TRANSONIC SOLUTIONS FOR THE MACH REFLECTION OF WEAK SHOCKS

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Abstract We present numerical solutions of the steady and unsteady transonic small disturbance equations that describe the Mach reflection of weak shock waves. The solutions contain a complex structure consisting of a sequence of triple points and tiny supersonic patches directly behind the leading triple point, formed by the reflection of weak shocks and expansion waves between the sonic line and the Mach shock. The presence of an expansion fan at each triple point resolves the von Neumann paradox. The numerical results and theoretical considerations suggest that there may be an infinite sequence of triple points in an inviscid weak shock Mach reflection.

Keywords: von Neumann paradox, weak shock Mach reflection, self-similar solutions

1. Introduction

For sufficiently weak shocks, the von Neumann theory of shock reflection shows that a standard triple point configuration, in which three shocks meet at a point, is impossible. However, experimental observations of weak shock reflections off a wedge show a pattern that resembles a single Mach reflection with a triple point. This apparent disagreement between theory and observation became known as the triple point, or von Neumann, paradox.

Guderley [1] proposed that there is a supersonic region behind the triple point of a steady weak shock Mach reflection and an additional expansion wave centered at the triple point, which provides a resolution of the paradox. This theory was not widely accepted for weak shock Mach reflection off a wedge because no evidence of a supersonic region or an expansion wave was found in experiments or numerical solutions. However, recent numerical solutions [2, 4, 5] have shown that Guderley's explanation is correct; the supersonic region was not detected previously because it is extremely small. The solutions in

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[2] were for a shock reflection problem for the unsteady transonic small disturbance (UTSD) equations that provides an asymptotic description of weak shock reflection, while the solutions in [4, 5] were for shock reflection problems for the Euler equations. These solutions show a supersonic region and indications of an expansion fan at the triple point, but none of them are sufficiently resolved to show the detailed structure of the flow in the supersonic region.

In [3], high-resolution numerical solutions of the UTSD shock reflection problem were obtained for a range of parameter values corresponding to Mach reflection. All of the solutions contain a supersonic region behind the triple point. This region consists of a sequence of supersonic patches formed by the reflection of shock and expansion waves between the sonic line and the Mach shock. Each reflected shock intersects the Mach shock, resulting in a sequence of triple points, instead of a single triple point. The presence of an expansion fan at each triple point resolves the von Neumann triple point paradox.

In this paper, we present solutions of the steady TSD equations for a problem that describes the stationary Mach reflection of weak shocks. The solutions contain a sequence of supersonic patches and triple points similar to, but more resolved than, those obtained in [3]. These results support the conjecture, suggested by theoretical considerations [2, 3], that there is an infinite sequence of triple points in an inviscid weak shock Mach reflection.

In addition, we present two new solutions of the UTSD shock reflection problem at parameter values that are closer to the detachment point than the ones in [3]. The supersonic regions in these solutions are remarkably small.

2. The steady shock reflection problem

We write the steady TSD equations in normalized form as

$$\left(\frac{1}{2}u^2\right)_x + v_y = 0, \quad u_y - v_x = 0, \tag{1}$$

where $u(x,y,t)$, $v(x,y,t)$ are proportional to the x, y fluid velocity components, respectively. Equation (1) is hyperbolic in $u < 0$, elliptic in $u > 0$, and changes type across the sonic line

$$u = 0. \tag{2}$$

The steady shock reflection problem we consider here consists of (1) in the region $0 \leq y \leq 1, -\infty < x < +\infty$ subject to boundary conditions (cf. Fig. 1)

$$u(x,y) = -1, \quad v(x,y) = 0 \qquad \text{if } x > \sigma(y), \tag{3}$$

$$v(x,0) = \widetilde{a} \quad \text{if } x_0 < x < x_1, \qquad u(x,0) = u_0 \quad \text{if } x < x_0, \tag{4}$$

$$v(x,1) = 0. \tag{5}$$

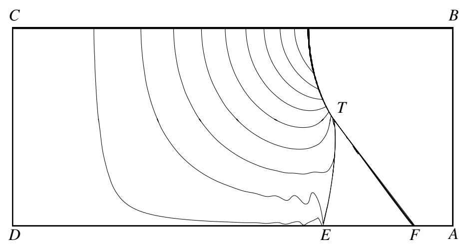


Figure 1. A diagram of the computational domain, showing global u-contours for the solution in Fig. 2(b). The boundary AD corresponds to $y = 0$ and BC to $y = 1$. The point $x = x_0$ lies between E and F, and $x = x_1$ at F. FT is the incident shock generated at the wedge corner F (the flow is from right to left), and T is the triple point.

Here, $\widetilde{a}$ and u_0 are constants, and $x = \sigma(y)$ is the location of the incident and Mach shocks. Physically, this problem corresponds to the reflection of a shock generated by a supersonic jet incident on a wedge located at $x_0 < x < x_1$, $y = 0$ off a rigid wall located at $y = 1$.

3. The numerical method

To solve (1), (3)–(5) numerically, we evolve a solution of the UTSD equation forward in time t until it converges to a steady state, using line relaxation. We write the unsteady TSD equation as

$$\varphi_{xt} + \left(\frac{1}{2}\varphi_x^2\right)_x + \varphi_{yy} = 0, \tag{6}$$

where $\varphi(x, y, t)$ is the velocity potential with $u = \varphi_x$, $v = \varphi_y$. We define a non-uniform grid x_i in the x direction and y_j in the y direction, where $x_{i+1} = x_i + \Delta x_{i+1/2}$ and $y_{j+1} = y_j + \Delta y_{j+1/2}$. We also define $(x_{i-1/2}, x_{i+1/2})$ as the neighborhood of the point x_i, with length $\Delta x_i = \frac{1}{2}(\Delta x_{i-1/2} + \Delta x_{i+1/2})$, where $x_{i+1/2} = \frac{1}{2}(x_{i+1} + x_i)$. Similar definitions apply for the non-uniform grid y_j. We denote an approximate solution of (6) by $\varphi^n_{i,j} \approx \varphi(x_i, y_j, n\Delta t)$, where Δt is a fixed time step, and discretize (6) in time t using

$$\frac{\varphi_x^{n+1} - \varphi_x^n}{\Delta t} + \varphi_{yy}^{n+1} + f(\varphi_x)_r^n = 0, \tag{7}$$

where the flux function f is defined by $f(u) = \frac{1}{2}u^2$. We solve (7) by sweeping from right to left in x, using the spatial discretization

$$\varphi^{n+1}_{i,j} - \Delta x_{i+1/2}\Delta t \left(\frac{\frac{\varphi_{i,j+1}-\varphi_{i,j}}{\Delta y_{j+1/2}} - \frac{\varphi_{i,j}-\varphi_{i,j-1}}{\Delta y_{j-1/2}}}{\Delta y_j} \right)^{n+1} = \varphi^{n+1}_{i+1,j} - \tag{8}$$

$$\varphi^n_{i+1,j} + \varphi^n_{i,j} + \Delta t \left(F(u_{i+1/2,j}, u_{i+3/2,j})^n - F(u_{i-1/2,j}, u_{i+1/2,j})^n \right).$$

Here, F is a numerical flux function, and $u_{i-1/2,j} = \frac{\varphi_{i,j}-\varphi_{i-1,j}}{\Delta x_{i-1/2}}$. We used a second-order flux limiter scheme, with a Lax-Wendroff flux as the higher order flux, and an Engquist-Osher flux as the lower order flux.

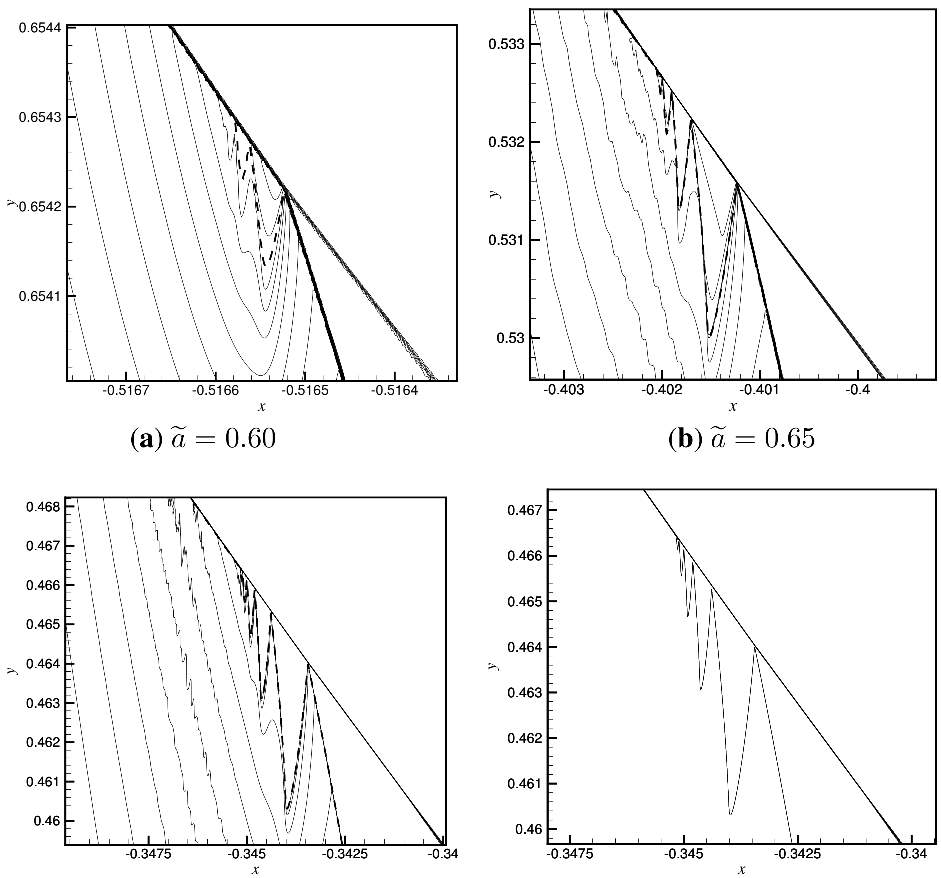
We compute solutions on the finite computational domain illustrated in Fig. 1. We use a grid that is exponentially stretched away from the triple point toward the boundaries. We impose the no-flow condition (5) on BC, and (4) on DF. In our computations we take u_0 equal to the value of u behind the incident shock FT. On the boundary FAB we impose the Dirichlet data (3). Since the flow is supersonic on CD, no boundary condition is required there.

4. Numerical results

Fig. 2(a)–(c) shows u-contour plots for solutions of the steady shock reflection problem (1), (3)–(5) with $\widetilde{a}$ equal to 0.6, 0.65, and 0.67. The dashed line is the sonic line (2), showing that all of the solutions contain a supersonic region directly behind the leading triple point, the size of which increases rapidly with $\widetilde{a}$. The slight thickening of the incident shock visible in Fig. 2(a) and in Fig. 1 is caused by the use of a stretched grid. The nonuniform grids are stretched by amounts between 0.5% and 1.0%, and the number of points in our largest grid is approximately 19×10^6.

In our most refined solution, with $\widetilde{a} = 0.67$, a sequence of triple points formed by the reflection of weak shock and expansion waves between the sonic line and the Mach shock is clearly visible. Fig. 2(d) is a plot of the sonic line alone, which shows the sequence of supersonic patches behind the leading triple point. The number of triple points and supersonic patches in the numerical solution increases with increasing resolution. As in the "transonic controversy" for shock-free flows over an airfoil, the smooth termination of a supersonic patch appears unlikely because of the overdetermination of boundary value problems for hyperbolic PDEs. This argument and the numerical results suggest that there may be an infinite sequence of triple points in the inviscid solution.

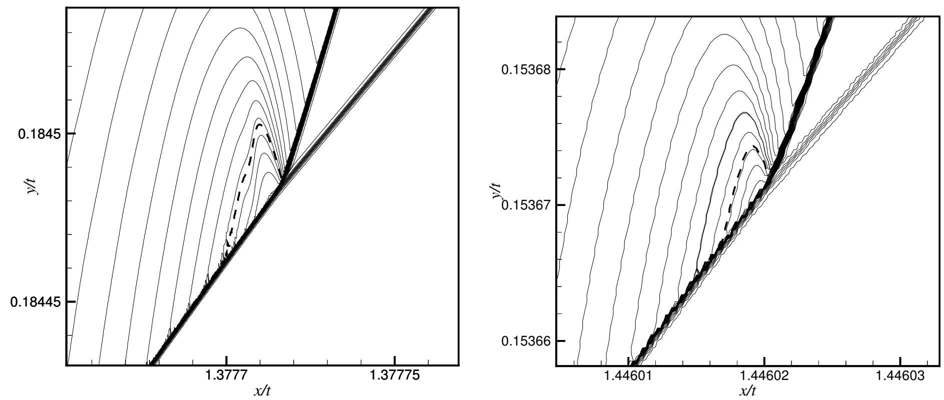
In Fig. 3 we present two new solutions of the UTSD shock reflection problem to augment those in [3]. There, it was shown that the size of the supersonic region decreased rapidly with increasing a, where a is a parameter that measures the inverse shock slope, and the largest value of a used was 0.8. In Fig. 3 plots of u-contours are shown for a equal to 0.85 and 0.9. The supersonic region for $a = 0.9$ is smaller than the region for $a = 0.5$ given in [3] by a linear factor of approximately 1600. The supersonic regions in Fig. 3 are so small that the relative resolutions of the solutions are too low to see the detailed structure of the flows inside the supersonic region.



(**a**) $\widetilde{a} = 0.60$ (**b**) $\widetilde{a} = 0.65$

(**c**) $\widetilde{a} = 0.67$ (**d**) $\widetilde{a} = 0.67$

Figure 2. Contour plots of u near the triple point for increasing values of $\widetilde{a}$ in (a)–(c), and in (d) a plot of the sonic line behind the incident and Mach shocks, for the solution in (c). The u-contour spacing is 0.01 in (a)–(c), and the dotted line is the sonic line. The regions shown contain the refined uniform grids, which have the following numbers of grid points: (a) 880×402; (b) 408×408; (c),(d) 2094×1392.



(**a**) $a = 0.85$ (**b**) $a = 0.9$

Figure 3. Contour plots of u near the triple point for two new values of a, for the unsteady TSD shock reflection problem in [3]. The u-contour spacing is 0.01 in (a)–(b). The dotted line is the sonic line. The regions shown contain the refined uniform grids, which have the following numbers of grid points: (a) 474×216; (b) 460×210.

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