Closed Maximality Principles and Large Cardinals

Gunter Fuchs
Institut für Mathematische Logik und Grundlagenforschung
Westfälische Wilhelms-Universität Münster

First European Set Theory Meeting, Bedlewo, 2007

July 12, 2007
Let’s view the universe and its possible generic extensions as a Kripke model for modal logic.
Question: CH?
Question: $\omega_1 > (\omega_1)^L$?
$\phi$ is forceably necessary.

MP says $\phi$ is true.
“$\phi$ is necessary” is forceably necessary.

MP says $\phi$ is necessary.
Write $\diamond \varphi$ to express that $\varphi$ holds in a forcing extension ($\varphi$ is forceable).

Note: This is the first order statement $\exists P \ P \models \varphi$. 
Write $\diamond \varphi$ to express that $\varphi$ holds in a forcing extension ($\varphi$ is forceable).

Note: This is the first order statement $\exists P \ P \models \varphi$.

$\square \varphi$ means that $\varphi$ holds in every forcing extension ($\varphi$ is necessary).

This is again a first order statement.
Write $\diamond \varphi$ to express that $\varphi$ holds in a forcing extension ($\varphi$ is forceable).

Note: This is the first order statement $\exists P \ P \models \varphi$.

$\Box \varphi$ means that $\varphi$ holds in every forcing extension ($\varphi$ is necessary).

This is again a first order statement.

So the statement $\diamond (\Box \varphi)$ makes sense.

It expresses that it is forceable that $\varphi$ is necessary, or in short, that $\varphi$ is forceably necessary.
Write \( \Diamond \varphi \) to express that \( \varphi \) holds in a forcing extension (\( \varphi \) is forceable).

Note: This is the first order statement \( \exists P \ P \models \varphi \).

\( \square \varphi \) means that \( \varphi \) holds in every forcing extension (\( \varphi \) is necessary).

This is again a first order statement.

So the statement \( \Diamond (\square \varphi) \) makes sense.

It expresses that it is forceable that \( \varphi \) is necessary, or in short, that \( \varphi \) is forceably necessary.

The Maximality Principle MP is the scheme consisting of the formulae

\[
(\Diamond \square \varphi) \implies \varphi,
\]

for every sentence \( \varphi \). It was introduced by Joel Hamkins, and a close relative was introduced earlier and independently by Stavi and Väänänen.
Possible modifications of MP:
1. Restrict to certain classes of forcings, such as: Proper, ccc, stationary-preserving, ...
Possible modifications of $\mathcal{MP}$:

1. Restrict to certain classes of forcings, such as: Proper, ccc, stationary-preserving, . . .

2. Allow parameters in the scheme $\Diamond \Box \varphi \implies \varphi$, i.e., boldface versions of the principles.
Possible modifications of MP:

1. Restrict to certain classes of forcings, such as: Proper, ccc, stationary-preserving, . . .

2. Allow parameters in the scheme $\Diamond \Box \varphi \implies \varphi$, i.e., boldface versions of the principles.

3. Necessary forms of the boldface principles.
Possible modifications of $\text{MP}$:

1. Restrict to certain classes of forcings, such as: Proper, ccc, stationary-preserving, . . .

2. Allow parameters in the scheme $\Diamond \Box \varphi \implies \varphi$, i.e., boldface versions of the principles.

3. Necessary forms of the boldface principles.

4. (Restrict to a subclass of formulae.)
Possible modifications of $\mathbf{MP}$:

1. Restrict to certain classes of forcings, such as: Proper, ccc, stationary-preserving, . . .

2. Allow parameters in the scheme $\Diamond\Box\varphi \implies \varphi$, i.e., boldface versions of the principles.

3. Necessary forms of the boldface principles.

4. (Restrict to a subclass of formulae.)

General form of the principle:

$$\text{MP}_{\Gamma}(X),$$

where $\Gamma$ is a class of partial orders and $X$ is the parameter set.
I looked at the case where $\Gamma$ is one of the following, for some fixed regular cardinal $\kappa$. 
I looked at the case where $\Gamma$ is one of the following, for some fixed regular cardinal $\kappa$.

1. The class of all $<\kappa$-closed forcings,
I looked at the case where $\Gamma$ is one of the following, for some fixed regular cardinal $\kappa$.

1. The class of all $<\kappa$-closed forcings,

2. the class of all $<\kappa$-directed-closed forcings,
I looked at the case where $\Gamma$ is one of the following, for some fixed regular cardinal $\kappa$.

1. The class of all $\mathcal{<}\kappa$-closed forcings,

2. the class of all $\mathcal{<}\kappa$-directed-closed forcings,

3. the class of all forcings of the form $\text{Col}(\kappa, \lambda)$ or $\text{Col}(\kappa, \mathcal{<}\lambda)$, for some $\lambda$. Call the class $\text{Col}(\kappa)$. 
I looked at the case where $\Gamma$ is one of the following, for some fixed regular cardinal $\kappa$.

1. The class of all $<_\kappa$-closed forcings,

2. the class of all $<_\kappa$-directed-closed forcings,

3. the class of all forcings of the form $\text{Col}(\kappa, \lambda)$ or $\text{Col}(\kappa, < \lambda)$, for some $\lambda$. Call the class $\text{Col}(\kappa)$.

Note: $\kappa = \omega$ is allowed!

The corresponding parameter set will usually be one of the following:

$$\emptyset, \ H_\kappa \cup \{\kappa\}, \ H_\kappa^+. $$
I looked at the case where $\Gamma$ is one of the following, for some fixed regular cardinal $\kappa$.

1. The class of all $<\kappa$-closed forcings,

2. the class of all $<\kappa$-directed-closed forcings,

3. the class of all forcings of the form $\text{Col}(\kappa, \lambda)$ or $\text{Col}(\kappa, < \lambda)$, for some $\lambda$. Call the class $\text{Col}(\kappa)$.

Note: $\kappa = \omega$ is allowed!

The corresponding parameter set will usually be one of the following:

$$\emptyset, \ H_\kappa \cup \{\kappa\}, \ H_\kappa^+.$$
Relationships

\[ \text{MP}_{\text{Col}(\kappa)}(H_\kappa \cup \{\kappa\}) \iff \text{MP}_{\text{Col}(\kappa)}(H_\kappa^+) \]

\[ \text{MP}_{<\kappa - \text{dir. cl.}}(H_\kappa \cup \{\kappa\}) \iff \text{MP}_{<\kappa - \text{dir. cl.}}(H_\kappa^+) \]

\[ \text{MP}_{<\kappa - \text{closed}}(H_\kappa \cup \{\kappa\}) \iff \text{MP}_{<\kappa - \text{closed}}(H_\kappa^+) \]
In general, none of the implications can be reversed.
Consistency Results

1. If $V_\delta \prec V$ and $\kappa < \delta$ is regular, then forcing with $Col(\kappa, < \delta)$ produces a model of $MP_{Col(\kappa)}(H_\kappa \cup \{\kappa\})$. 
Consistency Results

1. If $V_\delta \prec V$ and $\kappa < \delta$ is regular, then forcing with $\text{Col}(\kappa, <\delta)$ produces a model of $\text{MP}_{\text{Col}(\kappa)}(H_\kappa \cup \{\kappa\})$.

2. If in addition, $\delta$ is regular (i.e., inaccessible), then the extension will model $\text{MP}_{\text{Col}(\kappa)}(H_{\kappa^+})$. 
Consistency Results

1. If $V_\delta \prec V$ and $\kappa < \delta$ is regular, then forcing with $Col(\kappa, <\delta)$ produces a model of $\text{MP}_{Col(\kappa)}(H_\kappa \cup \{\kappa\})$.

2. If in addition, $\delta$ is regular (i.e., inaccessible), then the extension will model $\text{MP}_{Col(\kappa)}(H_\kappa^+)$.

3. Conversely, $\text{MP}_{<\kappa - \text{closed}}(H_\kappa \cup \{\kappa\})$ implies that $L_\delta \prec L$, where $\delta$ is the supremum of the ordinals which are definable in $\kappa$; it follows that $\delta \leq \kappa^+$. 
1. If $V_\delta \prec V$ and $\kappa < \delta$ is regular, then forcing with $\text{Col}(\kappa, <\delta)$ produces a model of $\text{MP}_{\text{Col}(\kappa)}(H_\kappa \cup \{\kappa\})$.

2. If in addition, $\delta$ is regular (i.e., inaccessible), then the extension will model $\text{MP}_{\text{Col}(\kappa)}(H_{\kappa^+})$.

3. Conversely, $\text{MP}_{<\kappa-\text{closed}}(H_\kappa \cup \{\kappa\})$ implies that $L_\delta \prec L$, where $\delta$ is the supremum of the ordinals which are definable in $\kappa$; it follows that $\delta \leq \kappa^+$.

4. $\text{MP}_{<\kappa-\text{closed}}(H_{\kappa^+})$ implies that $L_\delta \prec L$, where $\delta = \kappa^+$. 
Results on Consequences of the Principles

Here are some consequences of $\text{MP}_{\prec \kappa - \text{closed}}(X)$ with $\kappa \in X$:

1. $\diamondsuit_\kappa$ holds.
Results on Consequences of the Principles

Here are some consequences of $\text{MP}_{\kappa-\text{closed}}(X)$ with $\kappa \in X$:

1. $\diamondsuit_\kappa$ holds.

2. No tree in $X$ is a $\kappa$-Kurepa tree.
Results on Consequences of the Principles

Here are some consequences of \( \text{MP}_{\kappa-\text{closed}}(X) \) with \( \kappa \in X \):

1. \( \Diamond_{\kappa} \) holds.

2. No tree in \( X \) is a \( \kappa \)-Kurepa tree.

3. \( \kappa \)-closed generic \( \Sigma^1_2(H_\kappa) \)-absoluteness, with parameters from \( X \).
Results on Consequences of the Principles

Here are some consequences of $\text{MP}_{<\kappa-\text{closed}}(X)$ with $\kappa \in X$:

1. $\Diamond_\kappa$ holds.

2. No tree in $X$ is a $\kappa$-Kurepa tree.

3. $<\kappa$-closed generic $\Sigma^1_2(H_\kappa)$-absoluteness, with parameters from $X$.

4. . . .
Combining
Lemma 1. Assume $\text{MP}_{<\kappa}\text{-closed}(H_{\kappa^+})$. Let $\mathbb{P}$ be a $<\kappa^+$-closed notion of forcing. If $G$ is $\mathbb{P}$-generic, then in $V[G]$, $\text{MP}_{<\kappa}\text{-closed}(H_{\kappa^+})$ continues to hold. This remains true if “$<\kappa$-closed” is replaced with “$<\kappa$-directed-closed”.
Combining

Lemma 1. Assume $MP_{<\kappa-\text{closed}}(H_{\kappa^+})$. Let $\mathbb{P}$ be a $<\kappa^+$-closed notion of forcing. If $G$ is $\mathbb{P}$-generic, then in $V[G]$, $MP_{<\kappa-\text{closed}}(H_{\kappa^+})$ continues to hold. This remains true if “$<\kappa$-closed” is replaced with “$<\kappa$-directed-closed”.

So the closed maximality principles can be combined:

- Assume that $\kappa_0 < \delta_0 \leq \kappa_1 < \delta_1$ are regular, and $\delta_0$, as well as $\delta_1$ are fully reflecting.
Combining

**Lemma 1.** Assume $\text{MP}_{<\kappa-\text{closed}}(H_{\kappa^+})$. Let $\mathbb{P}$ be a $<\kappa^+-\text{closed}$ notion of forcing. If $G$ is $\mathbb{P}$-generic, then in $V[G]$, $\text{MP}_{<\kappa-\text{closed}}(H_{\kappa^+})$ continues to hold. This remains true if “$<\kappa$-closed” is replaced with “$<\kappa$-directed-closed”.

So the closed maximality principles can be combined:

- Assume that $\kappa_0 < \delta_0 \leq \kappa_1 < \delta_1$ are regular, and $\delta_0$, as well as $\delta_1$ are fully reflecting.

- Forcing with $\text{Col}(\kappa_0, < \delta_0)$ produces a model in which $\text{MP}_{<\kappa_0-\text{dir. cl.}}(H_{\kappa_0^+})$ holds.
Lemma 1. Assume $\text{MP}_{<\kappa-\text{closed}}(H_{\kappa^+})$. Let $\mathbb{P}$ be a $<\kappa^+$-closed notion of forcing. If $G$ is $\mathbb{P}$-generic, then in $V[G]$, $\text{MP}_{<\kappa-\text{closed}}(H_{\kappa^+})$ continues to hold. This remains true if “$<\kappa$-closed” is replaced with “$<\kappa$-directed-closed”.

So the closed maximality principles can be combined:

- Assume that $\kappa_0 < \delta_0 \leq \kappa_1 < \delta_1$ are regular, and $\delta_0$, as well as $\delta_1$ are fully reflecting.
- Forcing with $\text{Col}(\kappa_0, < \delta_0)$ produces a model in which $\text{MP}_{<\kappa_0-\text{dir. cl.}}(H_{\kappa_0^+})$ holds.
- Moreover, in that model, $\kappa_1 < \delta_1$ are regular, and $\delta_1$ is still fully reflecting.
Combining

**Lemma 1.** Assume $\text{MP}_{<\kappa^-\text{closed}}(H_{\kappa^+})$. Let $\mathbb{P}$ be a $<\kappa^+\text{-closed}$ notion of forcing. If $G$ is $\mathbb{P}$-generic, then in $V[G]$, $\text{MP}_{<\kappa^-\text{closed}}(H_{\kappa^+})$ continues to hold. This remains true if “$<\kappa^-\text{closed}”$ is replaced with “$<\kappa^-\text{directed-closed}”$.

So the the closed maximality principles can be combined:

- Assume that $\kappa_0 < \delta_0 \leq \kappa_1 < \delta_1$ are regular, and $\delta_0$, as well as $\delta_1$ are fully reflecting.

- Forcing with $\text{Col}(\kappa_0, < \delta_0)$ produces a model in which $\text{MP}_{<\kappa_0^-\text{dir. cl.}}(H_{\kappa_0^+})$ holds.

- Moreover, in that model, $\kappa_1 < \delta_1$ are regular, and $\delta_1$ is still fully reflecting.

- So since $\kappa_1 \geq \kappa_0^+$, further forcing with $\text{Col}(\kappa_1, < \delta_1)$ preserves $\text{MP}_{<\kappa_0^-\text{dir. cl.}}(H_{\kappa_0^+})$ and makes $\text{MP}_{<\kappa_1^-\text{dir. cl.}}(H_{\kappa_1^+})$ true, in addition.
Pushing this idea further, one can now produce models where $\text{MP}_{<\kappa} \text{ - dir. cl.} (H_{\kappa^+})$ holds simultaneously at each of the first $\alpha$ regular cardinals $\kappa$. 
Pushing this idea further, one can now produce models where $\text{MP}_{\kappa-\text{dir. cl.}}(H_{\kappa^+})$ holds simultaneously at each of the first $\alpha$ regular cardinals $\kappa$. In order to force such a model, assume

- $\text{ZFC}_A$. 
Pushing this idea further, one can now produce models where $\text{MP}_{\kappa-\text{dir.}} \text{ cl.}(H_{\kappa^+})$ holds simultaneously at each of the first $\alpha$ regular cardinals $\kappa$. In order to force such a model, assume

- $\text{ZFC}_A$.

- $A$ consists of inaccessible cardinals and has order-type $\alpha$. 
Pushing this idea further, one can now produce models where $\text{MP}_{\prec \kappa - \text{dir. cl.}}(H_{\kappa^+})$ holds simultaneously at each of the first $\alpha$ regular cardinals $\kappa$. In order to force such a model, assume

- $\text{ZFC}_A$.
- $A$ consists of inaccessible cardinals and has order-type $\alpha$.
- $A$ is discrete, that is, it contains no limit point of itself.
Pushing this idea further, one can now produce models where \( \text{MP}_{\kappa - \text{dir. cl.}}(H_{\kappa^+}) \) holds simultaneously at each of the first \( \alpha \) regular cardinals \( \kappa \). In order to force such a model, assume

- \( \text{ZFC}_A \).

- \( A \) consists of inaccessible cardinals and has order-type \( \alpha \).

- \( A \) is discrete, that is, it contains no limit point of itself.

- For every \( \delta \in A \), \( \langle V_\delta, \in \rangle \prec \langle V, \in \rangle \).
Pushing this idea further, one can now produce models where $\text{MP}_{<\kappa \text{-- dir. cl.}(H_{\kappa^+})}$ holds simultaneously at each of the first $\alpha$ regular cardinals $\kappa$. In order to force such a model, assume

- $\text{ZFC}_A$.
- $A$ consists of inaccessible cardinals and has order-type $\alpha$.
- $A$ is discrete, that is, it contains no limit point of itself.
- For every $\delta \in A$, $\langle V_\delta, \in \rangle \prec \langle V, \in \rangle$.

For $\delta \in A$, let $\bar{\delta}$ be the least regular cardinal which is greater than or equal to $\sup(A \cap \delta)$. The forcing which produces the desired model is then a reverse Easton iteration of collapses of the form $\text{Col}(\bar{\delta}, <\delta)$, for $\delta \in A$. Call this forcing iteration $\mathbb{P}_A$. 
An intriguing feature of the “combinatorics” of the boldface closed maximality principles:
An intriguing feature of the “combinatorics” of the boldface closed maximality principles:

• It is consistent to have them hold at the first $\alpha$ regular cardinals, but
An intriguing feature of the “combinatorics” of the boldface closed maximality principles:

- It is **consistent** to have them hold at the first $\alpha$ regular cardinals, but
- it is **inconsistent** to have them hold at arbitrarily large regular cardinals.
An intriguing feature of the “combinatorics” of the boldface closed maximality principles:

- It is **consistent** to have them hold **at the first** \( \alpha \) regular cardinals, but
- it is **inconsistent** to have them hold **at arbitrarily large** regular cardinals.

Why is the global combination impossible?
An intriguing feature of the “combinatorics” of the boldface closed maximality principles:

• It is consistent to have them hold at the first $\alpha$ regular cardinals, but

• it is inconsistent to have them hold at arbitrarily large regular cardinals.

Why is the global combination impossible?

**Definition 2.** A forcing has a strong closure point at a cardinal $\delta$ if it factors as $P \ast \dot{Q}$, where $P$ has size at most $\delta$ and $P$ forces that $\dot{Q}$ is $<\delta^{++}$-strategically closed.
An intriguing feature of the “combinatorics” of the boldface closed maximality principles:

- It is **consistent** to have them hold at the first $\alpha$ regular cardinals, but
- it is **inconsistent** to have them hold at arbitrarily large regular cardinals.

Why is the global combination impossible?

**Definition 2.** A forcing has a strong closure point at a cardinal $\delta$ if it factors as $P \times \dot{Q}$, where $\dot{Q}$ has size at most $\delta$ and $P$ forces that $\dot{Q}$ is $<\delta^{++}$-strategically closed.

The crucial point is the following:
Lemma 3. There is a formula $\psi(\cdot, \cdot)$ with the following property:

If $V = M[G]$, where $G$ is generic over $M$ for a forcing which has a strong closure point at $\delta$ then

$$M = \{x \mid V \models \psi(x, z)\},$$

where $z = \mathcal{P}(\delta^+)^M$.

This uses Hamkins’ approximation and cover properties and ideas of Reitz.
Now assume the closed Maximality Principle holds at arbitrarily large regular cardinals.
Now assume the closed Maximality Principle holds at arbitrarily large regular cardinals. Then $V$ has to be a forcing extension of an inner model, since this is forceably necessary (and first order expressible).
Now assume the closed Maximality Principle holds at arbitrarily large regular cardinals. Then \( V \) has to be a forcing extension of an inner model, since this is forceably necessary (and first order expressible). So let \( V = M[G] \), where \( G \) is generic over \( M \) for \( \mathbb{P} \). Let \( \delta \) be the cardinality of \( \mathbb{P} \). Let \( z = \mathcal{P}(\delta^+)^M \), and let \( \kappa \) be a regular cardinal greater than \( 2^{\delta^+} \), at which the closed Maximality Principle holds. Note that \( z \) is allowed as a parameter in the principle.
Now assume the closed Maximality Principle holds at arbitrarily large regular cardinals. Then \( V \) has to be a forcing extension of an inner model, since this is forceably necessary (and first order expressible). So let \( V = M[G] \), where \( G \) is generic over \( M \) for \( P \). Let \( \delta \) be the cardinality of \( P \). Let \( z = \mathcal{P}(\delta^+)^M \), and let \( \kappa \) be a regular cardinal greater than \( 2^{\delta^+} \), at which the closed Maximality Principle holds. Note that \( z \) is allowed as a parameter in the principle. Now the statement

\[
\kappa^+ > (\kappa^+)^M
\]

is \(<\kappa\)-closed-forceably necessary:
Now assume the closed Maximality Principle holds at arbitrarily large regular cardinals. Then $V$ has to be a forcing extension of an inner model, since this is forceably necessary (and first order expressible). So let $V = M[G]$, where $G$ is generic over $M$ for $P$. Let $\delta$ be the cardinality of $P$. Let $z = P(\delta^+)^M$, and let $\kappa$ be a regular cardinal greater than $2^{\delta^+}$, at which the closed Maximality Principle holds. Note that $z$ is allowed as a parameter in the principle. Now the statement

\[ "\kappa^+ > (\kappa^+)^M" \]

is $<\kappa$-closed-forceably necessary:

First collapse $(\kappa^+)^M$ to $\kappa$ over $V$. 
Now assume the closed Maximality Principle holds at arbitrarily large regular cardinals. Then $V$ has to be a forcing extension of an inner model, since this is forceably necessary (and first order expressible). So let $V = M[G]$, where $G$ is generic over $M$ for $P$. Let $\delta$ be the cardinality of $P$. Let $\kappa$ be a regular cardinal greater than $2^{\delta^+}$, at which the closed Maximality Principle holds. Note that $z$ is allowed as a parameter in the principle. Now the statement

"$\kappa^+ > (\kappa^+)^M$"

is $<\kappa$-closed-forceably necessary:

First collapse $(\kappa^+)^M$ to $\kappa$ over $V$.

The point is now that any further extension by $<\kappa$-closed forcing is a forcing extension of $M$ by a forcing which has a strong closure point at $\delta$, so that $M$ is defined by $\psi(\cdot, z)$ in any such extension.
Now assume the closed Maximality Principle holds at arbitrarily large regular cardinals. Then $V$ has to be a forcing extension of an inner model, since this is forceably necessary (and first order expressible). So let $V = M[G]$, where $G$ is generic over $M$ for $\mathbb{P}$. Let $\delta$ be the cardinality of $\mathbb{P}$. Let $\kappa$ be a regular cardinal greater than $2^{\delta^+}$, at which the closed Maximality Principle holds. Note that $z$ is allowed as a parameter in the principle. Now the statement

\[
\kappa^+ > (\kappa^+)^M
\]

is $\langle \kappa \rangle$-closed-forceably necessary:

First collapse $(\kappa^+)^M$ to $\kappa$ over $V$.

The point is now that any further extension by $\langle \kappa \rangle$-closed forcing is a forcing extension of $M$ by a forcing which has a strong closure point at $\delta$, so that $M$ is defined by $\psi(\cdot, z)$ in any such extension.

So $\kappa^+ > (\kappa^+)^M$, which is impossible, since $\mathbb{P}$ has size less than $\kappa$. 
A Natural Question

So I hope everybody will agree that the following question arises naturally:
A Natural Question

So I hope everybody will agree that the following question arises naturally:

**How large can a cardinal** $\kappa$ **be if** $\text{MP}_{<\bar{\kappa}}(H_{\bar{\kappa}^+})$ **holds at every regular** $\bar{\kappa} < \kappa$ **(or even at every regular** $\bar{\kappa} \leq \kappa$ **)?
A Natural Question

So I hope everybody will agree that the following question arises naturally:

**How large can a cardinal \( \kappa \) be if \( \text{MP}_{< \bar{\kappa} - \text{closed}}(H_{\bar{\kappa} +}) \) holds at every regular \( \bar{\kappa} < \kappa \) (or even at every regular \( \bar{\kappa} \leq \kappa \))?**

This leads to a study of techniques which lift embeddings of a model to generic extensions.
Suppose $\kappa$ is a large cardinal (as witnessed by certain kinds of embeddings) which is fully reflecting.
Suppose \( \kappa \) is a large cardinal (as witnessed by certain kinds of embeddings) which is fully reflecting.

1. Let \( j : M \rightarrow N \) be an embedding witnessing that \( \kappa \) is large.
Suppose $\kappa$ is a large cardinal (as witnessed by certain kinds of embeddings) which is fully reflecting.

1. Let $j : M \rightarrow N$ be an embedding witnessing that $\kappa$ is large.

2. Let $G$ be generic for some forcing $\mathbb{P}$ making MP true as often as wished.
Suppose $\kappa$ is a large cardinal (as witnessed by certain kinds of embeddings) which is fully reflecting.

1. Let $j : M \rightarrow N$ be an embedding witnessing that $\kappa$ is large.

2. Let $G$ be generic for some forcing $\mathbb{P}$ making $\text{MP}$ true as often as wished.

3. Find $G'$ which is $\dot{\mathbb{P}}_G$-generic, such that $j"G \subseteq G'$. Here, $j(\mathbb{P}) = \mathbb{P} \ast \dot{\mathbb{P}}'$.

   So $j$ extends to $j' : M[G] \rightarrow N[G']$. 

Suppose $\kappa$ is a large cardinal (as witnessed by certain kinds of embeddings) which is fully reflecting.

1. Let $j : M \rightarrow N$ be an embedding witnessing that $\kappa$ is large.

2. Let $G$ be generic for some forcing $\mathbb{P}$ making $MP$ true as often as wished.

3. Find $G'$ which is $\mathbb{P}'_G$-generic, such that $j \upharpoonright G \subseteq G'$. Here, $j(\mathbb{P}) = \mathbb{P} \ast \mathbb{P}'$.

   So $j$ extends to $j' : M[G] \rightarrow N[G']$.

4. Derive some object $\mathcal{F}$ from $j'$ which codes the relevant portion of the embedding.
Suppose $\kappa$ is a large cardinal (as witnessed by certain kinds of embeddings) which is fully reflecting.

1. Let $j : M \rightarrow N$ be an embedding witnessing that $\kappa$ is large.

2. Let $G$ be generic for some forcing $\mathbb{P}$ making $\text{MP}$ true as often as wished.

3. Find $G'$ which is $\mathbb{P}'_G$-generic, such that $j^{"G \subseteq G'}$. Here, $j(\mathbb{P}) = \mathbb{P} \ast \mathbb{P}'$.

   So $j$ extends to $j' : M[G] \rightarrow N[G']$.

4. Derive some object $\mathcal{F}$ from $j'$ which codes the relevant portion of the embedding.


Suppose $\kappa$ is a large cardinal (as witnessed by certain kinds of embeddings) which is fully reflecting.

1. Let $j : M \to N$ be an embedding witnessing that $\kappa$ is large.

2. Let $G$ be generic for some forcing $\mathbb{P}$ making $\text{MP}$ true as often as wished.

3. Find $G'$ which is $\dot{\mathbb{P}}'_{G}$-generic, such that $j^{\dot{\mathbb{P}}'}_{\mathbb{P}} G \subseteq G'$. Here, $j(\mathbb{P}) = \mathbb{P} \ast \dot{\mathbb{P}}'$.

   So $j$ extends to $j' : M[G] \to N[G']$.

4. Derive some object $\mathcal{F}$ from $j'$ which codes the relevant portion of the embedding.


6. $\pi : M[G] \to \mathcal{F} M'$ witnesses that $\kappa$ is large in $V[G]$. 
Lemma 4. Let $\kappa$ be weakly compact and $V_\kappa \prec V$. Then there is a forcing $P$ such that in any $P$-generic extension, $V[G]$, $\kappa$ is still weakly compact, and the boldface maximality principle for directed closed forcings holds at every regular cardinal $\bar{\kappa} \prec \kappa$. 
Lemma 4. Let $\kappa$ be weakly compact and $V_\kappa \prec V$. Then there is a forcing $\mathbb{P}$ such that in any $\mathbb{P}$-generic extension, $V[G]$, $\kappa$ is still weakly compact, and the boldface maximality principle for directed closed forcings holds at every regular cardinal $\bar{\kappa} < \kappa$.

In this case, we force with $\mathbb{P}_A$ to get $V[G]$. Given a transitive model in $V[G]$ which has size $\kappa$ there, pick a name for that model, and a transitive model $M$ of size $\kappa$, containing the name, that’s closed under $<\kappa$-sequences in $V$. 
Lemma 4. Let $\kappa$ be weakly compact and $V_\kappa \prec V$. Then there is a forcing $P$ such that in any $P$-generic extension, $V[G]$, $\kappa$ is still weakly compact, and the boldface maximality principle for directed closed forcings holds at every regular cardinal $\bar{\kappa} < \kappa$.

In this case, we force with $P_A$ to get $V[G]$. Given a transitive model in $V[G]$ which has size $\kappa$ there, pick a name for that model, and a transitive model $M$ of size $\kappa$, containing the name, that’s closed under $<\kappa$-sequences in $V$. Now lift a weakly compact embedding $j : M \rightarrow N$ to $j' : M[G] \rightarrow N[G']$. In this case, $G = j "G$ and $G' \in V[G]$, as the tail forcing is $<\kappa$-closed and $N$ has size $\kappa$. 
Lemma 4. Let $\kappa$ be weakly compact and $V_\kappa \prec V$. Then there is a forcing $P$ such that in any $P$-generic extension, $V[G]$, $\kappa$ is still weakly compact, and the boldface maximality principle for directed closed forcings holds at every regular cardinal $\bar{\kappa} < \kappa$.

In this case, we force with $P_A$ to get $V[G]$. Given a transitive model in $V[G]$ which has size $\kappa$ there, pick a name for that model, and a transitive model $M$ of size $\kappa$, containing the name, that’s closed under $<\kappa$-sequences in $V$. Now lift a weakly compact embedding $j : M \longrightarrow N$ to $j' : M[G] \longrightarrow N[G']$. In this case, $G = j''G$ and $G' \in V[G]$, as the tail forcing is $<\kappa$-closed and $N$ has size $\kappa$.

Note: This is an equiconsistency; we get the reflecting weakly compact back in $L$. 
Lemma 5. Let $\kappa$ be measurable, as witnessed by some normal ultrafilter $U$ on $\kappa$, $V_\kappa \prec V$, $2^\kappa = \kappa^+$ and $A \subseteq \kappa$ be such that
On an unbounded measure 0 set below a measurable

Lemma 5. Let $\kappa$ be measurable, as witnessed by some normal ultrafilter $U$ on $\kappa$, $V_\kappa \prec V$, $2^\kappa = \kappa^+$ and $A \subseteq \kappa$ be such that

1. $A$ consists of regular limit cardinals,
Lemma 5. Let $\kappa$ be measurable, as witnessed by some normal ultrafilter $U$ on $\kappa$, $V_\kappa \prec V$, $2^\kappa = \kappa^+$ and $A \subseteq \kappa$ be such that

1. $A$ consists of regular limit cardinals,

2. If $\kappa_0 < \kappa_1$, then there exists a $\rho \in (\kappa_0, \kappa_1]$ which is regular and reflecting.
Lemma 5. Let $\kappa$ be measurable, as witnessed by some normal ultrafilter $U$ on $\kappa$, $V_\kappa \prec V$, $2^\kappa = \kappa^+$ and $A \subseteq \kappa$ be such that

1. $A$ consists of regular limit cardinals,

2. If $\kappa_0 < \kappa_1$, then there exists a $\rho \in (\kappa_0, \kappa_1]$ which is regular and reflecting.

3. $A$ has $U$-measure 0.
Lemma 5. Let \( \kappa \) be measurable, as witnessed by some normal ultrafilter \( U \) on \( \kappa \), \( V_\kappa \prec V \), \( 2^\kappa = \kappa^+ \) and \( A \subseteq \kappa \) be such that

1. \( A \) consists of regular limit cardinals,
2. If \( \kappa_0 < \kappa_1 \), then there exists a \( \rho \in (\kappa_0, \kappa_1] \) which is regular and reflecting.
3. \( A \) has \( U \)-measure 0.

Then there is a forcing \( P \) such that if \( G \) is \( P \)-generic over \( V \), in \( V[G] \), the following hold:

1. \( MP_{<\bar{\kappa}} \text{– dir. cl.}(H_{\bar{\kappa}^+}) \) is true, for every \( \bar{\kappa} \in A \), and
Lemma 5. Let $\kappa$ be measurable, as witnessed by some normal ultrafilter $U$ on $\kappa$, $V_\kappa \prec V$, $2^\kappa = \kappa^+$ and $A \subseteq \kappa$ be such that

1. $A$ consists of regular limit cardinals,

2. If $\kappa_0 < \kappa_1$, then there exists a $\rho \in (\kappa_0, \kappa_1]$ which is regular and reflecting.

3. $A$ has $U$-measure 0.

Then there is a forcing $\mathbb{P}$ such that if $G$ is $\mathbb{P}$-generic over $V$, in $V[G]$, the following hold:

1. $\text{MP}_{<\kappa-\text{dir. cl.}}(H_{\kappa^+})$ is true, for every $\kappa \in A$, and

2. $\kappa$ is measurable.
In this case, let $U$ be a normal ultrafilter on $\kappa$, let $j : V \rightarrow N$ be the ultrapower by $U$, and let $\mathbb{P}$ force MP at all $\bar{\kappa} \in A$. Let $G$ be generic for $\mathbb{P}$. $N[G]$ is closed under $\kappa$-sequences and thinks that the tail forcing $\dot{Q}_G$, where $j(\mathbb{P}) = \mathbb{P} \ast \dot{Q}$, is $<\kappa^+$-closed, so that it is $<\kappa^+$-closed in $V[G]$. Since moreover, $\mathcal{P}(j(\mathbb{P})) \cap N[G]$ has size $\kappa^+$ in $V[G]$, it is possible to construct a generic $G'$ for $\dot{Q}_G$ over $N[G]$ in $V[G]$. Then $j$ lifts to $j' : V[G] \rightarrow N[G][G']$. \qed
Remark 6.

1. If $\text{MP}_{<\kappa-\text{closed}}(H_\kappa \cup \{\kappa\})$ holds and $\kappa$ is weakly compact, then $\kappa$’s weak compactness is indestructible under $<\kappa$-closed forcing.
Remark 6.

1. If $\text{MP}_{<\kappa-\text{closed}}(H_\kappa \cup \{\kappa\})$ holds and $\kappa$ is weakly compact, then $\kappa$’s weak compactness is indestructible under $<\kappa$-closed forcing. This is because $\kappa$ being weakly compact is a $\Pi^1_2$ property over $H_\kappa$, and under $\text{MP}_{<\kappa-\text{closed}}(H_\kappa \cup \{\kappa\})$, $<\kappa$-closed-generic $\Sigma^1_2(H_\kappa)$ absoluteness holds.
Remark 6.

1. If $\text{MP}_{<\kappa\text{-closed}}(H_\kappa \cup \{\kappa\})$ holds and $\kappa$ is weakly compact, then $\kappa$’s weak compactness is indestructible under $<\kappa$-closed forcing. This is because $\kappa$ being weakly compact is a $\Pi^1_2$ property over $H_\kappa$, and under $\text{MP}_{<\kappa\text{-closed}}(H_\kappa \cup \{\kappa\})$, $<\kappa$-closed-generic $\Sigma^1_2(H_\kappa)$ absoluteness holds.

2. Suppose $\kappa$ is measurable, $U$ is a normal ultrafilter on $\kappa$ and the set of $\bar{\kappa} < \kappa$ such that $\text{MP}_{<\bar{\kappa}\text{-closed}}(H_{\bar{\kappa}^+})$ holds has $U$-measure 1.
Remark 6.

1. If $\text{MP}_{<\kappa\text{-closed}}(H_\kappa \cup \{\kappa\})$ holds and $\kappa$ is weakly compact, then $\kappa$’s weak compactness is indestructible under $<\kappa$-closed forcing. This is because $\kappa$ being weakly compact is a $\Pi^1_2$ property over $H_\kappa$, and under $\text{MP}_{<\kappa\text{-closed}}(H_\kappa \cup \{\kappa\})$, $<\kappa$-closed-generic $\Sigma^1_2(H_\kappa)$ absoluteness holds.

2. Suppose $\kappa$ is measurable, $U$ is a normal ultrafilter on $\kappa$ and the set of $\bar{\kappa} < \kappa$ such that $\text{MP}_{<\bar{\kappa}\text{-closed}}(H_{\bar{\kappa}+})$ holds has $U$-measure 1. Let $M = \text{Ult}(V,U)$. 
Remark 6.

1. If $\text{MP}_{<\kappa\text{-closed}}(H_\kappa \cup \{\kappa\})$ holds and $\kappa$ is weakly compact, then $\kappa$’s weak compactness is indestructible under $<\kappa$-closed forcing. This is because $\kappa$ being weakly compact is a $\Pi^1_2$ property over $H_\kappa$, and under $\text{MP}_{<\kappa\text{-closed}}(H_\kappa \cup \{\kappa\})$, $<\kappa$-closed-generic $\Sigma^1_2(H_\kappa)$ absoluteness holds.

2. Suppose $\kappa$ is measurable, $U$ is a normal ultrafilter on $\kappa$ and the set of $\bar{\kappa} < \kappa$ such that $\text{MP}_{<\bar{\kappa}\text{-closed}}(H_{\bar{\kappa}+})$ holds has $U$-measure 1. Let $M = \text{Ult}(V,U)$. Then in $M$, $\text{MP}_{<\bar{\kappa}\text{-closed}}(H_{\bar{\kappa}+})$ holds, and $\kappa$ is weakly compact in $M$. 
Remark 6.

1. If $\text{MP}_{<\kappa-\text{closed}}(H_{\kappa} \cup \{\kappa\})$ holds and $\kappa$ is weakly compact, then $\kappa$’s weak compactness is indestructible under $<\kappa$-closed forcing. This is because $\kappa$ being weakly compact is a $\Pi^1_2$ property over $H_\kappa$, and under $\text{MP}_{<\kappa-\text{closed}}(H_{\kappa} \cup \{\kappa\})$, $<\kappa$-closed-generic $\Sigma^1_2(H_\kappa)$ absoluteness holds.

2. Suppose $\kappa$ is measurable, $U$ is a normal ultrafilter on $\kappa$ and the set of $\bar{\kappa} < \kappa$ such that $\text{MP}_{<\bar{\kappa}-\text{closed}}(H_{\bar{\kappa}+})$ holds has $U$-measure 1. Let $M = \text{Ult}(V,U)$. Then in $M$, $\text{MP}_{<\kappa-\text{closed}}(H_{\kappa+})$ holds, and $\kappa$ is weakly compact in $M$. So in $M$, $\kappa$ is an indestructible weakly compact cardinal.
Remark 6.

1. If $\text{MP}_{<\kappa-\text{closed}}(H_\kappa \cup \{\kappa\})$ holds and $\kappa$ is weakly compact, then $\kappa$'s weak compactness is indestructible under $<\kappa$-closed forcing. This is because $\kappa$ being weakly compact is a $\Pi^1_2$ property over $H_\kappa$, and under $\text{MP}_{<\kappa-\text{closed}}(H_\kappa \cup \{\kappa\})$, $<\kappa$-closed-generic $\Sigma^1_2(H_\kappa)$ absoluteness holds.

2. Suppose $\kappa$ is measurable, $U$ is a normal ultrafilter on $\kappa$ and the set of $\bar{\kappa} < \kappa$ such that $\text{MP}_{<\bar{\kappa}-\text{closed}}(H_{\bar{\kappa}+})$ holds has $U$-measure 1. Let $M = \text{Ult}(V, U)$. Then in $M$, $\text{MP}_{<\kappa-\text{closed}}(H_{\kappa+})$ holds, and $\kappa$ is weakly compact in $M$. So in $M$, $\kappa$ is an indestructible weakly compact cardinal. So, in $V$, the set of indestructible weakly compact cardinals below $\kappa$ has $U$-measure 1.
Remark 6.

1. If $\text{MP}_{<\kappa}\text{-closed}(H_\kappa \cup \{\kappa\})$ holds and $\kappa$ is weakly compact, then $\kappa$'s weak compactness is indestructible under $<\kappa$-closed forcing. This is because $\kappa$ being weakly compact is a $\Pi^1_2$ property over $H_\kappa$, and under $\text{MP}_{<\kappa}\text{-closed}(H_\kappa \cup \{\kappa\})$, $<\kappa$-closed-generic $\Sigma^1_2(H_\kappa)$ absoluteness holds.

2. Suppose $\kappa$ is measurable, $U$ is a normal ultrafilter on $\kappa$ and the set of $\bar{\kappa} < \kappa$ such that $\text{MP}_{<\bar{\kappa}}\text{-closed}(H_{\bar{\kappa}})$ holds has $U$-measure 1. Let $M = \text{Ult}(V, U)$. Then in $M$, $\text{MP}_{<\kappa}\text{-closed}(H_\kappa)$ holds, and $\kappa$ is weakly compact in $M$. So in $M$, $\kappa$ is an indestructible weakly compact cardinal. So, in $V$, the set of indestructible weakly compact cardinals below $\kappa$ has $U$-measure 1.

3. The strength of an indestructible weakly compact is at least that of a non-domestic mouse, by methods of Jensen, Schindler and Steel (cf. "Stacking Mice"), as was observed by Schindler and myself.
The previous remark shows an interesting aspect of the next lemma, because it provides a new way of producing an indestructible weakly compact cardinal, other than using the Laver preparation to make a supercompact cardinal indestructible.
The previous remark shows an interesting aspect of the next lemma, because it provides a new way of producing an indestructible weakly compact cardinal, other than using the Laver preparation to make a supercompact cardinal indestructible.

Before stating it, let’s make the following definition:
The previous remark shows an interesting aspect of the next lemma, because it provides a new way of producing an indestructible weakly compact cardinal, other than using the Laver preparation to make a supercompact cardinal indestructible.

Before stating it, let’s make the following definition:

**Definition 7.** Let $\kappa$ be a cardinal, $\gamma$ an ordinal, and $A$ a set. Then $\kappa$ is supercompact up to $\gamma$ wrt. $A$ if for every $\bar{\gamma} < \gamma$, there is a $\bar{\gamma}$-supercompact embedding $j$ of the universe, with critical point $\kappa$, such that $j(A) \cap V_{\bar{\gamma}} = A \cap V_{\bar{\gamma}}$ and $j(\kappa) > \bar{\gamma}$. 
The previous remark shows an interesting aspect of the next lemma, because it provides a new way of producing an indestructible weakly compact cardinal, other than using the Laver preparation to make a supercompact cardinal indestructible.

Before stating it, let’s make the following definition:

**Definition 7.** Let $\kappa$ be a cardinal, $\gamma$ an ordinal, and $A$ a set. Then $\kappa$ is supercompact up to $\gamma$ wrt. $A$ if for every $\bar{\gamma} < \gamma$, there is a $\bar{\gamma}$-supercompact embedding $j$ of the universe, with critical point $\kappa$, such that $j(A) \cap V_{\bar{\gamma}} = A \cap V_{\bar{\gamma}}$ and $j(\kappa) > \bar{\gamma}$.

The notion “almost huge to $\gamma$ wrt. $A$” is defined analogously. It is all just like in the case of strong cardinals.
Lemma 8. Assume that $\kappa < \rho$, $V_\kappa \prec V_\rho \prec V$ and $\kappa$ is supercompact up to $\rho + 1$ wrt. $A$, where $A = \{\bar{\rho} \mid \bar{\rho} \leq \rho \land V_{\bar{\rho}} \prec V \land \bar{\rho} \text{ is regular}\}$. Then there is a forcing $\mathbb{P}$ such that if $G$ is $\mathbb{P}$-generic over $V$, then in $V[G]$, $\kappa$ is measurable, $\text{MP}_{<\kappa - \text{dir. cl.}}(H_{\kappa^+})$ holds, and the set of $\lambda < \kappa$ which are regular and at which $\text{MP}_{<\lambda - \text{dir. cl.}}(H_{\lambda^+})$ holds has measure 1 wrt. any normal ultrafilter on $\kappa$ in $V[G]$. 
Lemma 8. Assume that $\kappa < \rho$, $V_\kappa \prec V_\rho \prec V$ and $\kappa$ is supercompact up to $\rho + 1$ wrt. $A$, where $A = \{\bar{\rho} \mid \bar{\rho} \leq \rho \land V_{\bar{\rho}} \prec V \land \bar{\rho}$ is regular.$\}$. Then there is a forcing $\mathbb{P}$ such that if $G$ is $\mathbb{P}$-generic over $V$, then in $V[G]$, $\kappa$ is measurable, $\text{MP}_{<\kappa-\text{dir. cl.}}(H_{\kappa^+})$ holds, and the set of $\lambda < \kappa$ which are regular and at which $\text{MP}_{<\lambda-\text{dir. cl.}}(H_{\lambda^+})$ holds has measure 1 wrt. any normal ultrafilter on $\kappa$ in $V[G]$.

Proof. The Silver argument works. Supercompactness wrt. $A$ is used in order to guarantee that $\mathbb{P}$ is an initial segment of $j(\mathbb{P})$, where $\mathbb{P}$ is the forcing iteration of length $\kappa + 1$ which forces the desired maximality principles.
Lemma 8. Assume that $\kappa < \rho$, $V_\kappa \prec V_\rho \prec V$ and $\kappa$ is supercompact up to $\rho + 1$ wrt. $A$, where $A = \{ \bar{\rho} \mid \bar{\rho} \leq \rho \land V_{\bar{\rho}} \prec V \land \bar{\rho} \text{ is regular.} \}$. Then there is a forcing $P$ such that if $G$ is $P$-generic over $V$, then in $V[G]$, $\kappa$ is measurable, $\text{MP}_{\kappa-\text{dir. cl.}}(H_{\kappa^+})$ holds, and the set of $\lambda < \kappa$ which are regular and at which $\text{MP}_{\lambda-\text{dir. cl.}}(H_{\lambda^+})$ holds has measure 1 wrt. any normal ultrafilter on $\kappa$ in $V[G]$.

Proof. The Silver argument works. Supercompactness wrt. $A$ is used in order to guarantee that $P$ is an initial segment of $j(P)$, where $P$ is the forcing iteration of length $\kappa + 1$ which forces the desired maximality principles. The gaps in the regular cardinals at which the principle holds are used in order to get the ultrafilter derived from the lifted embedding back in $V[G]$, and also for the master condition argument.
Lemma 9. Let $\kappa < \rho$, $\rho$ regular, $V_\rho \prec V$, $A = \{\bar{\rho} \mid \bar{\rho} \leq \rho \land V_{\bar{\rho}} \prec V\}$, and let $\kappa$ be $A$-supercompact to $\rho + 1$. Then there is a forcing $P$ which yields an extension $V[G]$ such that $\kappa$ is weakly compact in $V[G]$ and $\text{MP}_{\langle \kappa \rangle - \text{dir. cl.}}(H_{\kappa^+})$ holds at every regular $\bar{\kappa} \leq \kappa$. 

Up to (and including) a weakly compact
Lemma 9. Let $\kappa < \rho$, $\rho$ regular, $V_\rho \prec V$, $A = \{ \bar{\rho} \mid \bar{\rho} \leq \rho \land V_{\bar{\rho}} \prec V \}$, and let $\kappa$ be $A$-supercompact to $\rho + 1$. Then there is a forcing $\mathbb{P}$ which yields an extension $V[G]$ such that $\kappa$ is weakly compact in $V[G]$ and $\text{MP}_{<\bar{\kappa} - \text{dir. cl.}} \left( H_{\bar{\kappa}^+} \right)$ holds at every regular $\bar{\kappa} \leq \kappa$.

Proof. Let $\mathbb{P}$ be the length $\kappa + 1$ iteration forcing the desired Maximality Principles, and let $G$ be generic. Given a size $\kappa$ transitive model $M$ in the extension, it shows up in an extension of the form $V[G \restriction \kappa][\bar{G}]$, where $\bar{G}$ is the restriction of the last coordinate of $G$ to $\text{Col}(\kappa, < \bar{\rho})$, for some $\bar{\rho} < \rho$. Now a supercompact embedding $j : V \rightarrow P$ lifts to an embedding $j' : V[G \restriction \kappa][\bar{G}] \rightarrow P[G][H]$ ($H$ is generic over $V[G]$); the Silver master condition argument goes through because the size of the forcing on the left is less than the closure of the tail forcing. The ultrafilter derived from the restriction of $j'$ to $M$ is in $V[G]$, because it has size $\kappa < \rho$ and the tail forcing is $<\rho$-closed.
Lemma 10. Let $\kappa$ be almost huge up to $\rho + 2$ wrt. $A$, where

1. $\kappa < \rho \in A = \{\bar{\rho} \leq \rho \mid \bar{\rho} \text{ is regular and } V_{\bar{\rho}} \prec V\}$,

2. $\rho = \min(A \setminus (\kappa + 1))$.

Then there is a forcing extension of $V$ in which the following statements hold:

1. $\kappa$ is measurable and

2. $\text{MP}_{<\bar{\kappa} - \text{dir. cl.}}(H_{\bar{\kappa}^+})$ holds at every regular $\bar{\kappa} \leq \kappa$. 
Proof. Let $\mathbb{P}_{\kappa+1}$ be the notion of forcing making the desired Maximality Principles true.
Proof. Let $\mathbb{P}_{\kappa+1}$ be the notion of forcing making the desired Maximality Principles true. Force with $j(\mathbb{P}_\kappa)$ to obtain $V[G]$, where $j : V \rightarrow N$ is an almost huge embedding as in the Lemma, for $\rho + 1$. $j$ can be lifted to $j' : V[G \upharpoonright (\kappa + 1)] \rightarrow N[G][H]$, where $H$ is generic over $V[G]$ and contains a suitable master condition. Let $U$ be the ultrafilter derived from $j'$. Since $j(\kappa)$ is inaccessible in $V[G]$ and the tail forcing for which $H$ is generic is $<j(\kappa)$-closed, it follows that $U \in V[G]$. Since $V[G]$ and $V[G \upharpoonright (\kappa + 1)]$ have the same subsets of $\kappa$, $U$ is a normal ultrafilter on $\kappa$ from $V[G]$’s point of view.
Up to (and including) a (partial) supercompact

Lemma 11. Let $\kappa$ be almost huge wrt. $A = \{\rho \mid \rho$ is regular and $V_\rho \prec V\}$. 
Lemma 11. Let $\kappa$ be almost huge wrt. $A = \{ \rho \mid \rho$ is regular and $V_\rho \prec V \}$. (This is equivalent to saying that there is an almost huge embedding $j : V \rightarrow M$ with critical point $\kappa$, such that $V_{j(\kappa)} \prec V$.) Let $j : V \rightarrow M$ be almost huge wrt. $A$. Then there is a forcing extension of $V$ in which the following statements hold:

1. $\kappa$ is $<j(\kappa)$-supercompact, and

2. $\text{MP}_{<\bar{\kappa}-\text{dir. cl.}}(H_{\bar{\kappa}^+})$ holds at every regular $\bar{\kappa} \leq \kappa$. 
Lemma 11. Let $\kappa$ be almost huge wrt. $A = \{\rho \mid \rho$ is regular and $V_\rho \prec V\}$. (This is equivalent to saying that there is an almost huge embedding $j : V \rightarrow M$ with critical point $\kappa$, such that $V_j(\kappa) \prec V$.) Let $j : V \rightarrow M$ be almost huge wrt. $A$. Then there is a forcing extension of $V$ in which the following statements hold:

1. $\kappa$ is $<j(\kappa)$-supercompact, and

2. $\text{MP}_{<\kappa \text{-dir. cl.}}(H_{\kappa^+})$ holds at every regular $\bar{\kappa} \leq \kappa$.

Proof. Force with $j(\mathbb{P}_{\kappa+1})$, extend $j$ to an embedding from $V[G \upharpoonright \lambda]$ to $N[G][H]$, for arbitrarily large $\lambda < j(\kappa)$. $H$ is generic for a tail of $j(j(\mathbb{P})_\lambda)$. Derive a supercompactness measure and argue that it can be found in $V[G]$ and is a supercompactness measure there.
Lemma 11. Let $\kappa$ be almost huge wrt. $A = \{\rho \mid \rho \text{ is regular and } V_\rho \prec V\}$. (This is equivalent to saying that there is an almost huge embedding $j : V \rightarrow M$ with critical point $\kappa$, such that $V_{j(\kappa)} \prec V$.) Let $j : V \rightarrow M$ be almost huge wrt. $A$. Then there is a forcing extension of $V$ in which the following statements hold:

1. $\kappa$ is $<j(\kappa)$-supercompact, and

2. $\text{MP}_{<\kappa-\text{dir. cl.}}(H_{\bar{k}^+})$ holds at every regular $\bar{k} \leq \kappa$.

Proof. Force with $j(\mathbb{P}_{\kappa+1})$, extend $j$ to an embedding from $V[G \upharpoonright \lambda]$ to $N[G][H]$, for arbitrarily large $\lambda < j(\kappa)$. $H$ is generic for a tail of $j(j(\mathbb{P})_\lambda)$. Derive a supercompactness measure and argue that it can be found in $V[G]$ and is a supercompactness measure there.

I don’t know yet how to get a fully supercompact cardinal $\kappa$ such that the boldface closed maximality principles hold up to and including $\kappa$. 
Large Cardinals, Woodinized
Definition 12. A cardinal $\kappa$ is a Woodinized supercompact cardinal iff for every set $A \subseteq V_\kappa$, there is a cardinal $\tilde{\kappa} < \kappa$ which is supercompact up to $\kappa$ with respect to $A$. 
Definition 12. A cardinal $\kappa$ is a Woodinized supercompact cardinal iff for every set $A \subseteq V_\kappa$, there is a cardinal $\bar{\kappa} < \kappa$ which is supercompact up to $\kappa$ with respect to $A$. Woodinized almost huge cardinals are defined analogously.
Definition 12. A cardinal $\kappa$ is a Woodinized supercompact cardinal iff for every set $A \subseteq V_\kappa$, there is a cardinal $\bar{\kappa} < \kappa$ which is supercompact up to $\kappa$ with respect to $A$. Woodinized almost huge cardinals are defined analogously.

So a Woodin cardinal is just a Woodinized strong cardinal.
Definition 12. A cardinal \( \kappa \) is a Woodinized supercompact cardinal iff for every set \( A \subseteq V_\kappa \), there is a cardinal \( \bar{\kappa} < \kappa \) which is supercompact up to \( \kappa \) with respect to \( A \). Woodinized almost huge cardinals are defined analogously.

So a Woodin cardinal is just a Woodinized strong cardinal.

I am aiming at producing a model in which the boldface closed maximality principle holds up to a Woodin cardinal.
Definition 12. A cardinal $\kappa$ is a Woodinized supercompact cardinal iff for every set $A \subseteq V_\kappa$, there is a cardinal $\bar{\kappa} < \kappa$ which is supercompact up to $\kappa$ with respect to $A$. Woodinized almost huge cardinals are defined analogously.

So a Woodin cardinal is just a Woodinized strong cardinal.

I am aiming at producing a model in which the boldface closed maximality principle holds up to a Woodin cardinal. I seem to need strong assumptions, and get the result for a Woodinized supercompact cardinal.
Lemma 13. Let $\kappa$ have the following properties:
Lemma 13. Let $\kappa$ have the following properties:

1. $V_\kappa \prec V$, 
Lemma 13. Let $\kappa$ have the following properties:

1. $V_{\kappa} \prec V$,

2. $\kappa$ is a Woodinized almost huge cardinal.
Lemma 13. Let $\kappa$ have the following properties:

1. $V_\kappa \prec V$,

2. $\kappa$ is a Woodinized almost huge cardinal.

Then there is a forcing $\mathbb{P}$ such that if $G$ is $\mathbb{P}$-generic over $V$, then in $V[G]$, the following holds:

1. $\kappa$ is a Woodinized supercompact cardinal,
Lemma 13. Let $\kappa$ have the following properties:

1. $V_\kappa \prec V$,

2. $\kappa$ is a Woodinized almost huge cardinal.

Then there is a forcing $P$ such that if $G$ is $P$-generic over $V$, then in $V[G]$, the following holds:

1. $\kappa$ is a Woodinized supercompact cardinal,

2. $\text{MP}_{\prec \bar{\kappa} - \text{dir. cl.}} \left( H_\bar{\kappa}^+ \right)$ holds at every regular $\bar{\kappa} < \kappa$. 
Lemma 13. Let $\kappa$ have the following properties:

1. $V_\kappa \prec V$,

2. $\kappa$ is a Woodinized almost huge cardinal.

Then there is a forcing $\mathbb{P}$ such that if $G$ is $\mathbb{P}$-generic over $V$, then in $V[G]$, the following holds:

1. $\kappa$ is a Woodinized supercompact cardinal,

2. $\text{MP}_{<\bar{\kappa}}\text{-dir. cl.}(H_{\bar{\kappa}^+})$ holds at every regular $\bar{\kappa} < \kappa$.

Proof. Forcing with $\mathbb{P}_A$ does the trick, where $A$ is the set of fully reflecting cardinals below $\kappa$. 
Up to (and including) a Woodinized supercompact cardinal

Corollary 14. Let $\kappa$ have the properties of the previous lemma:
Up to (and including) a Woodinized supercompact cardinal

Corollary 14. Let $\kappa$ have the properties of the previous lemma:

1. $V_\kappa \prec V_\rho \prec V$, where $\rho$ is regular and $\rho > \kappa$, and
Corollary 14. Let $\kappa$ have the properties of the previous lemma:

1. $V_{\kappa} \prec V_{\rho} \prec V$, where $\rho$ is regular and $\rho > \kappa$, and

2. $\kappa$ is a Woodinized almost huge cardinal.
Corollary 14. Let $\kappa$ have the properties of the previous lemma:

1. $V_{\kappa} \prec V_{\rho} \prec V$, where $\rho$ is regular and $\rho > \kappa$, and

2. $\kappa$ is a Woodinized almost huge cardinal.

Then there is a forcing $\mathbb{P}$ such that if $G$ is $\mathbb{P}$-generic over $V$, then in $V[G]$, the following holds:
Corollary 14. Let $\kappa$ have the properties of the previous lemma:

1. $V_{\kappa} \prec V_{\rho} \prec V$, where $\rho$ is regular and $\rho > \kappa$, and

2. $\kappa$ is a Woodinized almost huge cardinal.

Then there is a forcing $\mathbb{P}$ such that if $G$ is $\mathbb{P}$-generic over $V$, then in $V[G]$, the following holds:

1. $\kappa$ is a Woodinized supercompact cardinal,
Corollary 14. Let $\kappa$ have the properties of the previous lemma:

1. $V_\kappa \prec V_\rho \prec V$, where $\rho$ is regular and $\rho > \kappa$, and
2. $\kappa$ is a Woodinized almost huge cardinal.

Then there is a forcing $\mathbb{P}$ such that if $G$ is $\mathbb{P}$-generic over $V$, then in $V[G]$, the following holds:

1. $\kappa$ is a Woodinized supercompact cardinal,
2. $\text{MP}_{<\kappa-\text{dir. cl.}}(H_{\kappa^+})$ holds at every regular $\kappa \leq \kappa$. 