Semi-Fredholm solvability and asymptotic expansions of singular solutions for Protter problems

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Abstract. We study four-dimensional boundary value problems for the nonhomogeneous wave equation, which are analogues of Darboux problems on the plane. It is known that the unique generalized solution may have a strong power-type singularity at only one point. This singularity is isolated at the vertex O of the boundary light characteristic cone and does not propagate along the cone. We find asymptotic expansion of the generalized solutions in negative powers of the distance to O. Some necessary and sufficient conditions for existence of bounded solutions are derived and additionally some a priori estimates for the singular solutions are given.

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INTRODUCTION

In the present paper, boundary value problems for the wave equation in $\mathbb{R}^4$

$$u_{x_1x_1} + u_{x_2x_2} + u_{x_3x_3} - u_{tt} = f(x,t) \tag{1}$$

with points $(x,t) = (x_1, x_2, x_3, t)$ are studied in the domain

$$\Omega = \{(x,t) : 0 < t < 1/2, t < \sqrt{x_1^2 + x_2^2 + x_3^2} < 1 - t\},$$

bounded by the two characteristic cones

$$\Sigma_1 = \{(x,t) : 0 < t < 1/2, \sqrt{x_1^2 + x_2^2 + x_3^2} = 1 - t\},$$

$$\Sigma_2 = \{(x,t) : 0 < t < 1/2, \sqrt{x_1^2 + x_2^2 + x_3^2} = t\}$$

and the ball $\Sigma_0 = \{t = 0, \sqrt{x_1^2 + x_2^2 + x_3^2} < 1\}$, centered at the origin $O : x = 0, t = 0$. The following BVPs were proposed by M. Protter [24] as a multi-dimensional analogue of the planar Darboux problems with boundary data prescribed on one characteristic and on the noncharacteristic segment:

Problem P1. Find a solution of the wave equation (1) in Ω which satisfies the boundary conditions

$$P1: \quad u|_{\Sigma_0} = 0, \quad u|_{\Sigma_1} = 0.$$

Problem P1*. Find a solution of the wave equation (1) in Ω which satisfies the adjoint boundary conditions

$$P1^*: \quad u|_{\Sigma_0} = 0, \quad u|_{\Sigma_2} = 0.$$

Problems P2 and **P2*.** Find a solution of the wave equation (1) in Ω which satisfies the boundary conditions

$$P2: \quad u_t|_{\Sigma_0} = 0, \quad u|_{\Sigma_1} = 0$$

or the adjoint boundary conditions

$$P2^*: \quad u_t|_{\Sigma_0} = 0, \quad u|_{\Sigma_2} = 0,$$

respectively.

In 1952 at a AMS conference in New York, Protter [24] formulated and studied the analogues of $P1$ and $P1^*$ in $\mathbb{R}^3$. Initially the expectation was that such BVPs are classical solvable for very smooth right-hand side functions. Contrary to this traditional belief, soon it became clear that unlike the plane Darboux problem, the Protter's problems are not well posed. The reason is that the adjoint homogeneous Problems $P1^*$ and $P2^*$ has infinite number of nontrivial classical solutions (Tong Kwang-Chang [25], Popivanov and Schneider [19], Khe Kan Cher [17]).

Throughout the last sixty years different authors adopted a variety of approaches to Protter problems. Garabedian [8] proved the uniqueness of a classical solution of Problem $P1$, but generally, problems $P1$ and $P2$ are not classically solvable. A necessary condition for the existence of a classical solution for Problem $P1$ or Problem $P2$ is the orthogonality of the right-hand side function f to all solutions of the corresponding homogeneous adjoint Problem $P1^*$ or Problem $P2^*$. Instead, avoiding infinite number of necessary conditions, Popivanov and Schneider [19] introduced the concept of generalized solution that allows the solution to have singularity on the inner cone Σ_2. Many authors studied these problems using different methods, e.g., Wiener-Hopf method, special Legendre functions, a priori estimates, nonlocal regularization, etc. (see [19] and references there, further [2, 7, 13, 17, 20, 21]). On the other hand some different multidimensional analogues of the classical Darboux problem are also known ([4, 5, 15]). The existence of bounded or unbounded solutions for the wave equation in $\mathbb{R}^3$ and $\mathbb{R}^4$, as well as for the Euler-Poisson-Darboux equation has been studied in [1, 2, 17, 13, 14, 10, 21]. Results on related multidimensional problems for mixed-type equations (also studied in [24]) can be found in [3, 5, 7, 11, 16, 17]. Concerning the nonexistence principle for nontrivial solution of semilinear mixed-type equations in multidimension case, we refer to [18].

From [19] it is known that for each $n \in \mathbb{N}$ there exists a right-hand side function $f \in C^n(\overline{\Omega})$ of the wave equation, for which the uniquely determined generalized solution has a strong power-type singularity like r^{-n} at the origin. Necessary and sufficient conditions for the existence of solutions with fixed order of singularity were derived for Problem $P1$ in [22] and for Problem $P2$ in [21]. Similarly, for the $\mathbb{R}^3$-analogues of Protter problems some results are presented in [20] (Problem $P1$) and [6] (Problem $P2$).

For Problem $P1$ in the case when the right-hand function f is a harmonic polynomial, formula for the asymptotic expansion of the singular solution could be found in [22]. The semi-Fredholm solvability of the Problem $P1$ is discussed in [23]. A comparison of various recent results for Protter problems is made in [6].

Regarding the Protter problems with lower order terms see [9] and references therein. Protter problems with more general boundary condition on Σ_0 are studied in [9, 10]. Some possible regularization methods can be found in [2, 7, 9].

In this paper we examine the behavior of the singular solutions of Problem $P2$. We give an exact asymptotic formula when the right-hand side function f is harmonic polynomial. Then, for the general case of smooth enough f, the necessary and sufficient condition for existence of bounded solutions is given.

GENERALIZED SOLUTIONS OF PROTTER PROBLEM

In order to construct the solutions of the homogenous Problem $P2^*$ we need the *spherical functions* Y_n^m in $\mathbb{R}^3$. They are defined usually on the unit sphere $S^2 := \{(x_1,x_2,x_3) : x_1^2+x_2^2+x_3^2 = 1\}$ in spherical polar coordinates and form a complete orthonormal system in $L_2(S^2)$ (see [12]). Expressed in Cartesian coordinates here, one can define them by

$$Y_n^{2m}(x_1,x_2,x_3) = C_n^m \frac{d^m}{dx_3^m} P_n(x_3)\,\mathrm{Im}\,\{(x_2+ix_1)^m\}, \text{ for } m = 1,...,n \tag{2}$$

and

$$Y_n^{2m+1}(x_1,x_2,x_3) = C_n^m \frac{d^m}{dx_3^m} P_n(x_3)\,\mathrm{Re}\,\{(x_2+ix_1)^m\}, \text{ for } m = 0,...,n, \tag{3}$$

where C_n^m are constants and P_n are the *Legendre polynomials*. For convenience in the discussions that follow, we extend the spherical functions out of S^2 radially, keeping the same notation Y_n^m for the extended function, i.e. $Y_n^m(x) := Y_n^m(x/|x|)$ for $x \in \mathbb{R}^3 \backslash \{O\}$.

Next, for $n,k \in \mathbb{N} \cup \{0\}$ define the functions

$$E_k^n(x,t) = \sum_{i=0}^{k} B_{k,i}^n \frac{(|x|^2-t^2)^{n-k-i}}{|x|^{n-2i+1}}, \tag{4}$$

where the coefficients are

$$B_{k,i}^n := (-1)^i \frac{(k-i+1)_i(n+1-k-i)_i}{i!(n-i+\frac{1}{2})_i}, \;\; B_{k,0}^n = 1,$$

with $(a)_i := a(a+1)\cdots(a+i-1)$ and $(a)_0 := 1$. Now we can construct the solutions of the homogenous Problem $P2^*$ as follows.

Lemma 1 ([21]) *The functions*

$$W_{k,m}^n(x,t) = E_k^n(x,t)Y_n^m(x)$$

are classical solutions of the homogeneous Problem $P2^*$ for $k = 0,1,\ldots,[n/2]-2$.

The corresponding solutions to $W^n_{0,m}$ for the three-dimensional case are known from [19], while the functions E^n_k expressed in terms of the Gauss hypergeometric function could be found in [17]. Naturally, a necessary condition for the existence of classical solution for the Problem $P2$ is the orthogonality of the right-hand side function f to all functions $W^n_{k,m}$ from Lemma 1. To avoid an infinite number of necessary conditions in the frame of classical solvability, we introduce *generalized solutions* for the Problem $P2$ (similarly to [19]).

Definition 1 ([21]) *A function $u = u(x,t)$ is called a generalized solution of the Problem $P2$ in Ω, if the following conditions are satisfied:*

1) $u \in C^1(\overline{\Omega}\backslash O)$, $u_t|_{\Sigma_0\backslash O} = 0, u|_{\Sigma_1} = 0$, and

2) the identity

$$\int_{\Omega} (u_t w_t - u_{x_1} w_{x_1} - u_{x_2} w_{x_2} - u_{x_3} w_{x_3} - fw) dxdt = 0$$

holds for all $w \in C^1(\overline{\Omega})$ such that $w_t|_{\Sigma_0} = 0$ and $w = 0$ in a neighborhood of Σ_2.

This definition allows the generalized solution to have singularity at the origin $O : x = 0, t = 0$. Further, our aim will be to study the behaviour at O of the generalized solutions of Problem $P2$ in the domain Ω.

ASYMPTOTIC EXPANSION FORMULA

In this section we fix the right-hand side function $f \in C^1(\overline{\Omega})$ of equation (1) as a harmonic polynomial of order l with $l \in \mathbb{N} \cup \{0\}$, i.e., having the representation

$$f(x,t) = \sum_{n=0}^{l} \sum_{m=1}^{2n+1} f^m_n(|x|,t) Y^m_n(x). \tag{5}$$

According to the results from [21] it is known that without any additional conditions imposed on f the corresponding *generalized solution* $u(x,t)$ may have a power type singularity at the origin. Here we will investigate more accurately the exact behaviour of the solution of Problem $P2$. It is governed by the parameters

$$\beta^n_{k,m} := \int_{\Omega} W^n_{k,m}(x,t) f(x,t) \, dxdt, \tag{6}$$

where $n = 0, \ldots, l$; $k = 0, \ldots, \left[\frac{n}{2}\right]$ and $m = 1, \ldots, 2n+1$. We find the asymptotic formula for the *generalized solution* of Problem $P2$.

Theorem 1 *Suppose that the right-hand side function $f \in C^1(\overline{\Omega})$ has the form (5). Then the unique generalized solution $u(x,t)$ of Problem $P2$ belongs to $C^2(\overline{\Omega}\backslash O)$ and has the following asymptotic expansion at the singular point O*

$$u(x,t) = \sum_{p=1}^{l+1} \left(|x|^2 + t^2\right)^{-p/2} F_p(x,t) + F(x,t),$$

where:
***(i)** the function* $F \in C^2(\overline{\Omega}\backslash O)$ *and satisfies the a priori estimate*

$$|F(x,t)| \leq C\,||f||_{C^1(\Omega)}\ , \quad (x,t) \in \Omega$$

with constant C *independent on* f *and* $||f||_{C^k(\Omega)} = \sum_{|\alpha|\leq k} \max_{\overline{\Omega}} |D^\alpha f(x,t)|$*;*
***(ii)** the functions* F_p*,* $p = 1,...,l+1$*, satisfy the equalities*

$$F_p(x,t) = \sum_{k=0}^{[(l-p+1)/2]} \sum_{m=1}^{2p+4k-1} \beta_{k,m}^{p+2k-1} F_{k,m}^{p+2k-1}(x,t)\ , \tag{7}$$

with functions $F_{k,m}^n \in C^2(\overline{\Omega}\backslash O)$ *bounded and independent on* f*;*
***(iii)** if at least one of the constants* $\beta_{k,m}^{p+2k-1}$ *in (7) is different from zero, then for the corresponding function* $F_p(x,t)$ *there exists a direction* $(\alpha,1) := (\alpha_1,\alpha_2,\alpha_3,1)$ *with* $(\alpha,1)t \in \Sigma_2$ *for* $0 < t < 1/2$*, such that*

$$\lim_{t\to+0} F_p(\alpha t,t) = c_p = const \neq 0.$$

Using this asymptotic expansion formula we see that the *generalized solution* could be bounded only if all parameters $\beta_{k,m}^n$ involved are zero.

Corollary 1 *Suppose that the right-hand side function* $f \in C^1(\overline{\Omega})$ *has the form (5) and satisfies the orthogonality conditions*

$$\int_\Omega W_{k,m}^n(x,t) f(x,t)\,dxdt \ = \ 0 \tag{8}$$

for all $n = 0,\ldots,l$*;* $k = 0,\ldots,\left[\frac{n}{2}\right]$ *and* $m = 1,\ldots,2n+1$*. Then the unique generalized solution* $u(x,t)$ *of Problem P2 belongs to* $C^2(\overline{\Omega}\backslash O)$*, is bounded and fulfills the a priori estimate*

$$\max_{\overline{\Omega}} |u| \leq C\,||f||_{C^1(\Omega)}\,.$$

The cases (ii) and (iii) of Theorem 1 describe the influence of the orthogonality conditions (8) on the exact behavior of the *generalized solution*. According to the case (iii), for fixed indexes (n,k,m), the corresponding condition (8) "controls" one power-type singularity.

On the other hand, without any orthogonality conditions on f, the following result is obtained as a consequence of Theorem 1.

Corollary 2 *The generalized solution* u *of Problem P2 with a right-hand side function* $f \in C^1(\overline{\Omega})$ *in the form (5) satisfies the a priori estimate*

$$|u(x,t)| \leq C\left(\max_{\overline{\Omega}} |f|\right)(|x|^2+t^2)^{-(l+1)/2}. \tag{9}$$

The estimate (9) is similar to known estimates for Protter problems in $\mathbb{R}^3$ ([19]) and in $\mathbb{R}^m$ ([1]). It is interesting that singularities of the *generalized solutions* are isolated at the origin and do not propagate in the direction of the bicharacteristics on the characteristic cone Σ_2. Traditionally, it is assumed that the wave equation, with sufficiently smooth right-hand side cannot have a solution with an isolated singular point (see Hörmander [11, Chapter 24.5]).

Let us point out some differences between the results presented here and the ones from [21] for Problem $P2$ in $\mathbb{R}^4$. First of all, the explicit asymptotic expansion here has no analogue in [21], where we have only the behavior of the singularities. Secondly, according to Corollary 1, if the orthogonality conditions are fulfilled, the *generalized* solution is bounded, while for the solution in [21, Theorem 1.1] still some logarithmic singularities are possible.

BOUNDED SOLUTIONS IN THE GENERAL CASE

Next, let us consider the more general case when the right-hand side function f is smooth, but it can not be expanded simply as a harmonic polynomial (5). Now, Lemma 1 shows that the Problem $P2$ is not Fredholm solvable. To explain the situation, consider the operator

$$T: \quad u_f \longmapsto f \in C^k(\overline{\Omega}),$$

where u_f is the unique classical solution to Protter Problem $P2$ for the right-hand side function f. According to Lemma 1 $\dim\operatorname{coker}(T) = \infty$. This means that T is not Fredholm operator for example in $C^k(\overline{\Omega})$, but one could expect it to be semi-Fredholm for appropriate $k \in \mathbb{N}$. A semi-Fredholm operator is a bounded operator that has a finite dimensional kernel or cokernel, and closed range. Here $\dim\ker(T) = 0$ since we have uniqueness result for the generalized solution of Problem $P2$.

We find the necessary and sufficient conditions for existence of bounded solution.

Theorem 2 *Let the function $f(x,t)$ belong to $C^9(\overline{\Omega})$. Then the necessary and sufficient conditions for existence of bounded generalized solution $u(x,t)$ of the Protter Problem $P2$ are*

$$\int_{\Omega} W^n_{k,m}(x,t) f(x,t)\, dxdt \;=\; 0, \qquad (10)$$

for all $n \in \mathbb{N}$, $k = 0, \dots, [n/2]$, $m = 1, \dots, 2n+1$.

Moreover, this generalized solution $u(x,t) \in C^1(\overline{\Omega}\backslash O)$ and satisfies the a priori estimates

$$|u(x,t)| \le C\,\|f\|_{C^7(\overline{\Omega})};$$

$$\sum_{i=1}^{3} |u_{x_i}(x,t)| + |u_t(x,t)| \le C(|x|^2+t^2)^{-3/2}\,\|f\|_{C^9(\overline{\Omega})}$$

where the constant C is independent of the function $f(x,t)$.

Naturally, the necessary orthogonality conditions (10) include the classical solutions of the homogenous Problem $P2^*$ given in Lemma 1. However, let us point out that there are also some other functions. For example, it is interesting that the conditions (10) include the case of even $n = 2k$. Notice that the functions $W^{2k}_{k,m}(x,t)$ are not classical solutions of the homogenous adjoint Problem $P2^*$. Actually, they satisfy the homogenous wave equation in Ω and $(W^{2k}_{k,m})_t$ vanish on Σ_0, but $W^{2k}_{k,m}$ is not zero on Σ_2. In addition, they have a singularity at the origin O like $|x|^{-1}$, however this singularity is integrable in the domain Ω.

From another perspective, Theorem 2 suggests that there are no other nontrivial classical solution of the homogenous adjoint Problem $P2^*$ except those listed in Lemma 1.

Let us mention some differences between Theorem 2 and the results from [23] for Problem $P1$. First, notice that in the case when right-hand side function f is harmonic polynomial of order l the solution of Problem $P2$ may have worse singularity (like $(|x|^2+t^2)^{-(l+1)/2}$, see Theorem 1) than the solution of Problem $P1$ (like $(|x|^2+t^2)^{-l/2}$, see [22]). Despite that, here we assume $f \in C^9(\overline{\Omega})$ while in [23, Theorem 1.1] smoother $f \in C^{10}$ is required. To achieve this, key role play the more accurate estimates found for the special functions involved in the representation formulas for the solution.

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