] Uri Abraham and Saharon Shelah. Isomorphism types of Aronszajn trees. *Israel Journal of Mathematics*, 50:75–113, 1985.

[DJ74] Keith J. Devlin and Håvard Johnsbråten. *The Souslin Problem.* Lecture Notes in Mathematics **405**. Springer, 1974.

[FH09] Gunter Fuchs and Joel David Hamkins. Degrees of rigidity for Souslin trees. *Journal of Symbolic Logic*, 74(2):423–454, Feb. 2009.

[GS64] Haim Gaifman and E. P. Specker. Isomorphism types of trees. *Proceedings of the American Mathematical Society*, 15:1–6, 1964.

[Jec72] Thomas Jech. Automorphisms of ω_1-trees. *Transactions of the American Mathematical Society*, 173:57–70, 1972.

[Jec74] Thomas Jech. Forcing with trees and ordinal definability. *Annals of Mathematical Logic*, 7:387–409, 1974.

[Jec03] Thomas Jech. *Set Theory: The Third Millenium Edition, Revised and Expanded.* Springer Monographs in Mathematics. Springer, 2003.

[Jen69] Ronald B. Jensen. Automorphism properties of Souslin continua. *Notices of the American Mathematical Society*, 16:576, 1969. Abstract #69T-E24.

[JS97] Renling Jin and Saharon Shelah. Can a small forcing create Kurepa trees? *Annals of Pure and Applied Logic*, 85:47–68, 1997.

[Tod80] Stevo Todorčević. Rigid Aronszajn trees. *Publications de l'Institut Mathématique (Beograd), Nouvelle Série*, 41(27):259–265, 1980.