Chapter 6

WEAK SHOCK REFLECTION

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Abstract An asymptotic analysis of the regular and Mach reflection of weak shocks leads to shock reflection problems for the unsteady transonic small disturbance equation. Numerical solutions of this equation resolve the von Neumann triple point paradox for weak shock Mach reflection. Related equations describe steady transonic shock reflections, weak shock focusing, and nonlinear hyperbolic waves at caustics.

Keywords: Shock reflection, von Neumann triple point paradox, nonlinear mixed-type PDEs, hyperbolic conservation laws

1. Introduction

Shock reflection is one of the simplest multi-dimensional processes involving shock waves. Nevertheless, despite long and intensive study, many features of shock reflection remain poorly understood. This fact is one indication of the difficulties posed by an analysis of multi-dimensional hyperbolic systems of conservation laws, such as the compressible Euler equations that model the flow of an inviscid compressible fluid [11]. Thus, shock reflection is important not only for its physical significance,

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but because it provides one point of entry into these long-standing mathematical problems.

In this paper, we will describe some studies of weak shock reflection that are based on asymptotic analysis. We discuss the construction of an asymptotic solution for regular reflection, the resolution of the von Neumann triple point paradox for Mach reflection, and steady Mach reflection. We also briefly review some related phenomena in shock focusing and nonlinear caustics.

We consider the basic two-dimensional shock reflection problem of a plane shock incident on a straight-sided wedge in an inviscid fluid. This problem is self-similar. The full reflection pattern forms the instant the shock hits the wedge, then expands linearly in time.

To be definite, we suppose that the shock is incident symmetrically on the wedge. Then, given the equation of state of the fluid and the state of the undisturbed fluid ahead of the incident shock, there are two parameters in this problem: the half-angle α of the wedge and the strength ϵ of the incident shock. For example, we may use as a measure of the shock strength $\epsilon = M^2 - 1$, where M is the Mach number of the incident shock.

At lower shock strengths or larger wedge angles, one observes a regular reflection (RR) similar to a linear wave reflection, except that the angles of incidence and reflection are not equal. At higher shock strengths or smaller wedge angles one observes a Mach reflection (MR) in which three shocks — the incident, reflected, and Mach shocks — meet at a point, called the triple point. A contact discontinuity, separating the higher entropy fluid that passes through the stronger Mach shock from the lower entropy fluid that passes through the incident and reflected shocks, also originates at the triple point. Various types of Mach reflection (e.g. simple, double, complex) have been observed and classified [2, 13]. A schematic diagram of the main regions is shown in Figure 6.1.

One principal difficulty in the analysis of shock reflection is the presence of a nonuniform diffracted wave behind the incident shock. An asymptotic analysis is possible when the diffracted wave is weak. Three different limits in which this happens are shown in Figure 6.1:

(a) the weak shock limit, $\epsilon \to 0$ with $\alpha > 0$ fixed, when regular reflection occurs;

(b) the thin wedge limit, $\alpha \to 0$ with $\epsilon > 0$ fixed, when Mach reflection occurs;

(c) the transitional thin wedge/weak shock limit, $\epsilon \to 0$, $\alpha \to 0$, with $\alpha = O(\epsilon^{1/2})$, when regular or Mach reflection occurs.

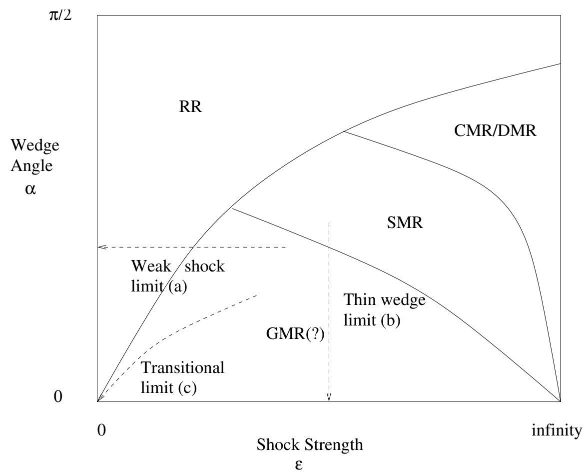

Figure 6.1. Schematic diagram of shock reflection patterns: RR = regular reflection; SMR = single Mach reflection; DMR = double Mach reflection; CMR = complex Mach reflection; GMR = Guderley Mach reflection.

The weak shock limit (a) was studied by Keller and Blank [31], Hunter and Keller [27], and others, and is discussed in §2. The thin wedge limit (b) was studied by Lighthill [32] and Ting and Ludloff [43], but in this paper we restrict our attention to weak shocks and will not discuss this work. The transitional limit (c) was studied by Hunter [23, 24], Hunter and Brio [25], and Morawetz [34], and leads to a shock reflection problem for the unsteady transonic small disturbance equation, discussed further in §3. In §4, we discuss steady shock reflections, and in §5, we discuss shock focusing.

2. Regular reflection

A first approximate solution for the regular reflection of a weak shock off a wedge may be obtained by linearization. The linearized solution is not, however, a uniformly valid leading order approximation throughout the flow-field. As indicated in Figure 6.2, it breaks down near the diffracted wavefront, where its radial derivative becomes infinite, and near the singular ray, where both its radial and tangential derivatives become infinite. To obtain a complete asymptotic solution, one needs to construct weakly nonlinear inner solutions in these regions and match them with the linearized outer solution. We outline the main ideas of this construction in the next three subsections.

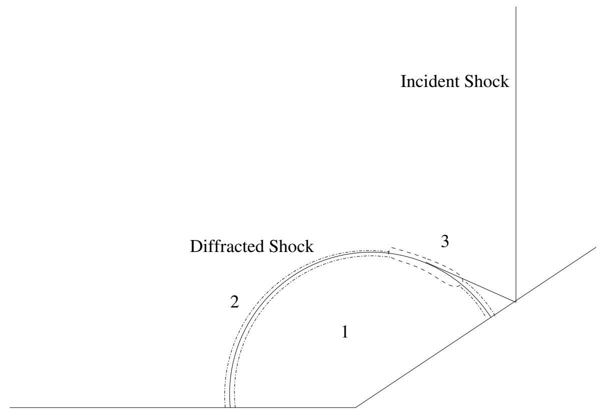

Figure 6.2. Regions for regular reflection: 1. linearized solution; 2. diffracted wavefront; 3. singular ray.

2.1 The linearized solution

To leading order in the shock strength, the flow in a weak shock reflection is irrotational and isentropic. The linearized compressible Euler equations therefore reduce to the acoustical wave equation for the pressure fluctuations p^l,

$$p_{tt}^l = c_0^2 \left(p_{xx}^l + p_{yy}^l \right). \tag{6.1}$$

Here, c_0 is the sound speed in the fluid ahead of the incident shock, t is the time variable and (x, y) are Cartesian space variables.

In self-similar coordinates

$$\xi = \frac{x}{c_0 t}, \qquad \eta = \frac{y}{c_0 t},$$

the wave equation reduces to a mixed type PDE

$$\left(\xi^2 - 1\right) p_{\xi\xi}^l + 2\xi\eta p_{\xi\eta}^l + \left(\eta^2 - 1\right) p_{\eta\eta}^l + 2\xi p_\xi^l + 2\eta p_\eta^l = 0. \tag{6.2}$$

We denote the self-similar radial coordinate by $\rho = \left(\xi^2 + \eta^2\right)^{1/2}$. Then (6.2) is hyperbolic in $\rho > 1$ and elliptic in $\rho < 1$. The shock reflection solution in the hyperbolic region is piecewise constant, and can be written down immediately. One then has to solve a degenerate elliptic boundary value problem in $\rho < 1$ subject to Neumann boundary conditions on the wedge and Dirichlet boundary conditions on the sonic circle $\rho = 1$, obtained from the solution in the hyperbolic region. This was done by Keller and Blank [31] using Busemann's conical flow method (Albert Blank was Joe's first PhD student). Busemann [4] had observed that the change of radial coordinates

$$\tilde{\rho} = \frac{1 - \left(1 - \rho^2\right)^{1/2}}{\rho}$$

reduces (6.2) to Laplace's equation in $\rho < 1$, which may be solved by standard methods.

2.2 The diffracted wavefront

We denote by (r, θ) polar coordinates in the spatial (x, y)-plane. The linearized solution p^l described in §2.1 has a square-root singularity near the diffracted wavefront $r = c_0 t$. One finds that

$$\frac{p^l}{\rho_0 c_0^2} \sim \epsilon k(\theta) \left(1 - \frac{r}{c_0 t}\right)^{1/2} \qquad \text{as } r \to c_0 t^-, \tag{6.3}$$

where ρ_0 is the density ahead of the incident shock, and $k(\theta)$ is an explicitly computable dimensionless function.

Following the ideas of weakly nonlinear geometrical acoustics [8, 26, 38], Hunter and Keller [27] obtained a weakly nonlinear inner solution p^d for the pressure fluctuations near the diffracted wavefront $r = c_0 t$ of the form[1]

$$\frac{p^d}{\rho_0 c_0^2} \sim \delta a \left(r, \theta, \frac{r - c_0 t}{\delta}\right),$$

where δ is a small parameter. Here, the function $a(r, \theta, \tau)$ satisfies a cylindrical inviscid Burgers equation

$$a_r + \left(\frac{\gamma + 1}{4} a^2\right)_\tau + \frac{1}{2r} a = 0, \tag{6.4}$$

where, for simplicity, we assume that the fluid is an ideal gas with constant ratio of specific heats γ. The polar angle θ occurs in this equation as a parameter.

The solution of (6.4) must match as $\tau \to -\infty$ with the inner expansion of the outer linearized solution given in (6.3), and it must vanish ahead of the diffracted wavefront as $\tau \to +\infty$. In view of the self-similarity of the problem, this matching condition is equivalent to an asymptotic initial condition as $r \to 0^+$. Matching implies that $\delta = \epsilon^2$ and

$$a(r, \theta, \tau) \sim \begin{cases} k(\theta)\sqrt{-\tau/r} & \text{if } \tau < 0, \\ 0 & \text{if } \tau > 0, \end{cases} \qquad \text{as } r \to 0^+. \tag{6.5}$$

[1] In fact, one must also make a Galilean transformation into a reference frame moving with the fluid flow behind the incident or reflected shock to account for the advection of the diffracted wavefront by this flow. In order to explain the main ideas, we neglect this complication here.

Solving (6.4) and (6.5), we find that when $k(\theta) > 0$, corresponding to a compressive diffracted wave, the diffracted wavefront is a shock (rather than the square-root singularity that occurs in the linearized theory). The pressure jump across the diffracted shock is given by

$$\left[\frac{p^d}{\rho_0 c_0^2}\right] \sim \frac{3(\gamma+1)}{4}\epsilon^2 k^2(\theta) \qquad \text{as } \epsilon \to 0^+.$$

Thus, the diffracted shock strength is second order in the incident shock strength and the linearized solution breaks down in a region of width $O(\epsilon^2)$ around the diffracted shock. When $k(\theta) < 0$, corresponding to an expansive diffracted wave, the diffracted wavefront is an acceleration wave across which the pressure is continuous and the derivative has a jump discontinuity (but remains bounded, unlike the linearized solution).

2.3 The singular ray

The diffracted wavefront expansion breaks down near the singular ray because $k(\theta)$ in (6.3) becomes infinite at the corresponding value of θ. Asymptotic expansions valid near the singular ray were constructed by Harabetian [17], Hunter [22], and Zahalak and Myers [46].

The method of matched asymptotic expansions shows that the breakdown occurs in a region of width of the order ϵ in the radial direction and $\epsilon^{1/2}$ in the tangential direction about the singular ray. In this case, a two-scale weakly nonlinear expansion of the compressible Euler equations is required to describe the solution, leading to a two dimensional version of the inviscid Burgers equation called the unsteady transonic small disturbance (UTSD) equation [22]. The normalized form of the UTSD equation is

$$u_t + \left(\frac{1}{2}u^2\right)_x + v_y = 0, \qquad u_y - v_x = 0. \tag{6.6}$$

Here, x and y are scaled Cartesian spatial coordinates near the singular ray, respectively normal and tangential to the wavefront, and t is a 'slow' time or radial variable analogous to r in (6.4). The dependent variables u, v are scaled x, y velocity fluctuations, and the pressure fluctuations are proportional to u, which is analogous to a in (6.4).

In self-similar coordinates

$$\xi = \frac{x}{t}, \qquad \eta = \frac{y}{t},$$

equation (6.6) reduces to

$$\left(\frac{1}{2}u^2 - \xi u\right)_\xi + (v - \eta u)_\eta + 2u = 0, \qquad u_\eta - v_\xi = 0,$$

which changes type across the sonic line

$$u = \xi + \eta^2/4. \tag{6.7}$$

We refer to a point where the equations are hyperbolic ($u < \xi + \eta^2/4$) as supersonic, and a point where the equations are elliptic ($u > \xi + \eta^2/4$) as subsonic.

The inner weakly nonlinear solution must match with the outer linearized solution. As before, because of self-similarity, the far-field matching condition as $x, y \to \infty$ is equivalent to an initial condition as $t \to 0^+$. Matching[2] implies that

$$u(x, y, 0) = \begin{cases} 1 & \text{if } x < 0 \text{ and } y < 0, \\ 0 & \text{otherwise.} \end{cases} \tag{6.8}$$

The flow perturbations ahead of the incident and diffracted shocks are zero, so we also have

$$v(x, y, t) \to 0 \qquad \text{as } x \to +\infty. \tag{6.9}$$

A numerical solution of this two-dimensional Riemann problem for the UTSD equation is shown in Figure 6.3. In it, one can see the curved diffracted shock merge with the planar reflected shock, and the appearance of an expansive parabolic wave front along the sonic line $\xi + \eta^2/4 = 1$ behind the reflected shock. Numerical solutions of this singular ray problem were obtained by Tabak and Rosales [41], and further analysis of this problem is given in Ting and Keller [44].

3. Mach reflection

For weak shocks, the transition from regular to Mach reflection occurs when the wedge is thin, specifically when $\alpha = O(\epsilon^{1/2}) \ll 1$. In this

[2]Unlike the case of (6.5), where the matching data is unbounded as $r \to 0^+$, the matching data for the singular ray problem has a finite, well-defined limit as $t \to 0^+$, so it may be imposed at $t = 0$. A more refined asymptotic initial condition is the requirement that the solution of the nonlinear problem matches as $t \to 0^+$ with the solution of the linearized problem. These initial conditions give the same numerical solutions, which explains why it is legitimate to impose the initial data (6.8) on the UTSD equation even though the discontinuity in y violates the scaling assumption (that y variations are slower than x variations) used to derive it.

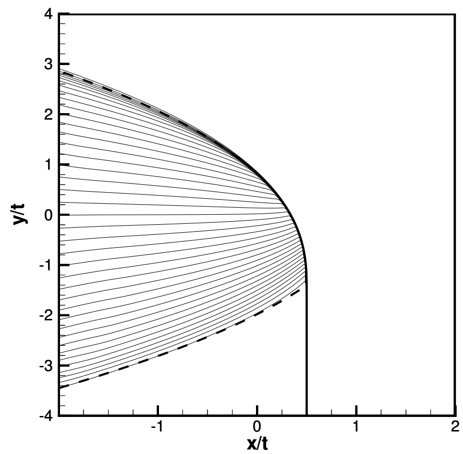

Figure 6.3. A numerical solution of the IBVP (6.6), (6.8)–(6.9) for a shock at a singular ray. The u-contour spacing is 0.025, and the sonic line (6.7) is dashed.

transitional limit, the singular ray lies inside the reflection region, and the method of matched asymptotic expansions gives an initial-boundary value problem for the UTSD equation that provides an asymptotic description of the transition from regular to Mach reflection for weak shocks. Numerical solutions of this problem answer a long-standing puzzle about weak shock Mach reflection called the von Neumann triple point paradox.

3.1 The von Neumann paradoxes

During the second world war, von Neumann carried out an extensive study of the jump conditions at the junction of shocks and contact discontinuities [36]. He suggested a number of criteria for the transition from regular to Mach reflection, and compared his theoretical results with observations. There was generally excellent agreement for strong shocks, but for weak shocks serious discrepancies were found, which became known as the 'von Neumann paradoxes'.

One discrepancy was that regular reflection is observed experimentally to persist into parameter regions where it is theoretically impossible. This persistance appears to be the result of a displacement effect of the viscous boundary layer on the wedge, which is not accounted for in von Neumann's theory. A second, more puzzling, discrepancy was that a pattern closely resembling simple Mach reflection is observed for weak shocks, but no standard triple point configuration is compatible with the jump relations across shocks and contact discontinuities. This discrepancy was called the 'triple point paradox' by Birkhoff, who states in §I.11 of [3] that "the predicted limits for triple shocks seem to differ grossly from those observed."

The resolution of the triple point paradox is that a remarkable type of Mach reflection occurs for weak shocks — we call it a Guderley Mach reflection (GMR).

Using numerical solutions of the UTSD equation, we show in [42] that there is a sequence of supersonic patches, shocks, expansion fans, and triple points in a tiny region behind the leading triple point (see §3.2). We conjecture that this sequence is infinite for an inviscid weak shock Mach reflection. At each triple point, there is an additional expansion fan, thus resolving the apparent conflict with von Neumann's theoretical arguments.[3] Furthermore, an infinite sequence of shrinking supersonic patches resolves theoretical difficulties connected with the transition from supersonic to subsonic flow at the rear of the supersonic region.

The existence of a supersonic patch and an expansion fan at the triple point of a weak shock Mach reflection was first proposed by Guderley [15, 16], although he did not give any evidence that this is what actually occurs, nor did he suggest that there is, in fact, a sequence of supersonic patches and triple points.[4]

Numerical solutions of weak shock Mach reflections with a supersonic region and an expansion fan behind the triple point were obtained by Hunter and Brio [25] for the unsteady transonic small disturbance equation, and by Vasil'ev and Kraiko [45] and Zakarian et. al. [47] for the full Euler equations. These solutions, however, were not sufficiently well-resolved to show the true nature of the solution.

3.2 The asymptotic shock reflection problem

In the transitional thin wedge/weak shock limit (c) in Figure 6.1, a similar expansion and matching procedure to the one described in §2.3 for singular rays leads to the following asymptotic shock reflection

[3]Von Neumann himself in Paragraph 10 of [37] states that additional waves must exist at the triple point and that an expansion fan is the most reasonable candidate. He concludes, however, that "The situation is far from clear mathematically. Possibly the nature of the neccessary singularity at [the triple point] is of a different kind."

[4]Guderley comments in §VI.9 of [16] only that: "Careful analysis (cf. Guderley [15]) indicates, however, that a singularity results at the [rear sonic point] or that also in this case a solution in Tricomi's sense is not possible. From the practical point of view the details of the structure of the flow field are greatly complicated by this singularity. In general the form of the flow developed above is correct." Unfortunately, we have been unable to obtain a copy of the Technical Report [15].

problem for the UTSD equation in $y > 0$ (see [24, 25, 34] and the review in [48]):

$$\begin{aligned} & u_t + \left(\frac{1}{2}u^2\right)_x + v_y = 0, \qquad u_y - v_x = 0, \\ & u(x, y, 0) = \begin{cases} 1 & \text{if } x < ay, \\ 0 & \text{if } x > ay, \end{cases} \\ & v(x, 0, t) = 0, \\ & v(x, y, t) \to 0 \qquad \text{as } x \to +\infty. \end{aligned} \tag{6.10}$$

Here, the parameter a is related to the wedge angle α and the incident shock Mach number M by

$$a = \frac{\alpha}{\sqrt{2(M^2 - 1)}},$$

which is held fixed in the transitional limit.

Numerical solutions of (6.10) for two values of a are shown in shown Figures 6.4–6.5. The left and right boundaries of the computational domain are curved because of the use of parabolic coordinates (see [42] for a discussion of the numerical scheme). We see a transition from regular to Mach reflection as a decreases though a critical value close to the detachment value $a = a_d$ below which regular reflection becomes impossible. The detachment value for the UTSD equation is given by [23]

$$a_d = \sqrt{2} \approx 1.414.$$

One can verify that this value agrees quantitatively with the weak shock limit of the detachment value for the full Euler equations given in [19].

A second value of a that is potentially significant for transition is the sonic value

$$a_s = \sqrt{1 + \frac{\sqrt{5}}{2}} \approx 1.455.$$

In the parameter range $a_d \le a \le a_s$, the state behind the reflected shock is subsonic. It is conceivable that the diffracted wave triggers a transition from regular to Mach reflection somewhere in this range. However, the sonic and detachment values are so close together that we have been unable to tell from numerical solutions where the transition occurs.

For some rigorous work on the UTSD shock reflection problem (6.10) in the regular reflection regime, see [5].

Figure 6.6, from [42], shows the sequence of supersonic patches, shocks, expansion fans, and triple points in a tiny region immediately behind the leading triple point in a Mach reflection. The change from supersonic to

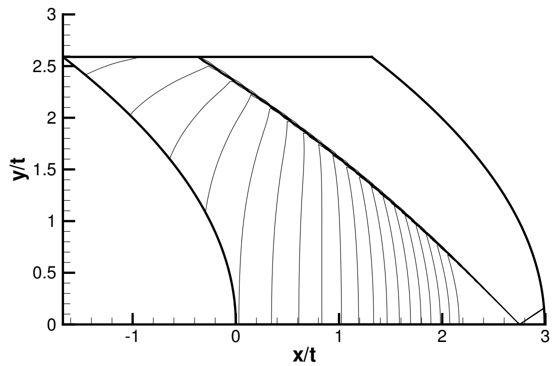

Figure 6.4. A supersonic regular reflection for $a = 1.5$. The u-contour spacing is 0.05. The reflection point is at $x/t = 2.75$, $y/t = 0$. The solution is piecewise constant in the hyperbolic region, and non-constant in the elliptic region that lies below the reflected shock and to the left of the parabolic sonic line (6.7) behind the reflection point.

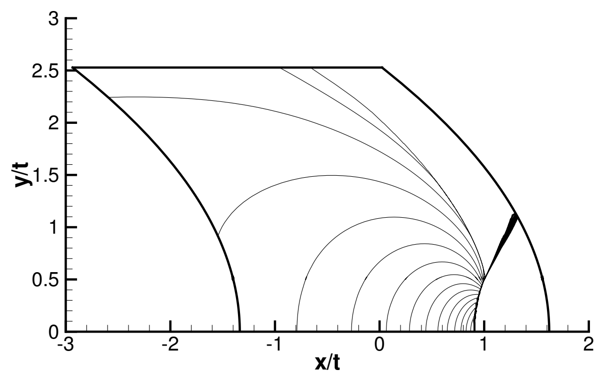

Figure 6.5. A Mach reflection for $a = 0.5$. The u-contour spacing is 0.05. The triple point is at $x/t \approx 1.008$, $y/t \approx 0.514$. The Mach shock curves down from the triple point and hits the wall at $x/t \approx 0.9$. The thickening of the incident shock as it moves up from the triple point toward the right computational boundary is a numerical effect caused by the use of a nonuniform grid that is stretched exponentially away from the triple point.

subsonic flow at the rear of the region by means of a subsonic triple point appears to be impossible because no expansion fan can then occur (see also [12]), and a continuous change across a sonic line is unlikely because it would result in an apparently overdetermined BVP for a mixed-type

PDE.[5] This argument suggests that the sequence of supersonic patches is infinite. (See [42] for further discussion.)

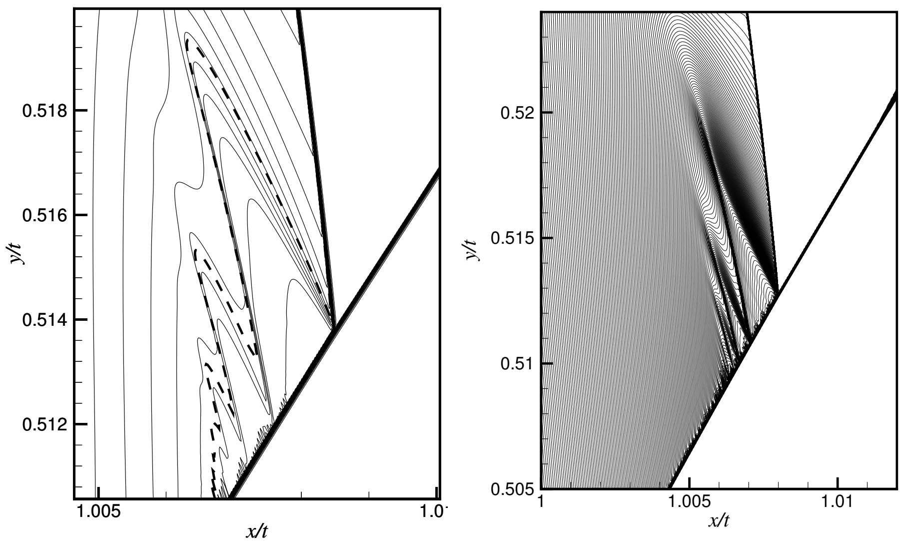

Figure 6.6. The sequence of shocks, fans, triple points, and supersonic patches immediately behind the leading triple point for $a = 0.5$. The u-contour spacing is 0.005 on the left, and 0.0005 on the right. The dashed line on the left is the numerically computed location of the sonic line (6.7).

These results show that a GMR occurs for very weak shocks and small wedge angles. It is reasonable to expect that one also occurs throughout the parameter region where both RR and SMR are impossible (see Figure 6.1), although there is presently no direct evidence for this.

A sequence of supersonic patches is likely to be a generic feature of solutions of mixed-type conservation laws and two-dimensional Riemann problems for hyperbolic systems of conservation laws. In the next section, we show that such sequences arise in steady weak shock Mach reflections. They also arise in the reflection of shocks off contact discontinuities in strong-shock Mach reflections (see Figure 22 in [20]).

4. Steady shock reflection and transonic flow

There is a close connection between the self-similar shock reflection problems discussed above, and the problems of steady transonic flow [9, 16]. The basic asymptotic equation that describes nearly sonic steady

[5]The hyperbolic region would be enclosed by a sonic line that lacks a 'gap'.

flows is the transonic small disturbance (TSD) equation

$$\left(\frac{1}{2}u^2\right)_x + v_y = 0, \qquad u_y - v_x = 0, \tag{6.11}$$

which is the steady form of (6.6). Here, (x, y) are spatial coordinates along and transverse to the sonic flow, and (u, v) are the corresponding velocity perturbations. Equation (6.11) is hyperbolic in $u < 0$, corresponding to supersonic flow, and elliptic in $u > 0$, corresponding to subsonic flow.

We consider the following boundary value problem for (6.11) in the region $x_L < x < x_R$, $0 < y < 1$:

$$\begin{aligned}
&u(x_R, y) = -1, \quad v(x_R, y) = 0 \qquad 0 < y < 1,\\
&v(x, 1) = 0 \qquad x_L < x < x_R,\\
&u(x, 0) = u_0 \qquad x_L < x < x_0,\\
&v(x, 0) = \tilde{a} \quad x_0 < x < x_1, \qquad v(x, 0) = 0 \quad x_1 < x < x_R.
\end{aligned} \tag{6.12}$$

This BVP is the small disturbance approximation for the problem of a slightly supersonic jet hitting a thin wedge in a channel and issuing into a higher pressure region. For appropriately chosen parameter values, the shock generated at the corner of the wedge undergoes a Mach reflection off the top wall of the channel.

In Figures 6.7–6.8, we show a numerical solution of (6.11)–(6.12) with

$$x_L = -2,\ x_0 = -0.3,\ x_1 = 0,\ x_R = 0.1,\ \tilde{a} = 0.67,\ u_0 = -0.09. \tag{6.13}$$

We again see the formation of a Guderley Mach reflection with a sequence of supersonic patches, shocks, expansion fans, and triple points. It is remarkable that the solution of a boundary value problem for such a simple-looking mixed-type system of conservation laws exhibits such complex behavior.

In regions where the mapping $(x, y) \mapsto (u, v)$ is invertible, the hodograph transformation — which exchanges the roles of the independent and dependent variables (x, y) and (u, v) — reduces (6.11) to a linear Tricomi equation for $y(u, v)$:

$$y_{uu} + u y_{vv} = 0.$$

A numerical plot of the boundaries of the supersonic region in the (u, v)-hodograph plane near the triple point, together with the corresponding shock and rarefaction curves [42], is shown in Figure 6.9.

There are recent rigorous results on the existence of transonic shocks for the TSD equation [5] and for the full potential equation [7]. An alter-

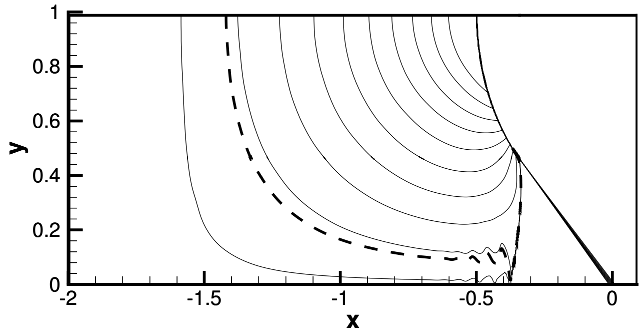

Figure 6.7. A solution of (6.11)–(6.13), showing a steady Mach reflection. The u-contour spacing is 0.1. The flow is from right to left. The incident shock is generated at the corner of the wedge ($x = 0$, $y = 0$) and undergoes a Mach reflection off the top wall of the channel ($y = 1$). The triple point is at $x \approx -0.343$, $y \approx 0.464$. The Mach shock curves up from the triple point, hitting the top wall at $x \approx -0.5$. The reflected shock curves down from the triple point, hitting the bottom boundary at $x \approx -0.38$, where it is reflected as an expansion wave. The dashed line is the numerically computed location of the sonic line $u = 0$. The Mach shock, the reflected shock, and the sonic line enclose a large subsonic region behind the Mach reflection. The flow accelerates back to supersonic to the left of the rear sonic line. In addition, as shown in Figure 6.8, there is a tiny supersonic region immediately behind the leading triple point.

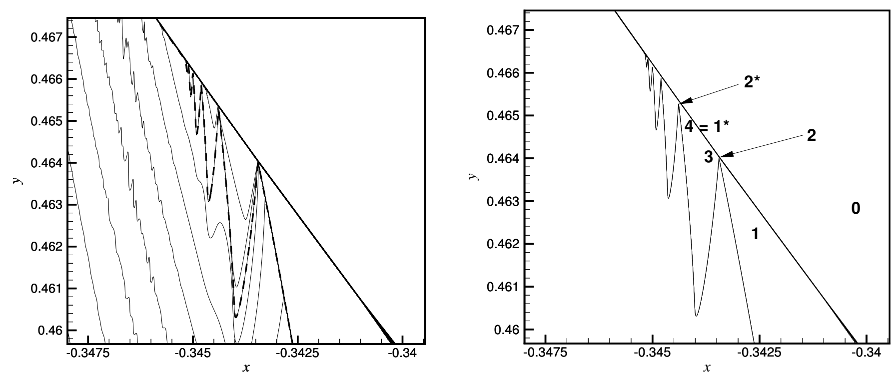

Figure 6.8. The local triple point structure for the steady Mach reflection shown in Figure 6.7. The u-contour spacing is 0.01 on the left, and the dashed line is the sonic line. The right picture shows the sonic line only. Five supersonic patches can be seen.

native approach is to adapt the theory of compensated compactness [11] to treat mixed type conservation laws [35], but this appears to require more a priori estimates for (6.11) than are presently available.

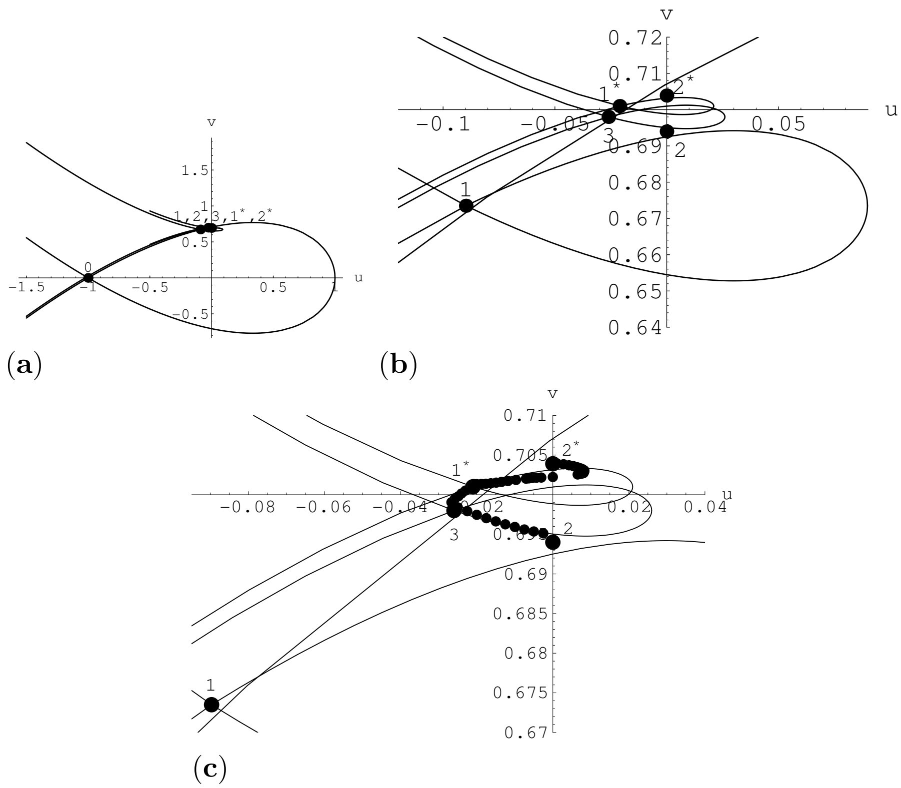

Figure 6.9. A numerical plot of the local triple point structure in the hodograph plane for the solution shown in Figure 6.8. The plots in (a)–(b) show the theoretical shock and rarefaction curves through the numerically computed values of (u, v) at the first and second triple points at the points labeled in Figure 6.8. The dots in (c) are obtained directly from the numerical data for the first supersonic patch. They consist of the values of (u, v) behind the Mach shock, the values on either side of the reflected shock, and the values on the expansion fan originating at the triple point. The resulting curves bound the supersonic patch in the hodograph plane, and are in excellent agreement with the theoretical curves.

5. Shock focusing and nonlinear caustics

Shock focusing is an unsteady, non-self-similar problem that is closely related to shock reflection. The beautiful experiments of Sturtevant and Kulkarny [40] on weak shock focusing show a transition from a ‘fish-tail’ wavefront pattern, typical of linear theory, to a nonlinear Mach-shock pattern as the strength of the shock is increased. Nonlinear effects are always important, however, in the immediate vicinity of the focal point, even for very weak shocks. This transition is analogous to the transition from regular to Mach reflection.[6]

Cramer and Seebass [10] derived the UTSD equation as an asymptotic description of nearly planar weak shock focusing. Numerical solutions show that it captures the transition that is observed in experiments [41]. Some typical solutions are shown in Figures 6.10. The initial data corresponds to a shock of constant strength (as measured by the jump in u) with a Gaussian shape:

$$u(x,y,0) = \begin{cases} \frac{1}{2}u_0 & \text{if } x < s(y), \\ -\frac{1}{2}u_0 & \text{if } x > s(y), \end{cases} \qquad s(y) = -\frac{1}{4}e^{-y^2/(0.4)^2}. \tag{6.14}$$

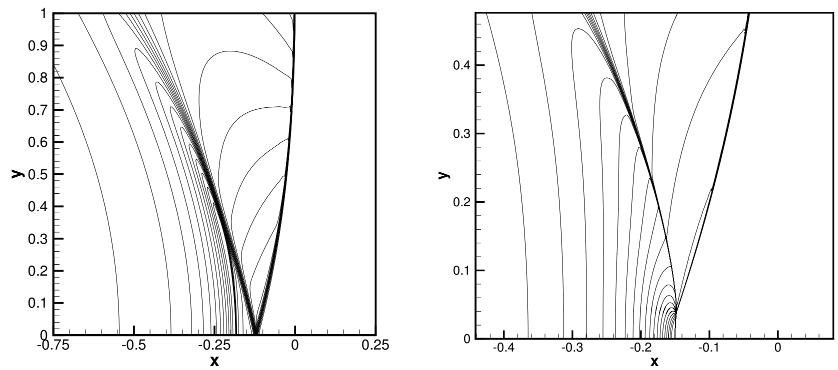

Figure 6.10. Solutions of the UTSD equation (6.6) with initial data (6.14). The left solution (with $u_0 = 0.05$, $t = 1$, and a u-contour spacing of 0.0025) shows a linear focusing pattern. The right solution (with $u_0 = 0.02$, $t = 0.65$, and a u-contour spacing of 0.01) shows a nonlinear focusing pattern with a triple point.

Although we have not been able so far to resolve the local structure of the solution near the triple point numerically, presumably there is a sequence of fans, shocks, and triple points similar to the sequence that occurs in the self-similar and steady shock reflection patterns. It is not clear, however, whether or not the sequence is infinite.

5.1 Nonlinear caustics

Weakly nonlinear hyperbolic waves at caustics were studied by Hunter and Keller [28], who derived weakly nonlinear caustic expansions that generalize the expansions of Ludwig [33] for linear hyperbolic waves. In

[6] The reflection problem is a special case of this focusing problem, corresponding to an initially '<-shaped' shock.

the simplest case of a smooth convex caustic, the result is a nonlinear Tricomi equation derived by Guiraud [14] and Hayes [18] for the compressible Euler equations, whose normalized form is

$$\left(\frac{1}{2}u^2 + yu\right)_x + v_y = 0, \qquad u_y - v_x = 0. \tag{6.15}$$

This equation is hyperbolic in $u > y$ and elliptic in $u < y$.

One conclusion of the analysis in [28] is that the passage of a smooth weakly nonlinear hyperbolic wave through a caustic may be treated to leading order in the wave amplitude by the use of linear theory. One must, however, use nonlinear theory in the neighborhood of a shock front at a caustic since, according to linear theory, the strength of a jump discontinuity becomes infinite (see also the discussion in [39]). Numerical solutions of (6.15) for shocks at a smooth convex caustic, together with experimental comparisons, are given in [1]. The strength of a shock at a smooth convex caustic appears to remain bounded because the shock accelerates as it grows stronger, preventing its focusing. Thus, nonlinearity has a regularizing effect on the solution.

Beyond a caustic, there are multiple wave fields, and one has to understand how they interact. This application was one of Joe's main motivations for proposing a study of the interaction of weakly nonlinear hyperbolic waves as a thesis problem to the first author of this paper [21]. As the discussion above shows, the interaction of weak shocks can be remarkably complex.

Further difficulties arise in analyzing the interaction of high-frequency, oscillatory waves. Fourier analysis gives a precise correspondence between the propagation of singularities and high-frequency oscillations for linear waves, but there are significant differences for nonlinear waves. In particular, the interaction of weakly nonlinear oscillatory hyperbolic waves beyond a caustic is not simple to describe. The waves are necessarily 'incoherent' in the terminology of [29] and this can lead to subtle phenomena, such as the 'hidden focusing' of formally lower-order waves that are generated by the interaction of the leading order waves [30]. Thus, many challenges remain in the development of a better understanding of the propagation, focusing, and interaction of nonlinear hyperbolic waves.

Acknowledgments

The topics described in this paper are ones, among many others, to which Joe Keller has made (and continues to make) fundamental contributions. It is a great pleasure to dedicate this paper to Joe on the occasion of his 80th birthday.

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