On the Vlasov Limit for Systems of Nonlinearly Coupled Oscillators without Noise

Carlo Lancellotti
Department of Mathematics, City University of New York–CSI, Staten Island, New York

Abstract: We discuss the application to systems of coupled nonlinear oscillators of methods originally developed for the Vlasov-Poisson equations of plasma kinetic theory. These methods lead to a simple but rigorous derivation of the infinite-oscillator limit for the well-known Kuramoto model, and also to a compact proof of global existence and uniqueness of solutions to the resulting Kuramoto-Sakaguchi equations. Vlasov-limit techniques can also be adapted to prove a central limit theorem for statistical ensembles of systems of oscillators. However, we emphasize that the noiseless Kuramoto model itself is strictly deterministic, and that the Kuramoto-Sakaguchi equations can be formulated without any reference to a stochastic distribution of oscillator frequencies.

Keywords: Kuramoto model, mean field limit, coupled oscillators, kinetic theory

1. THE KURAMOTO–SAKAGUCHI MODEL

Large systems of coupled nonlinear oscillators constitute an important family of mathematical models and have found many applications in a broad range of fields, from mathematical biology to electronics. Of particular interest are the models that exhibit spontaneous self-synchronization, i.e., the ability to develop synchronized oscillatory patterns—via nonlinear coupling—starting
from generic, nonsynchronous initial conditions. The oldest and best-known model that exhibits self-synchronization is due to Kuramoto (1975). If \( \theta_i(t) \in [0, 2\pi] \) denotes the phase of the \( i \)th oscillator at a time \( t \), and \( \omega_i \in \mathbb{R} \) is its natural frequency, the Kuramoto model is governed by \( N \) ordinary differential equations (ODEs)

\[
\dot{\theta}_i = \omega_i + \frac{K}{N} \sum_{j=1}^{N} \sin(\theta_j - \theta_i) \quad i = 1, \ldots, N,
\]

where \( K \geq 0 \) is the coupling strength and the frequencies are usually assumed to be randomly distributed according to a symmetric, one-humped density function \( g(\omega) \). Since each oscillator is coupled in the same way to all others, this represents a mean-field model for the set of oscillators, and it is natural to ask whether it possesses a meaningful Vlasov limit. Sakaguchi (1988) proposed that as \( N \to \infty \) the individual oscillators should be replaced by a continuum of oscillators distributed in \( \theta \) over \([0, 2\pi]\), according to a density function \( g(\theta, \omega, t) \). For each \( \omega \in \mathbb{R} \), this function must satisfy the normalization condition \( \int_0^{2\pi} g(\theta, \omega, t) \, d\theta = 1 \) and the continuity equation

\[
\frac{\partial \varrho}{\partial t} + \frac{\partial}{\partial \theta}(\nu \varrho) = 0,
\]

where the velocity \( \nu(\theta, \omega, t) \) is given by the continuous version of Equation (1):

\[
\nu(\theta, \omega, t) = \omega + K \int_0^{2\pi} \int_{\mathbb{R}} \sin(\theta' - \theta) g(\theta', \omega', t) g(\omega) \, d\omega' \, d\theta'.
\]

The combination of Equations (2) and (3) is usually called the Kuramoto-Sakaguchi equation, and has received much attention in the literature (Strogatz and Mirollo 1991; Bonilla et al. 1992; Crawford 1994; Crawford and Davies 1999; Balmforth and Sassi 2000; Strogatz 2000; Acebrón et al. 2005). However, its derivation from the discrete Kuramoto model seems to have been left at the formal level (e.g., see Appendix B in Crawford and Davies 1999), and we have found no notice in the literature of the fact that the derivation of Equations (2) and (3) from Equation (1) can be made fully rigorous simply by applying the elegant Vlasov-limit theory of Neunzert (Neunzert 1978, 1984; Spohn 1991). We should mention that the stochastic version of the Kuramoto model (obtained by adding a random noise term in Equation (1)) has received a rigorous treatment (Dai Pra and den Hollander 1996) as an example of a McKean–Vlasov stochastic process (McKean 1966). In fact, McKeans–Vlasov results also apply to the Kuramoto model without noise but with an ensemble of initial conditions (see Section 3). Neunzert’s method, on the other hand, leads to the Kuramoto–Sakaguchi equations within a completely deterministic framework. His approach, which was originally developed in view of the well-known Vlasov–Poisson
equations for plasmas and gravitating systems (see also Braun and Hepp 1977; Dobrushin 1979), is remarkably general, provided that the equations under consideration satisfy certain boundedness and Lipschitz-continuity assumptions. Whereas these assumptions lead to some unwelcome restrictions in the many-particle case (because they are incompatible with the non-mollified Coulomb/Newton interparticle potential), they are fulfilled naturally in the many-oscillator case, where the coupling is usually modeled by well-behaved functions like the sine in the Kuramoto equations.

We remark that from a technical point of view the application of the Vlasov-limit methods to the Kuramoto model is quite straightforward. Nevertheless, from the perspective of the Kuramoto theory itself it is an elegant and rigorous way to settle a few mathematical questions, besides the $N \to \infty$ limit itself:

1. By showing that Neunzert’s general theory applies to this problem, we also prove global existence and uniqueness of either weak or classical solutions to the Kuramoto-Sakaguchi equation (depending on the smoothness of the initial condition). To our knowledge, existence and uniqueness results have been established (Lavrentiev and Spigler 2000) only for the dissipative version of the Kuramoto–Sakaguchi equation, which is obtained by adding a diffusive term $D \partial^2 \varphi / \partial \theta^2$ on the right-hand side in Equation (2). The existence and uniqueness proof in Lavrentiev and Spigler (2000) relies heavily on the parabolic nature of the equation and does not appear to extend easily to the nondissipative case ($D = 0$). On the other hand, in our approach existence and uniqueness results for the “noiseless” Kuramoto-Sakaguchi equation come essentially for free. It may be worth recalling that this model is believed to possess singular solutions (Balmforth and Sassi 2000); our global existence results imply that such solutions cannot arise in finite time from smooth initial conditions.

2. Also, a careful discussion of the $N \to \infty$ limit helps clarify the role of the frequency distribution $g(\omega)$ in the Kuramoto model. Clearly, $g(\omega)$ does not play any role in the dynamics of any individual set on $N$ oscillators with assigned frequencies, since the Kuramoto model without noise is strictly deterministic. Of course, one can consider a statistical ensemble of initial conditions (including an ensemble of independent and identically distributed [i.i.d] random frequencies $\omega_k$), but this is not relevant to either the evolution of a given system of oscillators or the derivation of the Kuramoto–Sakaguchi equation, which is also a deterministic evolution equation for a density in phase space (as opposed to an ensemble probability density). The function $g(\omega)$ does arise in the $N \to \infty$ limit, but only a posteriori as the time-independent marginal of the oscillator density $\rho(\theta, \omega, t) = g(\omega)g(\theta, \omega, t)$. In fact, we will obtain the Kuramoto-Sakaguchi equation as an equation for $\rho$ (Equation (6)), in which $g$ does not appear at all and is just “buried” in the choice of
the initial condition \( \rho(\theta, \omega, 0) \). This clarification may be helpful to newcomers to the literature, where \( g(\omega) \) is often introduced as a constitutive part of the Kuramoto model. This gives the impression that the theory has an intrinsically stochastic character, which it does not.

3. Having carried out the mean-field limit for the (deterministic) Kuramoto model, we are then able to study statistical ensembles and transfer some probabilistic results that have been obtained in the Vlasov case (law of large numbers, propagation of chaos, central limit theorem, etc.). In particular, by adapting to the Kuramoto case the Braun–Hepp central limit theorem (Braun and Hepp 1977), we establish \( O(N^{-1/2}) \) behavior for the fluctuations for the Kuramoto model, a question that has been brought up in the physics literature (see, e.g., Strogatz 2000; Acebrón et al. 2005) and that was already answered in a more general framework by Dai Pra and den Hollander (1996).

The article is naturally divided in two parts. In the first part we present the Vlasov limit for the Kuramoto model in the purely deterministic setup, and we obtain existence and uniqueness results for the Kuramoto-Sakaguchi equation. In the second part we move to the probabilistic context and we present results on the evolution of statistical ensembles of systems of oscillators.

2. DETERMINISTIC VLASOV LIMIT

Because the oscillators’ frequencies do not change in time, the Kuramoto–Sakaguchi equation has traditionally been written in terms of the function \( \varphi(\theta, \omega, t) \), understood as a density with respect to the phase variable \( \theta \in [0, 2\pi] \) for each fixed \( \omega \) and \( t \). Then, the density in \( \omega \) has to be assigned by a separate function \( g(\omega) \). Here, both for the sake of symmetry and in view of more general models where the oscillators’ frequencies are time-dependent (Acebrón and Spigler 1998), we will consider the function \( \rho(\theta, \omega, t) \), the density with respect to both variables \( (\theta, \omega) \) on the slab \([0, 2\pi] \times \mathbb{R}\) with the normalization \( \int_0^{2\pi} \int_{\mathbb{R}} \rho(\theta, \omega, t)d\omega d\theta = 1 \). Note that, in probabilistic language, \( \varphi(\theta, \omega, t) \) is the conditional density for \( \theta \) given \( \omega \), since \( \varphi(\theta, \omega, t) = \rho(\theta, \omega, t)/g(\omega) \) and \( \int_0^{2\pi} \varphi(\theta, \omega, t)d\theta = 1 \), which implies that \( g(\omega) \) is the (time-independent) marginal

\[
g(\omega) = \int_0^{2\pi} \rho(\theta, \omega, t)d\theta. \tag{4}
\]

Multiplying Equation (2) by \( g(\omega) \) gives the continuity equation for \( \rho \)

\[
\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial \theta}(\nu \rho) = 0. \tag{5}
\]
We will consider a more general form of the equation for \( v(\theta, \omega, t) \), Equation (3), namely,

\[
v(\theta, \omega, t) = \omega + K \int_0^{2\pi} \int_\mathbb{R} f(\theta - \theta') \rho(\theta', \omega', t) \, d\omega' \, d\theta'
\]

where \( f \) is a given \( 2\pi \)-periodic odd function that models the coupling between pairs of oscillators; clearly, Equation (3) is recovered for \( f(x) = \sin x \). From now on, \( f \) will be assumed to be bounded and Lipschitz continuous over \([0, 2\pi]\). To emphasize that we are working in the \( \theta-\omega \) slab, we write the generalized version of the Kuramoto ODEs in terms of the pairs \((\theta_i(t), \omega_i(t))\), \( i = 1, \ldots, N \):

\[
\dot{\theta}_i = \omega_i + \frac{K}{N} \sum_{j=1}^{N} f(\theta_j - \theta_i) \quad \theta_i(0) = \theta_i^{(0)}
\]

\[
\dot{\omega}_i = 0 \quad \omega_i(0) = \omega_i^{(0)}
\]

The \( \omega_i(t) \), of course, are just going to be constant functions of \( t \). The generalized equations, Equations (5)–(6) and Equations (7), will still be called, respectively, the Kuramoto–Sakaguchi and Kuramoto equations.

It order to compare their solutions, we reformulate both problems in terms of probability measures on \( \mathcal{B}(G) \), the Borel \( \sigma \)-algebra of subsets of \( G = [0, 2\pi] \times \mathbb{R} \). For finite \( N \) this is done by associating with each configuration \([\theta_1, \ldots, \theta_N, \omega_1, \ldots, \omega_N]\) of the oscillators the discrete (empirical) measure

\[
\mu^{(N)}_t = \frac{1}{N} \sum_{i=1}^{N} \delta(P_i(t))
\]

where \( P_i(t) = (\theta_i(t), \omega_i(t)) \) is the state of the \( i \)th oscillator at time \( t \) and \( \delta \) is the usual Dirac measure in \( G \). In the continuous case we simply interpret the distribution function \( \rho(\theta, \omega, t) \) in the Kuramoto–Sakaguchi equation as the density (with respect to the Lebesgue measure on \( G \)) of a probability measure \( \mu \), such that

\[
\mu_t(M) = \int_M \rho(\theta, \omega, t) \, d\theta \, d\omega
\]

for every Borel set \( M \subset \mathcal{B}(G) \). Then, the mean-field limit will be carried out within the set \( \mathcal{M} \) of all probability measures on \( \mathcal{B}(G) \), endowed with the bounded Lipschitz metric

\[
d(\mu, \nu) = \sup_{\phi \in \mathcal{L}} \left| \int_G \phi d\mu - \int_G \phi d\nu \right|
\]

where \( \mathcal{L} = \{ \phi: G \to [0, 1], |\phi(P) - \phi(Q)| \leq \|P - Q\| \} \) (\( \| \cdot \| \) denotes the usual Euclidean norm in \( \mathbb{R}^2 \)). As is well known, this metric generates the weak convergence in \( \mathcal{M} \), which will be denoted by the symbol \( \overset{w}{\rightarrow} \).
Remark 1. We recall that the use of probability measures in the Vlasov limit is just a matter of normalization of the initial densities, and does not have any real probabilistic meaning. In fact, the limit itself does not require any statistical assumptions on the initial conditions for finite N's, which only have to well approximate the smooth initial condition for the Kuramoto-Sakaguchi equation as $N \to \infty$. In this sense, assuming a statistical distribution of initial data for the Kuramoto model is (in the words of H. Spohn for the Vlasov case; Spohn 1991, p. 81) “some sort of luxury,” which will be briefly discussed in the next section. Here, as anticipated, we will construct a purely deterministic theory, in which $g(\omega)$ is just the marginal of $\rho(\theta, \omega, t)$ in Equation (4).

Just like in the Vlasov-Poisson context, both the Kuramoto and Kuramoto-Sakaguchi equations will be obtained as special cases of Neunzert’s abstract continuity equation in $\mathcal{M}$

$$\mu_0 = \mu_t \circ T_{t,0}[\mu]$$

(11)

Here $\mu$ represents a $\mathcal{M}$-valued function of $t \in [0, T]$, and the subscripts are used to denote the values of $\mu$ at different times; $T_{t,s}[\mu]$ is some appropriate “flow” on $G$, i.e., a family of Borel-measurable bijective mappings on $G$ that satisfy (a) $T_{s,s} = \text{Id}$ and (b) $T_{t,s} \circ T_{s,r} = T_{t,r}$; $T_{t,s}$ “moves” the point $P$ at time $s$ to the point $T_{t,s}P$ at time $t$. The nonlinearity of the problem is contained in the functional dependence of $T_{t,s}[\mu]$ on the measure $\mu$ itself, assumed to be in the space $C_M$ of all weakly continuous time-dependent measures (meaning that $\int \psi d\mu_t$ must be a continuous function of $t$ for any $\psi \in C_b(G)$, the continuous and bounded functions on $G$). The flow $T_{t,s}[\mu]$ will be taken to be differentiable and to be determined by $\mu$ through a velocity field $V[\mu]$, in the sense that $P(t) = T_{t,s}Q$ is a solution to the initial value problem

$$\frac{dP}{dt} = V[\mu](P, t), \quad P(s) = Q.$$ 

(12)

In this framework Neunzert (1978, 1984) proved the following general results.

Theorem 1. Let the velocity field $V[\mu](P, t)$ satisfy the following hypotheses:

(a). $V[\mu](P, t)$ is a continuous function on $G \times [0, T]$ and satisfies the (uniform) Lipschitz condition

$$\|V[\mu](P, t) - V[\mu](Q, t)\| \leq L\|P - Q\|$$

(13)

for all $t \in [0, T], P, Q \in G$, and $\mu \in C_M$.

$^1$Neunzert’s proof applies to any domain $G$ in $\mathbb{R}^n$. 


(b). For each fixed $P$ and $t$, $V[\mu](P, t)$ as a function on $C_M$ also is Lipschitz continuous, i.e.,

$$\|V[\mu](P, t) - V[v](P, t)\| \leq Md(\mu, v).$$

(14)

Then, if $T_{t,s}[\mu]$ is the flow generated by $V[\mu]$,

1. For every $\mu_0 \in M$ Equation (11) has a unique solution $\mu \in C_M$.
2. If $\mu_0$ is absolutely continuous with density $\rho_0 \in L^1(G)$, then $\mu$ is also absolutely continuous, and its density $\rho$ is a weak solution to

$$\frac{\partial \rho}{\partial t} + \text{div}(\rho V) = 0 \quad \rho(P, 0) = \rho_0(P).$$

(15)

3. If $\mu_0, v_0 \in M$ and if $\mu, v \in C_M$ are the corresponding solutions to Equation (11), then

$$d(\mu_t, v_t) \leq e^{(L+M)t}d(\mu_0, v_0)$$

(16)

where $L$ and $M$ are the constants that appear in Equations (13) and (14).

Remark 2.

1. The first hypothesis ensures that Equation (12) has a unique solution on $[0, T]$, i.e., $V[\mu]$ generates $T_{t,s}[\mu]$ uniquely.
2. Weak solvability is defined here (Neunzert 1978) in terms of the space of test functions that are continuously differentiable and compactly supported in $G \times [0, T]$. See Definition 1 below for the specific Kuramoto-Sakaguchi case.

In order to show that, for the appropriate choices of the initial measures, both the Kuramoto and Kuramoto–Sakaguchi equations are special cases of Equation (11), we introduce the velocity field

$$V[\mu](\theta, \omega, t) = \left(\omega + K \int_{G} f(\theta - \theta')\mu_t(d\theta', d\omega')\right).$$

(17)

We proceed as follows: first, we prove that Neunzert’s general theorem applies to this case. Then, we show that for this velocity field Equation (11) yields (unique) solutions to either the Kuramoto or Kuramoto–Sakaguchi equations, depending on the choice of the initial condition, and that the expected limit behavior occurs as $N \to \infty$.

Lemma 1. When $f$ is bounded and Lipschitz continuous on $[0, 2\pi]$, the velocity field $V[\mu](\theta, \omega, t)$ in Equation (17) satisfies the hypotheses of Theorem 1.
Proof. (a) Since \( \mu \) is weakly continuous and \( f \) is Lipschitz continuous, \( V[\mu] \) is continuous in \( G \times [0, T] \), and for each given \( \mu \),

\[
\|V[\mu](\theta_1, \omega_1, t) - V[\mu](\theta_2, \omega_2, t)\|
\leq |\omega_1 - \omega_2| + K \int_G |f(\theta' - \theta_1) - f(\theta' - \theta_2)| \mu_t(d\theta', d\omega')
\leq |\omega_1 - \omega_2| + K \int_G |\theta_1 - \theta_2| \mu_t(d\theta', d\omega')
= |\omega_1 - \omega_2| + KL_f |\theta_1 - \theta_2| \leq L \| (\omega_1, \theta_1) - (\omega_2, \theta_2) \|
\]

(18)

where \( L_f \) is the Lipschitz constant for \( f \) and \( L = \sqrt{2} \max(1, KL_f) \).

(b) For \((\theta, \omega)\) and \( t \) fixed

\[
\|V[\mu](\theta, \omega, t) - V[v](\theta, \omega, t)\| = K \int_G [f(\theta' - \theta)\mu_t(d\theta', d\omega')
- f(\theta - \theta)\omega_t(d\theta', d\omega')] .
\]

(19)

This expression is formally similar to the difference that appears in the definition of the metric \( d(\mu, v) \), Equation (10), except that \( f(\theta' - \theta) \) (as a function of \( \theta' \), with \( \theta \) fixed) is not necessarily in the set \( \mathcal{L} \). Like in the Vlasov–Poisson theory (Neunzert 1978), this difficulty is easily overcome by noticing that even if \( f \) may not be in \( \mathcal{L} \), the modified function \((2BL_f)^{-1}(f + B)\) will always be in \( \mathcal{L} \) since

\[
\frac{f(\theta' - \theta) + B}{2BL_f} \leq \frac{1}{L_f} \left| \frac{f(\theta' - \theta_1) + B}{2BL_f} - \frac{f(\theta' - \theta_2) + B}{2BL_f} \right| \leq \frac{1}{2B}
\]

(20)

(one can always choose \( B \geq \frac{1}{2}, L_f \geq 1 \)). Thus, we rewrite Equation (19) in the form

\[
2KBL_f \left| \int_G \frac{f(\theta' - \theta) + B}{2BL_f} \mu_t(d\theta', d\omega') - \int_G \frac{f(\theta - \theta) + B}{2BL_f} \omega_t(d\theta', d\omega') \right| .
\]

(21)

which is indeed \( \leq Md(\mu, v) \), with \( M = 2KBL_f \).

From the properties of the velocity field \( V[\mu] \) it follows that all of Neunzert’s results apply to the Kuramoto–Sakaguchi equation. In order to consider a larger class of initial conditions we make precise what we mean by weak solutions.

Definition 1. The density function \( \rho(\theta, \omega, t) \) is a weak solution to the Kuramoto–Sakaguchi equation if it is weakly continuous in \( t \) and for all
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\( \phi \in C^1_c(G \times [0, T]) \) (C\(^1\) with support in \( K \times \{0, T\} \), \( K \) compact in \( G \)) the equation

\[
\int_G [p \phi]_{\theta=0} d\mu_\theta + K \int_0^T \left[ \int_G \left( p \frac{\partial \phi}{\partial \theta} + v \phi \right) \right] \, d\mu_\theta \, dt = 0 \tag{22}
\]

is satisfied.

Now we are ready for the following.

**Theorem 2.** When \( f \) is bounded and Lipschitz continuous the initial value problem for the Kuramoto-Sakaguchi equation has a unique weak solution in \( L^1(G) \).

**Proof.** This follows directly from Neunzert’s results 1 and 2 in Theorem 1, since for \( V[\mu] \) given by Equation (17) and for an absolutely continuous \( \mu_\theta \) with density \( \rho \)

\[
\text{div}(pV) = \frac{\partial}{\partial \theta} \left[ (\omega + K \int_G f(\phi - \theta) \rho(\theta, \omega, t) d\theta d\omega) \rho \right] = \frac{\partial}{\partial \theta} (v \rho) \tag{23}
\]

Thus, in this case Equation (15) is given precisely by the Kuramoto-Sakaguchi equation.

**Remark 3.** If \( \rho(\theta, \omega, 0) \) and \( f(\theta) \) are sufficiently smooth (say, \( C^1 \)) then the same argument leads to existence and uniqueness of a classical solution.

The last theorem is based on the fact that the Kuramoto-Sakaguchi equation has been shown to be a special case of Neunzert’s abstract continuity equation, Equation (11). A similar identification can be carried out in the finite \( N \) case, via the following.

**Lemma 2.** If \( P_j(t) = (\theta_j(t), \omega_j(t)) \), \( i = 1, \ldots, N \) is the solution to the (generalized) Kuramoto initial value problem, Equations (7), then

\[
\mu^{(N)}_i = \frac{1}{N} \sum_{j=1}^N \delta(P_j(t)) \tag{24}
\]

is the solution to Equation (11) with \( T[\mu^{(N)}] \) determined by \( V[\mu^{(N)}] \) according to Equations (12) and (17), and with initial density

\[
\mu^{(N)}_0 = \frac{1}{N} \sum_{j=1}^N \delta(P_j(0)) \quad P_j(0) = (\theta^{(0)}_j, \omega^{(0)}_j). \tag{25}
\]

**Proof.** Since \( f \) is globally Lipschitz continuous, Equations (7a) and (7b) have a unique solution for \( t \in [0, T] \), and the associated measure \( \mu^{(N)}_i \) is weakly
continuous in $t$. Then, for each fixed $i = 1, \ldots, N$, $P_i(t) = (\theta_i(t), \omega_i(t))$ solves Equation (12) with $P_i(0) = (\theta_i^{(0)}, \omega_i^{(0)})$, since

$$V[\mu^N](\theta, \omega, t) = \left( \omega_i + K \int_G f(\theta' - \theta) \mu_i^N(d\theta', d\omega') \right)$$

$$= \left( \omega_i + \frac{K}{N} \sum_{j=1}^{N} f(\theta_j - \theta_i) \right).$$

(26)

By definition then $P_i(t) = T_{t,o}[\mu_i^N]P_i(0)$, which implies $\delta(P_i(t)) \circ T_{t,o}[\mu_i^N] = \delta(P_j(0))$, which implies $\mu_i^N \circ T_{t,o}[\mu_i^N] = \mu_i^N$, i.e., Equation (11). □

Having thus identified solutions to the discrete Kuramoto equation with measures that solve Equation (11), we are ready to conclude that the Kuramoto-Sakaguchi equation is the correct limit as $N \to \infty$.

**Theorem 3.** Let $\mu_0^N \in \mathcal{M}$ be a sequence of discrete measures of the form in Equation (25) such that $\mu_0^N \rightharpoonup \mu_0$, where $\mu_0 \in \mathcal{M}$ is absolutely continuous. Let $\mu^N$ and $\mu$ be the (unique) solutions to Equation (11) associated, respectively, with the initial conditions $\mu_0^N$ and $\mu_0$. Then $\mu^N \rightharpoonup \mu$ as $N \to \infty$, for all $t \in [0,T]$.

**Proof.** This follows immediately from Neunzert’s estimate, Equation (16). □

**Remark 4.**

1. Of course, the density associated with $\mu$ is the unique solution to the Kuramoto-Sakaguchi equation from Theorem 2.
2. Note that it is always possible to find a sequence $\mu_0^N \rightharpoonup \mu_0$ because discrete measures are dense in $\mathcal{M}$.

**3. PROPAGATION OF CHAOS AND FLUCTUATIONS**

Even if the $N \to \infty$ limit for solutions to the Kuramoto model has been carried out in terms of probability measures on $G$, all the development so far has been completely deterministic. One can also consider a genuinely probabilistic setup by assigning a statistical superposition of $N$-oscillator initial conditions (in particular, of oscillator frequencies) and studying the time evolution of an ensemble probability density on the $N$-oscillator phase space $G^N$. Like in the Vlasov case, this is not necessary in order to justify the kinetic (Kuramoto-Sakaguchi) equation, and does not require any fundamentally new ideas,
because the Kuramoto dynamics itself remains deterministic and the evolution of the probability density on $G^N$ is simply induced by the flow $T_{t,s}$ acting on each pure state. Moreover, the development is very similar to the corresponding theory for the Vlasov equation. Thus, we only state the main results and point to the relevant references in the Vlasov literature for the proofs.

Let $\Omega = G^N$ be the space of $G$-valued sequences $\pi = (P_1, \ldots, P_N, \ldots)$, $P_i = (\theta_i, \omega_i)$, endowed with the $\sigma$-algebra $\mathcal{B}$ generated by the finite-dimensional sets. We denote by $\pi_N = (P_1, \ldots, P_N)$ the projection of $\pi$ on $G^N$ and we consider the evolution of symmetric product measures $\mu_0^{(N)} = \prod_1^N \mu_0$, $\mu_0 \in \mathcal{M}$. The complete sequence $\mu_0^{(N)}(d\pi_N)$, $N = 1, 2, \ldots$, defines an infinite product measure $\tilde{\mu}_0(d\pi)$ on $(\Omega, \mathcal{B})$, and for each finite $N$ the measure $\mu_0^{(N)}$ at time zero leads to the measure at time $t$

$$\mu_t^{(N)} = \mu_0^{(N)} \circ T_{t,s}^{(N)}[\mu_0^{(N)}]$$

where $T_{t,s}^{(N)}$ is the flow on $G^N$ induced by the flow $T_{t,s}[\mu_0^{(N)}]$ on $G$ (i.e., $T_{t,s}^{(N)}$ maps each component $P_i$ of $\pi_N \in G^N$ to $T_{t,s}[\mu_0^{(N)}]P_i$, $\mu_t^{(N)}$ being the empirical measure in Equation (24).

**Theorem 4.** Let $\mathcal{C}(G)$ be the continuous and bounded real functions on $G$.

1. **Law of large numbers:** For all $g \in \mathcal{C}(G)$ and $t > 0$,

$$\lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} g(\theta_i(t), \omega_i(t)) = \int_G g(\theta, \omega) \mu_t(d\theta d\omega)$$

with probability 1 in $(\Omega, \mathcal{B}, \tilde{\mu}_0)$ (i.e., $\mu_t^{(N)} \to \mu_t$ a.s. with respect to $\tilde{\mu}_0$). Here $\mu_t$ is the solution to the Kuramoto-Sakaguchi equation for measures (Equation (11)) together with Equations (12) and (17), with initial condition $\mu_0$.

2. **Propagation of chaos:** For all $n = 1, 2, \ldots$ and $g_1, \ldots, g_n \in \mathcal{C}(G)$,

$$\lim_{N \to \infty} \int_G \prod_{i=1}^{n} g_i(\theta_i, \omega_i) \mu_t^{(N,n)}(d\pi_n) = \prod_{i=1}^{n} \int_G g_i(\theta, \omega) \mu_t(d\theta d\omega)$$

where $\mu_t^{(N,n)}(d\pi_n)$ denotes the $n$-th marginal of $\mu_t^{(N)}(d\pi_N)$.

3. **Central limit theorem:** Consider the fluctuation field

$$\xi_N(g, t) = \sqrt{N} \left[ \frac{1}{N} \sum_{i=1}^{N} g(\theta_i(t), \omega_i(t)) - \int_G g(\theta, \omega) \mu_t(\theta, \omega) \right]$$

For initial probabilities on $\Omega$ that do not factorize, the $N \to \infty$ limit will lead to a Kuramoto-Sakaguchi hierarchy analogous to the well-known Vlasov hierarchy (e.g., see Spohn 1981).
and assume that $f(\theta)$ has three bounded derivatives. Then

$$\lim_{N \to \infty} \int \mu_0(d\pi)e^{i\xi_0(\theta, \omega)} = \int \mu_0(d\pi)e^{i\xi(\theta, \omega)}$$

(31)

where $\xi(g, t)$ is a Gaussian random variable.

Proof. The law of large numbers is proved along the lines of the Vlasov proof due to Braun and Hepp (1977, Theorem 3.3). For a somewhat streamlined argument see also Golse (2003). Propagation of chaos follows, as discussed in Gottlieb (1998). We remark that propagation of chaos for the Kuramoto model is a special case of a general result due to McKean (1966). The proof of the central limit theorem is identical to the original Vlasov argument by Braun and Hepp (1977); see also the discussion of Vlasov fluctuations in Chapter 7 of Spohn (1991).

Of course, whenever $\mu_0$ is absolutely continuous with density $\rho_0(\theta, \omega, \mu, t)$ is absolutely continuous with density $\rho(\theta, \omega, t)$ and $\rho$ is a solution to the Kuramoto–Sakaguchi equation (weak or classical, depending on the smoothness of $\rho_0$). As is well known, pointwise convergence of the characteristic functionals in Equation (31) means that $\xi_N \to \xi$. By adapting Braun and Hepp’s proof to our problem it is easy to see that the Gaussian random variable $\xi(g, t)$ satisfies $d\xi = L(t)\xi dt$, where $L(t)$ is the linearized Kuramoto–Sakaguchi operator\(^4\)

$$(L(t)\phi)(\theta, \omega) = -\frac{\partial(v\phi)}{\partial \theta} - K \frac{\partial}{\partial \theta} \left[ \rho(\theta, \omega, t) \int f(\theta - \theta')\phi(\theta', \omega')d\theta' d\omega' \right]$$

(32)

Here $v$ is the velocity determined by $\rho$ via Equation (6) and $\xi(g, 0)$ is the Gaussian random variable associated with the central limit theorem at $t = 0$; see Spohn (1991), and Braun and Hepp (1977) for further details.

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\(^3\)Actually, in this theorem Braun and Hepp also proved propagation of chaos, but only for “purely atomic initial data”: see Gottlieb (1998).

\(^4\)For absolutely continuous $\mu_0$. The extension to any $\mu_0 \in \mathcal{M}$ is straightforward.
REFERENCES


