PATH PROPERTIES OF CAUCHY'S PRINCIPAL VALUES RELATED TO LOCAL TIME

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Abstract: Sample path properties of Cauchy's principal values of Brownian and random walk local times are studied.

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1. Introduction and main results

Recently a collection of papers appeared in Yor (1997) that deal with principal values related to Brownian motion (cf. Part B of Yor 1997).

Let $W(\cdot)$ be a standard Wiener process (Brownian motion) with local time $L(x,t)$, i.e. for any Borel function $f \geq 0$ and $t \geq 0$,

$$\int_0^t f(W(s))\, ds = \int_{\mathbf{R}} f(x) L(x,t)\, dx.$$

Define the integral $\int_0^t ds/W^\alpha(s)$ (notation: $z^\alpha = |z|^\alpha \operatorname{sgn}(z)$) in the sense of Cauchy's principal value:

$$Y_\alpha(t) \stackrel{\text{def}}{=} \int_0^t \frac{ds}{W^\alpha(s)} = \int_0^\infty \frac{L(x,t) - L(-x,t)}{x^\alpha}\, dx. \tag{1.1}$$

It is well-known (McKean 1962, Borodin 1985) that the local time in the space variable is Hölder continuous of order $(1/2 - \varepsilon)$ for any $\varepsilon > 0$, which ensures the finiteness (for each $t \geq 0$) of the second integral in (1.1) whenever $\alpha < 3/2$. We shall from now on assume this condition. Strictly speaking, the first integral is defined as Cauchy's principal value for $1 \leq \alpha < 3/2$ (cf., e.g., Sections 1.1-1.3 in Alili 1997 and references therein), and as Riemann integral for $\alpha < 1$.

The aim of our paper is to study the almost sure (a.s.) path behaviour of the process $Y_\alpha(\cdot)$. For a significant first step along these lines we refer to the paper by Hu and Shi (1997), who proved the local, as well as the global, laws of the iterated logarithm (LIL) for $Y_1(\cdot)$. Namely, they established the following results:

$$\limsup_{t\to\infty} \frac{Y_1(t)}{\sqrt{t \log\log t}} = 2\sqrt{2}\,, \qquad \text{a.s.} \tag{1.2}$$

and

$$\limsup_{h\to 0} \frac{Y_1(h)}{\sqrt{h \log\log(1/h)}} = 2\sqrt{2}\,, \qquad \text{a.s.} \tag{1.3}$$

We are interested in studying the modulus of continuity and large increment properties (including the LIL) of $Y_\alpha(\cdot)$, as well as appropriate properties of a simple symmetric random walk along these lines. Due however to lack of precise distributional properties of $Y_\alpha(\cdot)$, when $\alpha \neq 1$, we could not obtain the desirable exact constants, though the rates we establish here are optimal.

We present now our main results. First we prove the upper bounds for the LIL, large increments and modulus of continuity. Concerning (1.6) of Theorem 1.1, we assume that a_T is a non-decreasing function of T, such that $0 < a_T \leq T$ and a_T/T is non-increasing.

Theorem 1.1. *For $0 < \alpha < 3/2$ we have*

$$\limsup_{t\to\infty} \frac{|Y_\alpha(t)|}{t^{1-\alpha/2}(\log\log t)^{\alpha/2}} \le c_1(\alpha), \qquad \text{a.s.} \tag{1.4}$$

$$\limsup_{t\to\infty} \frac{\sup_{0\le t\le T-a_T}\sup_{0\le s\le a_T}|Y_\alpha(t+s)-Y_\alpha(t)|}{a_T^{1-\alpha/2}\left(\log(T/a_T)+\log\log T\right)^{\alpha/2}} \le c_1(\alpha), \qquad \text{a.s.} \tag{1.5}$$

$$\limsup_{h\to 0} \frac{|Y_\alpha(h)|}{h^{1-\alpha/2}(\log\log(1/h))^{\alpha/2}} \le c_1(\alpha), \qquad \text{a.s.} \tag{1.6}$$

$$\limsup_{h\to 0} \frac{\sup_{0\le t\le 1-h}\sup_{0\le s\le h}|Y_\alpha(t+s)-Y_\alpha(t)|}{h^{1-\alpha/2}\left(\log(1/h)\right)^{\alpha/2}} \le c_1(\alpha), \qquad \text{a.s.} \tag{1.7}$$

Here, the constant $c_1(\alpha)$ is given by

$$c_1(\alpha) = \frac{3\cdot 2^{5\alpha/6}}{\alpha^{2\alpha/3}(3-2\alpha)^{1-\alpha/3}(2-\alpha)^{\alpha/3}}. \tag{1.8}$$

Concerning the constant in LIL, we have the following result

Theorem 1.2. *For $0 < \alpha < 3/2$, there exists a finite positive constant $c_2(\alpha)$ such that*

$$\limsup_{t\to\infty} \frac{|Y_\alpha(t)|}{t^{1-\alpha/2}(\log\log t)^{\alpha/2}} = c_2(\alpha) \in \left[2^{3\alpha/2}\,\Gamma(2-\alpha),\, c_1(\alpha)\right], \qquad \text{a.s.} \tag{1.9}$$

The LIL holds true also for random walks via an invariance principle. Let S_i, $i = 1, 2, \ldots$ be a simple symmetric random walk on the line, starting from 0, and let $\xi(x, n) = \#\{i : 1 \le i \le n, S_i = x\}$ be its local time. Define

$$G_\alpha(n) \stackrel{\text{def}}{=} \sum_{k=1}^{n} \frac{\mathbf{1}_{[S_k\ne 0]}}{S_k^\alpha} = \sum_{i=1}^{\infty} \frac{\xi(i,n)-\xi(-i,n)}{i^\alpha}. \tag{1.10}$$

We prove the following invariance principle.

Theorem 1.3. *On a suitable probability space one can define a Wiener process $\{W(t),\, t \ge 0\}$ and a simple symmetric random walk $\{S_n,\, n = 1, 2, \ldots\}$ such that for any $0 < \alpha < 3/2$ and sufficiently small $\varepsilon > 0$ we have*

$$|Y_\alpha([t]) - G_\alpha([t])| = o(t^{1-\alpha/2-\varepsilon}), \qquad \text{a.s.,} \tag{1.11}$$

as $t \to \infty$.

As a consequence of our Theorem 1.3, the LILs (1.2), (1.4) and (1.9) remain true if Y_α is replaced by G_α.

As it is easily seen, Y_α is not defined for $\alpha \geq 3/2$. In this case, we consider instead the process

$$Z_\alpha(t) \stackrel{\text{def}}{=} \int_0^t \frac{\mathbf{1}_{[|W(s)|\geq 1]}}{W^\alpha(s)}\, ds = \int_1^\infty \frac{L(x,t) - L(-x,t)}{x^\alpha}\, dx. \tag{1.12}$$

This is a "nice" additive functional, for which the strong approximation of Csáki et al. (1992) holds. The limit process associated with such functionals is $V(t) = B(L(t))$, where $B(\cdot)$ is a standard Wiener process and $L(\cdot)$ is a Wiener local time at zero, independent of B, a version of iterated Brownian motion (cf., e.g., Csáki et al. 1992).

Theorem 1.4. *If $\alpha > 3/2$, then on a rich enough probability space one can define a Wiener process $\{W(t),\, t \geq 0\}$ and a process $\{V(t),\, t \geq 0\}$ such that for $\varepsilon > 0$ small enough*

$$Z_\alpha(t) = \sigma V(t) + o(t^{1/4-\varepsilon}), \qquad \text{a.s.}, \tag{1.13}$$

as $t \to \infty$, where

$$\sigma^2 \stackrel{\text{def}}{=} \frac{16}{(\alpha-1)(2\alpha-3)}. \tag{1.14}$$

For the random walk case we have

Theorem 1.5. *If $\alpha > 3/2$, then on a rich enough probability space one can define a simple symmetric random walk $\{S_n,\, n = 1, 2, \ldots\}$ and a process $\{V(t),\, t \geq 0\}$ such that for $\varepsilon > 0$ small enough*

$$G_\alpha(t) = \sigma_0 V(t) + o(t^{1/4-\varepsilon}), \qquad \text{a.s.}, \tag{1.15}$$

as $t \to \infty$, where

$$\sigma_0^2 \stackrel{\text{def}}{=} 16 \sum_{\ell=1}^\infty \sum_{k=\ell+1}^\infty \frac{1}{k^\alpha \ell^{\alpha-1}} + 8 \sum_{k=1}^\infty \frac{1}{k^{2\alpha-1}} - 2 \sum_{k=1}^\infty \frac{1}{k^{2\alpha}}. \tag{1.16}$$

The rest of the paper is organized as follows. In Section 2, we briefly describe our method. Depending on whether we are interested in the process Y_α or in its increments, we shall use slightly different approaches, which are discussed in separated subsections for the sake of clarity. Theorems 1.1, 1.2 and 1.3 are proved in Sections 3, 4 and 5 respectively. Finally, Section 6 is devoted to the proof of Theorems 1.4 and 1.5.

Throughout the paper, some universal (finite and positive) constants are denoted by the letter c with subscripts. When they depend on some (possibly multi-dimensional) parameter p, they are denoted by $c(p)$ with subscripts.

Our use of "almost surely" is not systematic.

2. Preliminaries

We describe the main lines of our approach that will be used in the proofs in the sequel. Throughout, we assume $\alpha < 3/2$. We shall be using somewhat different approaches for the process Y_α and its increments. Our basic tools are: for Y_α, a martingale inequality due to Barlow and Yor (1982); and for the increments of Y_α, Tanaka's formula together with some elementary stochastic calculus.

2.1. The process Y_α

For any $b > 0$, consider the decomposition

$$Y_\alpha(t) = \int_0^b \frac{L(x,t) - L(-x,t)}{x^\alpha}\, dx + \int_b^\infty \frac{L(x,t) - L(-x,t)}{x^\alpha}\, dx. \tag{2.1}$$

It is easy to estimate the second expression on the right hand side. Indeed, by the occupation time formula,

$$\left| \int_b^\infty \frac{L(x,t) - L(-x,t)}{x^\alpha}\, dx \right| \le \frac{1}{b^\alpha} \int_{\mathbf{R}} L(y,t)\, dy = \frac{t}{b^\alpha}, \tag{2.2}$$

so that, for any $b > 0$,

$$|Y_\alpha(t)| \le \left| \int_0^b \frac{L(x,t) - L(-x,t)}{x^\alpha}\, dx \right| + \frac{t}{b^\alpha}. \tag{2.3}$$

To treat the integral expression on the right hand side, we first recall the following useful inequality.

Fact 2.1 (Barlow and Yor 1982). *For any $t > 0$, $\varepsilon \in (0, 1/2]$ and $\gamma \ge 1$,*

$$\mathbf{E} \left(\sup_{0\le s\le t} \sup_{x\neq y} \frac{|L_s^x - L_s^y|}{|x-y|^{1/2-\varepsilon}} \right)^\gamma \le c_3\, t^{(1+2\varepsilon)\gamma/4}, \tag{2.4}$$

where $c_3 = c_3(\gamma, \varepsilon)$.

Fact 2.1 allows us to control the almost sure asymptotics (when t is large) of expressions like $\sup_{x\neq y} |L_t^x - L_t^y|/|x-y|^{1/2-\varepsilon}$. Indeed, let $\nu \in [0, 1/2)$ and let $\varepsilon > 0$. By Chebyshev's inequality and (2.4), for any $\gamma \ge 1$ and $n \ge 1$,

$$\mathbf{P} \left(\sup_{0\le s\le n} \sup_{x\neq y} \frac{|L_s^x - L_s^y|}{|x-y|^\nu} > n^{(1-\nu)/2+\varepsilon} \right) \le c_4(\gamma, \nu, \varepsilon)\, n^{-\gamma\varepsilon}.$$

Take $\gamma = 2/\varepsilon$ and use the Borel–Cantelli lemma to see that

$$\sup_{0\le s\le n}\sup_{x\neq y}\frac{|L_s^x - L_s^y|}{|x-y|^\nu} = \mathcal{O}(n^{(1-\nu)/2+\varepsilon}), \qquad \text{a.s.}$$

Since $t \mapsto \sup_{0\le s\le t}\sup_{x\neq y}|L_s^x - L_s^y|/|x-y|^\nu$ is non-decreasing, and since ε can be arbitrarily small, we have proved the following result.

Lemma 2.2. *For any $\nu \in [0, 1/2)$ and $\varepsilon > 0$, when $t \to \infty$,*

$$\sup_{x\neq y}\frac{|L_t^x - L_t^y|}{|x-y|^\nu} = o(t^{(1-\nu)/2+\varepsilon}), \qquad \text{a.s.} \tag{2.5}$$

This lemma will allow us to obtain useful estimates for the integral expression on the right hand side of (2.3). For example, since $\alpha < 3/2$, we can choose $\nu \in [0, 1/2)$ to be as close to $1/2$ as possible, such that $\alpha - \nu < 1$. As a consequence, for any fixed $b > 0$ and $\varepsilon > 0$, when $t \to \infty$,

$$\int_0^b \frac{L(x,t) - L(-x,t)}{x^\alpha}\,dx = o(t^{1/4+\varepsilon}), \qquad \text{a.s.} \tag{2.6}$$

2.2. The increments of Y_α

In Section 3, we shall be interested in the increments of Y_α. Let us first fix $b > 0$ and recall (2.1) here:

$$Y_\alpha(t) = \int_0^b \frac{L(x,t) - L(-x,t)}{x^\alpha}\,dx + \int_b^\infty \frac{L(x,t) - L(-x,t)}{x^\alpha}\,dx.$$

We now use Tanaka's formula for the first term on the right hand side:

$$L(x,t) = |W(t) - x| - |x| - \int_0^t \operatorname{sgn}(W(s) - x)\,dW(s),$$

(with the usual notation in Tanaka's formula: $\operatorname{sgn}(0) = -1$). Hence,

$$\begin{aligned}
&\int_0^b \frac{L(x,t) - L(-x,t)}{x^\alpha}\,dx \\
&= \int_0^b \frac{A(x, W(t))}{x^\alpha}\,dx + \int_0^b \frac{2\int_0^t \mathbf{1}_{[|W(s)|\le x]}\,dW(s)}{x^\alpha}\,dx \\
&= \int_0^b \frac{A(x, W(t))}{x^\alpha}\,dx + 2\int_0^t \mathbf{1}_{[|W(s)|\le b]}\frac{b^{1-\alpha} - |W(s)|^{1-\alpha}}{1-\alpha}\,dW(s),
\end{aligned} \tag{2.7}$$

where

$$A(x, W(t)) \stackrel{\text{def}}{=} |W(t) - x| - |W(t) + x|. \tag{2.8}$$

It is easily seen that

$$|A(x, W(t))| \leq 2x. \tag{2.9}$$

For the second term on the right hand side of (2.7), we may apply the Dambis–Dubins–Schwarz theorem for continuous local martingales (cf., e.g., Theorem V.1.6 of Revuz and Yor 1994) to conclude that for the stochastic integral $\int_0^t f(W(s))\, dW(s)$, there exists a standard Wiener process $B(\cdot)$ such that

$$\int_0^t f(W(s))\, dW(s) = B(U(t)), \tag{2.10}$$

where

$$U(t) \stackrel{\text{def}}{=} \int_0^t f^2(W(s))\, ds.$$

Moreover, $U(t)$ is a stopping time for $B(\cdot)$, hence $\widetilde{B}(s) \stackrel{\text{def}}{=} B(U(t) + s) - B(U(t)),\ s \geq 0$, is also a standard Wiener process for fixed t.

To study the increments of $Y_\alpha(\cdot)$, we have

$$\begin{aligned} Y_\alpha(t+h) - Y_\alpha(t) &= \int_0^b \frac{A(x, W(t+h)) - A(x, W(t))}{x^\alpha}\, dx \\ &+ B(U(t+h)) - B(U(t)) \\ &+ \int_b^\infty \frac{L(x, t+h) - L(x,t) - (L(-x, t+h) - L(-x, t))}{x^\alpha}\, dx. \end{aligned} \tag{2.11}$$

Here,

$$\begin{aligned} U(t) &= 4 \int_0^t \mathbf{1}_{[|W(s)| \leq b]} \left(\frac{b^{1-\alpha} - |W(s)|^{1-\alpha}}{1-\alpha} \right)^2 ds \\ &= 4 \int_{-b}^b \left(\frac{b^{1-\alpha} - |x|^{1-\alpha}}{1-\alpha} \right)^2 L(x,t)\, dx. \end{aligned} \tag{2.12}$$

Hence

$$\begin{aligned} U(t+h) - U(t) &= 4 \int_{-b}^b \left(\frac{b^{1-\alpha} - |x|^{1-\alpha}}{1-\alpha} \right)^2 (L(x, t+h) - L(x,t))\, dx \\ &\leq \sup_x (L(x, t+h) - L(x,t))\, 4 \int_{-b}^b \left(\frac{b^{1-\alpha} - |x|^{1-\alpha}}{1-\alpha} \right)^2 dx \\ &= \sup_x (L(x, t+h) - L(x,t))\, c_5(\alpha)\, b^{3-2\alpha}, \end{aligned}$$

where

$$c_5(\alpha) = 4\int_{-1}^{1}\left(\frac{1-|u|^{1-\alpha}}{1-\alpha}\right)^2 du = \frac{16}{(2-\alpha)(3-2\alpha)}. \tag{2.13}$$

Furthermore, denoting

$$\widetilde{B}^*(t) \stackrel{\text{def}}{=} \sup_{0\le s\le t} |\widetilde{B}(s)|$$

and

$$L^*(h,t) \stackrel{\text{def}}{=} \sup_x (L(x,t+h) - L(x,t)),$$

we obtain

$$|B(U(t+h)) - B(U(t))| \le \widetilde{B}^*\left(c_5(\alpha)\, b^{3-2\alpha}\, L^*(h,t)\right). \tag{2.14}$$

For the third term in (2.11) we use the estimate

$$\begin{aligned} &\left|\int_b^\infty \frac{L(x,t+h) - L(x,t) - (L(-x,t+h) - L(-x,t))}{x^\alpha}\, dx\right| \\ &\le \frac{1}{b^\alpha}\int_{-\infty}^{\infty} (L(x,t+h) - L(x,t))\, dx = \frac{h}{b^\alpha}. \end{aligned} \tag{2.15}$$

Putting (2.11), (2.9), (2.14) and (2.15) together, we finally get that, for any $b > 0$,

$$|Y_\alpha(t+h) - Y_\alpha(t)| \le \left(\frac{4b^2}{2-\alpha} + h\right)\frac{1}{b^\alpha} + \widetilde{B}^*\left(c_5(\alpha)\, b^{3-2\alpha}\, L^*(h,t)\right). \tag{2.16}$$

Note that repeating the same procedure in the case $\alpha = 1$ with obvious modifications, one can easily see that the final conclusion (2.16) holds true in this case as well.

3. Proof of Theorem 1.1

The main step in proving Theorem 1.1 is the following upper bound.

Lemma 3.1. *For $0 < \alpha < 3/2$ and $\delta > 0$ there exist positive constants $c_6 = c_6(\alpha,\delta)$ and $\lambda_0 = \lambda_0(\alpha,\delta)$ such that for any $t > 0$, $h > 0$ and $\lambda \ge \lambda_0$,*

$$\mathbf{P}\left(|Y_\alpha(t+h) - Y_\alpha(t)| > \lambda h^{1-\alpha/2}\right) \le c_6 \exp(-c_7\, \lambda^{2/\alpha}), \tag{3.1}$$

where

$$c_7 = c_7(\alpha,\delta) = \frac{\alpha^{4/3}(3-2\alpha)^{2/\alpha-2/3}(2-\alpha)^{2/3}}{3^{2/\alpha}2^{5/3}(1+\delta)^{2/\alpha-1}}.$$

Proof. It follows from (2.16) that, for any $b > 0$,

$$\begin{aligned} &\mathbf{P}(|Y_\alpha(t+h) - Y_\alpha(t)| > \lambda h^{1-\alpha/2}) \\ \le\ &\mathbf{P}\left(\widetilde{B}^*(c_5(\alpha)\, b^{3-2\alpha}\, L^*(h,t)) \ge \lambda h^{1-\alpha/2} - \left(\frac{4b^2}{2-\alpha} + h\right)\frac{1}{b^\alpha}\right). \end{aligned} \tag{3.2}$$

Let $\delta > 0$ be arbitrarily small, and choose

$$b = h^{1/2}\left(\frac{3(1+\delta)}{(3-2\alpha)\lambda}\right)^{1/\alpha}, \tag{3.3}$$

so that

$$\lambda h^{1-\alpha/2} - (1+\delta)\frac{h}{b^\alpha} = \frac{2\alpha\lambda h^{1-\alpha/2}}{3}. \tag{3.4}$$

Define λ_0 by

$$\lambda_0 = \frac{3(1+\delta)4^{\alpha/2}}{(3-2\alpha)\delta^{\alpha/2}}.$$

Then for $\lambda \ge \lambda_0$,

$$\frac{4b^2}{2-\alpha} \le \delta h. \tag{3.5}$$

Collecting (3.2), (3.5) and (3.4) yields that for any $A > 0$,

$$\begin{aligned} &\mathbf{P}(|Y_\alpha(t+h) - Y_\alpha(t)| > \lambda h^{1-\alpha/2}) \\ \le\ &\mathbf{P}\left(\widetilde{B}^*(c_5(\alpha)\, b^{3-2\alpha}\, L^*(h,t)) > \lambda h^{1-\alpha/2} - (1+\delta)\frac{h}{b^\alpha}\right) \\ =\ &\mathbf{P}\left(\widetilde{B}^*(c_5(\alpha)\, b^{3-2\alpha}\, L^*(h,t)) > \frac{2\alpha\lambda h^{1-\alpha/2}}{3}\right) \\ \le\ &\mathbf{P}(L^*(h,t) > Ah^{1/2}) + \mathbf{P}\left(\widetilde{B}^*(c_5(\alpha)\, b^{3-2\alpha}\, Ah^{1/2}) > \frac{2\alpha\lambda h^{1-\alpha/2}}{3}\right). \end{aligned}$$

By the usual estimate for Gaussian tails,

$$\mathbf{P}(\widetilde{B}^*(t) \ge x) \le 4\exp\left(-\frac{x^2}{2t}\right). \tag{3.6}$$

On the other hand, for any $\delta > 0$, $t > 0$ and $\lambda > 0$,

$$\mathbf{P}\left(\sup_{x\in\mathbf{R}} L(x,t) \ge \lambda\right) \le c_8(\delta)\, \exp\left(-\frac{\lambda^2}{2(1+\delta)t}\right), \tag{3.7}$$

cf. Kesten (1965) and Csáki (1989). For forthcoming applications, we also recall the LIL for the maximum local time:

$$\limsup_{t\to\infty} (2t\log\log t)^{-1/2} \sup_{x\in\mathbf{R}} L(x,t) = 1, \qquad \text{a.s.} \tag{3.8}$$

Since $L^*(h,t)$ has the same distribution as $\sup_{x\in\mathbf{R}} L(x,h)$, we have

$$\mathbf{P}(L^*(h,t) \geq A h^{1/2}) \leq c_8(\delta) \exp\left(-\frac{A^2}{2(1+\delta)} \right).$$

Hence

$$\begin{aligned} & \mathbf{P}(|Y_\alpha(t+h) - Y_\alpha(t)| > \lambda h^{1-\alpha/2}) \\ \leq \;\; & c_9(\delta) \left(\exp\left(-\frac{A^2}{2(1+\delta)} \right) + \exp\left(-\frac{2\alpha^2\lambda^2 h^{3/2-\alpha}}{9c_5(\alpha)\, b^{3-2\alpha}\, A} \right) \right). \end{aligned}$$

Putting (3.3) and choosing A such that the two exponents above should be equal, we get (3.1). □

Proof of Theorem 1.1. Follows from Csáki and Csörgő (1992, Theorems 3.1 and 3.2) and Lemma 3.1. □

4. Proof of Theorem 1.2

For brevity, we write throughout the section

$$h(t) = \Big(\frac{t}{\log\log t} \Big)^{1/2}.$$

The proof of Theorem 1.2 is based on the following

Fact 4.1 (Donsker and Varadhan 1977). *Let*

$$\widehat{L}_t(x) = (t\log\log t)^{-1/2} L\left(x\, h(t),\, t\right), \qquad x \in \mathbf{R},$$

and let $\mathcal{A}$ denote the space of all uniformly continuous subprobability densities on $\mathbf{R}$, endowed with the topology of uniform convergence on bounded intervals. Then for any continuous functional Φ on $\mathcal{A}$,

$$\limsup_{t\to\infty} \Phi(\widehat{L}_t) = \sup_{f\in\mathcal{A},\, I(f)\leq 1} \Phi(f), \qquad \text{a.s.,} \tag{4.1}$$

where

$$I(f) = \frac{1}{8}\int_{\mathbf{R}} \frac{(f'(y))^2}{f(y)}\,dy.$$

Proof of Theorem 1.2. Let $\delta \in (0, 1/2)$. Observe that

$$\begin{aligned} Y_\alpha(t) &= \Big(\int_0^{\delta h(t)} + \int_{\delta h(t)}^{h(t)/\delta} + \int_{h(t)/\delta}^{\infty}\Big) \frac{L(x,t) - L(-x,t)}{x^\alpha}\,dx \\ &= \Big(\int_0^{\delta h(t)} + \int_{h(t)/\delta}^{\infty}\Big) \frac{L(x,t) - L(-x,t)}{x^\alpha}\,dx \\ &\qquad + t^{1-\alpha/2}(\log\log t)^{\alpha/2} \int_\delta^{1/\delta} \frac{\widehat{L}_t(x) - \widehat{L}_t(-x)}{x^\alpha}\,dx. \end{aligned}$$

For any fixed $\delta \in (0, 1/2)$, we can apply (4.1) to see that, almost surely,

$$\limsup_{t\to\infty} \int_\delta^{1/\delta} \frac{\widehat{L}_t(x) - \widehat{L}_t(-x)}{x^\alpha}\,dx = \sup_{f\in\mathcal{A},\, I(f)\le 1} \int_\delta^{1/\delta} \frac{f(x) - f(-x)}{x^\alpha}\,dx.$$

Let $q > 1$ and take the function

$$f(x) = \begin{cases} p\, x^q \exp(-rx), & \text{if } x \ge 0, \\ \\ p\, |x|^q \exp(-\widetilde{r}\,|x|), & \text{if } x < 0, \end{cases}$$

for some positive constants p, r and $\widetilde{r}$. It is easily checked that if

$$\frac{p\,\Gamma(q+1)}{r^{q+1}} + \frac{p\,\Gamma(q+1)}{\widetilde{r}^{\,q+1}} \le 1, \tag{4.2}$$

then f is a continuous subprobability density function. Furthermore, whenenever

$$\frac{p\,\Gamma(q)}{r^{q-1}} + \frac{p\,\Gamma(q)}{\widetilde{r}^{\,q-1}} \le 8, \tag{4.3}$$

we have $I(f) \le 1$. Therefore, under (4.2) and (4.3),

$$\limsup_{t\to\infty} \int_\delta^{1/\delta} \frac{\widehat{L}_t(x) - \widehat{L}_t(-x)}{x^\alpha}\,dx \ge \int_\delta^{1/\delta} p\, x^{q-\alpha}\,(e^{-rx} - e^{-\widetilde{r}\,x})\,dx.$$

By the dominated convergence theorem, we can send $\widetilde{r}$ to infinity to see that, for all $\delta \in (0, 1/2)$ and all positive p and r and $q > 1$ satisfying $p\,\Gamma(q+1) < r^{q+1}$ and $p\,\Gamma(q) < 8r^{q-1}$,

$$\limsup_{t\to\infty} \int_\delta^{1/\delta} \frac{\widehat{L}_t(x) - \widehat{L}_t(-x)}{x^\alpha}\,dx \ge \int_\delta^{1/\delta} p\, x^{q-\alpha} e^{-rx}\,dx, \qquad \text{a.s.}$$

Since δ can be as small as possible, this leads to: for $q > 1$, $p\,\Gamma(q+1) < r^{q+1}$ and $p\,\Gamma(q) < 8r^{q-1}$,

$$\liminf_{\delta\to 0}\ \limsup_{t\to\infty} \int_\delta^{1/\delta} \frac{\widehat{L}_t(x) - \widehat{L}_t(-x)}{x^\alpha}\, dx \geq \frac{p\,\Gamma(q-\alpha+1)}{r^{q-\alpha+1}}, \qquad \text{a.s.}$$

Let first $q \to 1^+$, and then send p and r respectively to 8 and $\sqrt{8}$, to arrive at:

$$\liminf_{\delta\to 0}\ \limsup_{t\to\infty} \int_\delta^{1/\delta} \frac{\widehat{L}_t(x) - \widehat{L}_t(-x)}{x^\alpha}\, dx \geq 2^{3\alpha/2}\,\Gamma(2-\alpha), \qquad \text{a.s.}$$

Write for brevity

$$\phi_\alpha(t) = t^{1-\alpha/2}(\log\log t)^{\alpha/2}. \tag{4.4}$$

If we could furthermore show that almost surely,

$$\lim_{\delta\to 0}\ \limsup_{t\to\infty} \frac{1}{\phi_\alpha(t)} \int_0^{\delta h(t)} \frac{L(x,t) - L(-x,t)}{x^\alpha}\, dx = 0, \tag{4.5}$$

$$\lim_{\delta\to 0}\ \limsup_{t\to\infty} \frac{1}{\phi_\alpha(t)} \int_{h(t)/\delta}^{\infty} \frac{L(x,t) - L(-x,t)}{x^\alpha}\, dx = 0, \tag{4.6}$$

then the proof of Theorem 1.2 would be completed. The rest of the section is devoted to the verification of (4.5) and (4.6). □

Proof of (4.6). Follows immediately from (2.2) (taking $b = h(t)/\delta$ there) and the definition of the function $h(\cdot)$. □

Proof of (4.5). Assume $\alpha \neq 1$ for the moment. Recall (2.7):

$$\begin{aligned} &\int_0^b \frac{L(x,t) - L(-x,t)}{x^\alpha}\, dx \\ &= \int_0^b \frac{A(x,W(t))}{x^\alpha}\, dx + 2\int_0^t \mathbf{1}_{[|W(s)|\leq b]} \frac{b^{1-\alpha} - |W(s)|^{1-\alpha}}{1-\alpha}\, dW(s), \end{aligned} \tag{4.7}$$

with $|A(x,W(t))| \leq 2x$ for any $t > 0$ and $x > 0$. The first term on the right hand side is easily controlled, since its modulus is smaller than or equal to a constant multiple of $b^{2-\alpha}$. To study the second term, let us write

$$\Lambda(t,b) \overset{\text{def}}{=} \int_0^t \mathbf{1}_{[|W(s)|\leq b]} \frac{b^{1-\alpha} - |W(s)|^{1-\alpha}}{1-\alpha}\, dW(s).$$

The proof of (4.5) will be completed as soon as we show

$$\lim_{\delta\to 0}\ \limsup_{t\to\infty} \frac{\Lambda(t,\delta h(t))}{\phi_\alpha(t)} = 0, \qquad \text{a.s.} \tag{4.8}$$

For each $b>0$, $t\mapsto \Lambda(t,b)$ is a Brownian time change as we briefly described in Section 2. However, to check (4.8), we have to consider the situation when b depends on t. Obviously, the process $t\mapsto \Lambda(t,\delta h(t))$ is no longer a local martingale (it is even not clear whether it is a semimartingale). So we have to handle it with care.

Fix $b>0$ for the moment. Then by (2.10) and (2.12), there exists a Brownian motion B such that $\Lambda(t,b)=B(U(t))$ for all $t\ge 0$, where

$$\begin{aligned} U(t) &= 4\int_{-b}^{b}\left(\frac{b^{1-\alpha}-|x|^{1-\alpha}}{1-\alpha}\right)^2 L(x,t)\,dx \\ &\le 4c_5(\alpha)\,b^{3-2\alpha}\sup_{x\in\mathbf{R}} L(x,t), \end{aligned} \tag{4.9}$$

(L being the local time of B. For the value of the constant $c_5(\alpha)$, cf. (2.13)). Note that both the Brownian motion B and its clock U depend on b. Also, (4.9) holds even in the case $\alpha=1$ (so from now on we only make the general assumption $\alpha<3/2$).

We have, for any $y>0$ and $z>0$,

$$\begin{aligned} \mathbf{P}\left(\sup_{0\le s\le t}|\Lambda(s,b)|>y\right) &\le \mathbf{P}\left(\sup_{0\le s\le t}|B(U(s))|>y\right) \\ &\le \mathbf{P}\left(\sup_{x\in\mathbf{R}} L(x,t)>z\right)+\left(\sup_{0\le u\le 4c_5(\alpha)\,b^{3-2\alpha}\,z}|B(u)|>y\right) \\ &\le c_{10}\,\exp\left(-\frac{z^2}{3t}\right)+4\exp\left(-\frac{y^2}{8c_5(\alpha)\,b^{3-2\alpha}\,z}\right), \end{aligned}$$

(we have used (3.7) and (3.6) in the last inequality). Take $z=(ty^2/b^{3-2\alpha})^{1/3}$ to see that

$$\mathbf{P}\left(\sup_{0\le s\le t}|\Lambda(s,b)|>y\right)\le (c_{10}+4)\exp\left(-c_{11}(\alpha)\,\frac{y^{4/3}}{b^{2-4\alpha/3}t^{1/3}}\right).$$

Now define $t_n=(1+\delta)^n$. Applying the Borel–Cantelli lemma gives that almost surely for all large n,

$$\sup_{0\le s\le t_n}|\Lambda(s,\delta h(t_n))|\le c_{12}(\alpha)\,\delta^{3/2-\alpha}\phi_\alpha(t_n), \tag{4.10}$$

where $\phi_\alpha(\cdot)$ is as in (4.4).

On the other hand, by (4.7) and the inequality $|A(x,W(t))| \le 2x$, we have, for $r \in [t_{n-1}, t_n]$,

$$
\begin{aligned}
&|\Lambda(r,\delta h(t_n)) - \Lambda(r,\delta h(r))| \\
&\qquad \le 2\int_0^{\delta h(t_n)} x^{1-\alpha}\,dx + \frac{1}{2}\int_{\delta h(r)}^{\delta h(t_n)} \frac{|L(x,r)-L(-x,r)|}{x^\alpha}\,dx \\
&\qquad \le \frac{2}{2-\alpha}(\delta h(t_n))^{2-\alpha} + \sup_{x\in\mathbf{R}} L(x,t_n)\int_{\delta h(t_{n-1})}^{\delta h(t_n)} \frac{dx}{x^\alpha}.
\end{aligned}
$$

For all large n and all $x \in [\delta h(t_{n-1}), \delta h(t_n)]$, we have $x^{-\alpha} \le 2(\delta h(t_n))^{-\alpha}$. Moreover, since $t \mapsto h'(t)$ is decreasing (for large t), we have, by the mean value theorem, when n is sufficiently large,

$$
\begin{aligned}
\delta h(t_n) - \delta h(t_{n-1}) &\le \delta(t_n - t_{n-1})h'(t_{n-1}) \\
&\le \frac{\delta(t_n - t_{n-1})}{2\sqrt{t_{n-1}\log\log t_{n-1}}} = \frac{\delta^2}{2\sqrt{1+\delta}}\frac{\sqrt{t_n}}{\sqrt{\log\log t_{n-1}}} \\
&\le \frac{\delta^2}{\sqrt{1+\delta}}\frac{\sqrt{t_n}}{\sqrt{\log\log t_n}}.
\end{aligned}
$$

Thus

$$
\int_{\delta h(t_{n-1})}^{\delta h(t_n)} \frac{dx}{x^\alpha} \le \frac{2\delta^{2-\alpha}}{\sqrt{1+\delta}}\, t_n^{1/2-\alpha/2}\,(\log\log t_n)^{\alpha/2-1/2}.
$$

In view of (3.8), we obtain, almost surely when $n \to \infty$,

$$
\begin{aligned}
&\sup_{r\in[t_{n-1},t_n]} |\Lambda(r,\delta h(t_n)) - \Lambda(r,\delta h(r))| \\
\le\ & \frac{2}{2-\alpha}(\delta h(t_n))^{2-\alpha} + \frac{2\delta^{2-\alpha}}{\sqrt{1+\delta}}\,(2^{1/2}+o(1))\,\phi_\alpha(t_n).
\end{aligned}
$$

Taking into account of (4.10), we have, with probability one,

$$
\limsup_{n\to\infty} \frac{1}{\phi_\alpha(t_n)} \sup_{r\in[t_{n-1},t_n]} |\Lambda(r,\delta h(r))| \le c_{12}(\alpha)\,\delta^{3/2-\alpha} + \frac{8^{1/2}\,\delta^{2-\alpha}}{\sqrt{1+\delta}}.
$$

Consequently,

$$
\limsup_{t\to\infty} \frac{1}{\phi_\alpha(t)}\,|\Lambda(t,\delta h(t))| \le c_{12}(\alpha)\,\delta^{3/2-\alpha} + \frac{8^{1/2}\,\delta^{2-\alpha}}{\sqrt{1+\delta}}, \qquad \text{a.s.}
$$

Since $\alpha < 3/2$, this clearly yields (4.8), hence completes the proof of (4.5). $\square$

5. Proof of Theorem 1.3

Throughout the section, $\{S_k;\ k=1,2,\cdots\}$ denotes a simple symmetric random walk on $\mathbf{Z}$ starting from 0, with local time $\xi(x,n)$, and $\{W(t);\ t\ge 0\}$ denotes a Wiener process with local time $L(x,t)$. The proof of Theorem 1.3 is based on a strong invariance principle for local time due to Révész (1981).

Fact 5.1 (Révész 1981). *On a suitable probability space we have, for any $\varepsilon>0$, when $n\to\infty$,*

$$\sup_x |\xi(x,n)-L(x,n)| = o(n^{1/4+\varepsilon}), \qquad \text{a.s.}$$

Proof of Theorem 1.3. Without loss of generality, we may and will suppose that t is an integer (the $[\cdot]$ sign for integer part is thus omitted).

Let $\varepsilon>0$. For convenience, we write

$$\gamma = \frac{(1-\alpha)^+}{2}.$$

Note that $\gamma < 3/4-\alpha/2$ whenever $\alpha<3/2$. The reason for which we have introduced γ is that, for several times we shall use the following relations:

$$\int_1^{t^{1/2+\varepsilon}} \frac{dx}{x^\alpha} = o(t^{\gamma+\varepsilon}), \qquad \sum_{i=1}^{t^{1/2+\varepsilon}} \frac{1}{i^\alpha} = o(t^{\gamma+\varepsilon}). \tag{5.1}$$

The usual LIL for random walk (cf. e.g. Révész 1990, p. 35) says that

$$\limsup_{n\to\infty} (2n\log\log n)^{-1/2} \max_{1\le k\le n} |S_k| = 1, \qquad \text{a.s.}$$

Thus, almost surely for all $\varepsilon>0$ and large t, $\xi(i,t)=0$ whenever $|i|>t^{1/2+\varepsilon}$. Accordingly,

$$\begin{aligned}
\sum_{k=1}^t \frac{1}{S_k^\alpha}\mathbf{1}_{[S_k\neq 0]} &= \sum_{i=1}^\infty \frac{\xi(i,t)-\xi(-i,t)}{i^\alpha} \\
&= \sum_{i=1}^{t^{1/2+\varepsilon}} \frac{\xi(i,t)-\xi(-i,t)}{i^\alpha} \\
&= \sum_{i=1}^{t^{1/2+\varepsilon}} \frac{(\xi(i,t)-L(i,t))-(\xi(-i,t))-L(-i,t))}{i^\alpha} \\
&\qquad + \sum_{i=1}^{t^{1/2+\varepsilon}} \frac{L(i,t)-L(-i,t)}{i^\alpha}.
\end{aligned} \tag{5.2}$$

By Fact 5.1 and (5.1), as $t \to \infty$, the first summation term on the right hand side of (5.2) is

$$= \sum_{i=1}^{t^{1/2+\varepsilon}} \frac{o(t^{1/4+\varepsilon})}{i^\alpha} = o(t^{1/4+\gamma+2\varepsilon}), \qquad \text{a.s.}$$

Since $\gamma < 3/4 - \alpha/2$, we can choose ε sufficiently small so that $1/4 + \gamma + 2\varepsilon < 1 - \alpha/2 - \varepsilon$. Therefore, the proof of Theorem 1.3 is reduced to showing the following estimate: almost surely, when t goes to infinity,

$$\int_0^\infty \frac{L(x,t) - L(-x,t)}{x^\alpha}\, dx - \sum_{i=1}^{t^{1/2+\varepsilon}} \frac{L(i,t) - L(-i,t)}{i^\alpha} = o\left(t^{1-\alpha/2-\varepsilon}\right),$$

or equivalently (since $L(x,t) = 0$ for all large t and $|x| > t^{1/2+\varepsilon}$),

$$I(t) = o\left(t^{1-\alpha/2-\varepsilon}\right), \qquad \text{a.s.,} \tag{5.3}$$

where

$$I(t) = \int_0^{1+t^{1/2+\varepsilon}} \frac{L(x,t) - L(-x,t)}{x^\alpha}\, dx - \sum_{i=1}^{t^{1/2+\varepsilon}} \frac{L(i,t) - L(-i,t)}{i^\alpha}.$$

(We have written $1 + t^{1/2+\varepsilon}$ instead of $t^{1/2+\varepsilon}$ in order to simply writings later).

Observe that

$$\begin{aligned} I(t) &= \sum_{i=1}^{t^{1/2+\varepsilon}} \int_i^{i+1} \frac{L(x,t) - L(-x,t) - L(i,t) + L(-i,t)}{x^\alpha} dx \\ &\quad + \int_0^1 \frac{L(x,t) - L(-x,t)}{x^\alpha} dx \\ &\quad + \sum_{i=1}^{t^{1/2+\varepsilon}} (L(i,t) - L(-i,t)) \int_i^{i+1} \left(\frac{1}{x^\alpha} - \frac{1}{i^\alpha}\right) dx \\ &\stackrel{\text{def}}{=} I_1(t) + I_2(t) + I_3(t). \end{aligned}$$

According to Bass and Griffin (1985),

$$\sup_{|x-y|\le 1} |L(x,t) - L(y,t)| = o(t^{1/4+\varepsilon}), \quad \text{a.s.,}$$

as $t \to \infty$. Therefore,

$$
\begin{aligned}
|I_1(t)| &\le o\left(t^{1/4+\varepsilon}\right) \sum_{i=1}^{t^{1/2+\varepsilon}} \int_i^{i+1} \frac{dx}{x^\alpha} \\
&= o\left(t^{1/4+\varepsilon}\right) \int_1^{t^{1/2+\varepsilon}} \frac{dx}{x^\alpha} \\
&= o\left(t^{1/4+\gamma+2\varepsilon}\right), \qquad \text{a.s.},
\end{aligned}
$$

where we have used (5.1) in the last identity.

The expression $I_2(t)$ was already estimated in Section 2. Indeed, according to (2.6),

$$
I_2(t) = o\left(t^{1/4+\varepsilon}\right), \qquad \text{a.s.}
$$

Since $\gamma < 3/4 - \alpha/2$ and $\min(1/2, \alpha) > \alpha - 1$, the proof of (5.3) (thus of Theorem 1.3) will be completed as soon as we prove the following estimate: for any $0 < \nu < \min(1/2, \alpha)$,

$$
I_3(t) = o(t^{(1-\nu)/2+\varepsilon}), \qquad \text{a.s.} \tag{5.4}
$$

To check (5.4), let us note that

$$
\int_i^{i+1} \left(\frac{1}{i^\alpha} - \frac{1}{x^\alpha}\right) dx \in (0, \frac{c_{13}(\alpha)}{i^{\alpha+1}}),
$$

for some constant $c_{13}(\alpha) < \infty$. By (2.5), for any $0 < \nu < \min(1/2, \alpha)$,

$$
\sup_{i \ge 1} \frac{|L(i,t) - L(-i,t)|}{i^\nu} = o(t^{(1-\nu)/2+\varepsilon}), \qquad \text{a.s.}
$$

Therefore,

$$
\begin{aligned}
|I_3(t)| &\le c_{13}(\alpha) \sup_{i \ge 1} \frac{|L(i,t) - L(-i,t)|}{i^\nu} \sum_{i=1}^{\infty} \frac{1}{i^{1+\alpha-\nu}} \\
&= c_{14}(\alpha, \nu) \sup_{i \ge 1} \frac{|L(i,t) - L(-i,t)|}{i^\nu},
\end{aligned}
$$

proving (5.4). □

6. Proof of Theorems 1.4 and 1.5

In the case $\alpha > 2$, the theorems follow from Csáki et al. (1992), where for the additive functionals $\sum_{i=1}^n f(S_i)$ and $\int_0^t g(W(s))\,ds$, resp. it was assumed that

$$\sum_{x=-\infty}^{\infty} |x|^{1+\delta}\,|f(x)| < \infty \tag{6.1}$$

and

$$\int_{-\infty}^{\infty} |x|^{1+\delta}\,|g(x)|\,dx < \infty, \tag{6.2}$$

resp., for some $\delta > 0$.

It is easy to see that these conditions are satisfied for Z_α and G_α when $\alpha > 2$. We however claim that the conditions (6.1) and (6.2), resp. can be replaced by the weaker conditions

$$\sum_{x=-\infty}^{\infty} |x|^{1/2+\delta}\,|f(x)| < \infty \tag{6.3}$$

and

$$\int_{-\infty}^{\infty} |x|^{1/2+\delta}\,|g(x)|\,dx < \infty, \tag{6.4}$$

resp. for some $\delta > 0$.

In order to see this we have to show that (6.3) implies

$$\mathbf{E}|\sum_{i=1}^{\rho} f(S_i)|^{2+\delta} < \infty, \tag{6.5}$$

where ρ is the time of the first return to zero of the random walk, since it follows from the proof of the strong approximation in Csáki et al. (1992), that (6.5) is sufficient for the conclusion to hold. To prove (6.5), we use

$$\mathbf{E}(\xi(x,\rho))^m \le c_{15}(m)\,(1+|x|)^{m-1},$$

(cf. (2.23) in Csáki et al. 1992) and the triangle inequality, to obtain (writing c_{16} for $c_{15}(2+\delta)$):

$$\begin{aligned}
\left(\mathbf{E}|\sum_{i=1}^{\rho} f(S_i)|^{2+\delta}\right)^{1/(2+\delta)} &\le \sum_{x\in\mathbf{Z}} |f(x)|\left(\mathbf{E}(\xi(x,\rho))^{2+\delta}\right)^{1/(2+\delta)}\\
&\le c_{16}\sum_{x\in\mathbf{Z}} |f(x)|(1+|x|^{(1+\delta)/(2+\delta)})\\
&\le c_{16}\sum_{x\in\mathbf{Z}} |f(x)|(1+|x|^{1/2+\delta}) < \infty.
\end{aligned}$$

Now one can see that (6.3) is satisfied whenever $\alpha > 3/2$, hence the conclusion of Theorem 1 in Csáki et al. (1992) holds, which gives Theorem 1.4.

Similar proof holds for the Wiener case, i.e. for Theorem 1.5.

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