Spherical geometry and Euler’s polyhedral formula

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Euclid was a Greek mathematician in Alexandria around 300 BC. Euclid is often referred to as the Father of Geometry.
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Euclid’s *Elements* is referred to as the most influential work in mathematics.
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Book 13 of the Elements constructs the five regular Platonic solids i.e. the tetrahedron, cube etc.
Euclid’s Postulates

1. A straight line segment can be drawn joining any two points.

2. Any straight line segment can be extended indefinitely in a straight line.

3. Given any straight line segment, a circle can be drawn having the segment as radius and one endpoint as center.

4. All right angles are congruent.

5. Parallel Postulate: In a plane, given a line and a point not on it, at most one line parallel to the given line can be drawn through the point.
Geometry

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Formally, a geometry is defined as a complete locally homogeneous Riemannian manifold (i.e. way to measure distances which is same everywhere).
Parallel Postulate and Non-Euclidean geometries

- “No lines” gives Spherical geometry (positively curved)
- “Infinitely many lines” gives Hyperbolic geometry (negatively curved)
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The possible 2-dimensional geometries are Euclidean, spherical and hyperbolic.

The possible 3-dimensional geometries include Euclidean, hyperbolic, and spherical, but also include five other types.
Set: The sphere $S^2$ is the unit sphere in $\mathbb{R}^3$ i.e.
$S^2 = \{(x, y, z) \in \mathbb{R}^3| \ x^2 + y^2 + z^2 = 1 \}$. A point $P \in S^2$ can be thought of as the unit vector $\vec{OP}$. 
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Lines: A line on the sphere is a great circle i.e. a circle which divides the sphere in half. In other words, a great circle is the intersection of $S^2$ with a plane passing through the origin.
Spherical geometry

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This line is given by the intersection of $S^2$ with the plane passing through the origin and the two given points.
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You can similarly verify the other three Euclid’s postulates for geometry.
Proposition

Let $P, Q \in S^2$ and let $\theta$ be the angle between the vectors $\vec{OP}$ and $\vec{OQ}$. The length of the shorter line segment $PQ$ is $\theta$. 

Remark: In geometry, length of a line segment between two points is the shortest distance between the points.
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Proof: Look at the plane determined by the origin and points \( P \) and \( Q \). The length of an arc of the unit circle which subtends an angle \( \theta \) is \( \theta \).
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Angles: The angle between two lines at an intersection point is the angle between their respective planes.

A region bounded by two lines is called a diangle or lune.

The opposite angles at a vertex, and angles are both the vertices are equal. Opposite diangles bounded by two lines are congruent to each other.
Let $\theta$ be the angle of a diangle. Then the area of diangle is $2\theta$.

Proof: The area of the diangle is proportional to its angle. Since the area of the sphere, which is a pair of diangles, each of angles $\pi$, is $4\pi$, the area of the diangle is $2\theta$.

Alternatively, one can compute this area directly as the area of a surface of revolution of the curve $z = \sqrt{1 - y^2}$ by an angle $\theta$. This area is given by the integral

$$\int_{-1}^{1} z \sqrt{1 + \left(z'\right)^2} \, dy.$$
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A **spherical polygon** is a polygon on $S^2$ whose sides are parts of lines on $S^2$. 

Example: Spherical triangles

**Question:** What are the angles of the green triangle?
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Girard’s Theorem: Area of a spherical triangle

Theorem

The area of a spherical triangle with angles $\alpha$, $\beta$ and $\gamma$ is $\alpha + \beta + \gamma - \pi$. 
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![Diagram of a spherical triangle](image-url)
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Let $R_{AD}$, $R_{BE}$ and $R_{CF}$ denote pairs of diangles as shown. Then $\triangle ABC$ and $\triangle DEF$ each gets counted in every diangle.
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$$\text{Area}(S^2) = \text{Area}(R_{AD}) + \text{Area}(R_{BE}) + \text{Area}(R_{CF}) - 4X$$

$$4\pi = 4\alpha + 4\beta + 4\gamma - 4X$$

$$X = \alpha + \beta + \gamma - \pi$$
Spherical Pythagorean Theorem

In a spherical right angle triangle, let $c$ denote the length of the side opposite to the right angle, and $a, b$ denote the lengths of the other two sides, then

$$\cos a \cos b = \cos c.$$
A prime meridian, based at the Royal Observatory, Greenwich, in London, was established in 1851. Greenwich Mean Time (GMT) is the mean solar time at the Royal Observatory in Greenwich, London. By 1884, over two-thirds of all ships and tonnage used it as the reference meridian on their charts and maps.
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Application 3: Map Projections

The Mercator projection is a cylindrical map projection presented by the cartographer Gerardus Mercator in 1569. It became the standard map projection for nautical purposes.
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The Mercator projection portrays Greenland as larger than Australia; in actuality, Australia is more than three and a half times larger than Greenland.

Google Maps uses a close variant of the Mercator projection, and therefore cannot accurately show areas around the poles.
The Gall-Peters projection, named after James Gall and Arno Peters, is a cylindrical equal-area projection.

It achieved considerable notoriety in the late 20th century as the centerpiece of a controversy surrounding the political implications of map design.
Leonhard Euler was a Swiss mathematician who made enormous contributions to a wide range of fields in mathematics.
A polyhedron is a solid in $\mathbb{R}^3$ whose faces are polygons.
Convex Polyhedron

A polyhedron is a solid in $\mathbb{R}^3$ whose faces are polygons.

A polyhedron $P$ is convex if the line segment joining any two points in $P$ is entirely contained in $P$. 
Euler’s Polyhedral Formula

Euler’s Formula

Let $P$ be a convex polyhedron. Let $v$ be the number of vertices, $e$ be the number of edges and $f$ be the number of faces of $P$. Then $v - e + f = 2$. 
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Examples

- **Tetrahedron**
  - $v = 4$, $e = 6$, $f = 4$

- **Cube**
  - $v = 8$, $e = 12$, $f = 6$

- **Octahedron**
  - $v = 6$, $e = 12$, $f = 8$
Euler’s Polyhedral Formula

Euler mentioned his result in a letter to Goldbach (of Goldbach’s Conjecture fame) in 1750. However Euler did not give the first correct proof of his formula.

It appears to have been the French mathematician Adrian Marie Legendre (1752-1833) who gave the first proof using Spherical Geometry.

Adrien-Marie Legendre (1752-1833)
Corollary

Let $R$ be a spherical polygon with $n$ vertices and $n$ sides with interior angles $\alpha_1, \ldots, \alpha_n$. Then $\text{Area}(R) = \alpha_1 + \ldots + \alpha_n - (n-2)\pi$. 
Corollary

Let $R$ be a spherical polygon with $n$ vertices and $n$ sides with interior angles $\alpha_1, \ldots, \alpha_n$. Then $\text{Area}(R) = \alpha_1 + \ldots + \alpha_n - (n - 2)\pi$.

**Proof:** Any polygon with $n$ sides for $n \geq 4$ can be divided into $n - 2$ triangles.

The result follows as the angles of these triangles add up to the interior angles of the polygon.■
Let $P$ be a convex polyhedron in $\mathbb{R}^3$. We can "blow air" to make (boundary of) $P$ spherical.
Application 4: Proof of Euler’s Polyhedral Formula

Let $P$ be a convex polyhedron in $\mathbb{R}^3$. We can “blow air” to make (boundary of) $P$ spherical.
Let $v, e$ and $f$ denote the number of vertices, edges and faces of $P$ respectively. Let $R_1, \ldots, R_f$ be the spherical polygons on $S^2$.

Since their union is $S^2$, $\text{Area}(R_1) + \ldots + \text{Area}(R_f) = \text{Area}(S^2)$.

Let $n_i$ be the number of edges of $R_i$ and $\alpha_{ij}$ for $j = 1, \ldots, n_i$ be its interior angles.

\[
\sum_{i=1}^{f} \left( \sum_{j=1}^{n_i} \alpha_{ij} - n_i \pi + 2\pi \right) = 4\pi.
\]

\[
\sum_{i=1}^{f} \sum_{j=1}^{n_i} \alpha_{ij} - \sum_{i=1}^{f} n_i \pi + \sum_{i=1}^{f} 2\pi = 4\pi.
\]
Since every edge is shared by two polygons

\[ \sum_{i=1}^{f} n_i \pi = 2\pi e. \]

Since the sum of angles at every vertex is \(2\pi\)

\[ \sum_{i=1}^{f} \sum_{j=1}^{n_i} \alpha_{ij} = 2\pi v. \]

Hence \(2\pi v - 2\pi e + 2\pi f = 4\pi\) that is \(v - e + f = 2.\) \[\blacksquare\]
Why Five?

A **platonic solid** is a polyhedron all of whose vertices have the same degree and all of its faces are congruent to the same regular polygon.

<table>
<thead>
<tr>
<th>Solid</th>
<th>Vertices ($v$)</th>
<th>Edges ($e$)</th>
<th>Faces ($f$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tetrahedron</td>
<td>4</td>
<td>6</td>
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</tr>
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Chapter 13 in Euclid’s Elements proved that there are only **five** platonic solids. Let us see why.

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By Euler’s Theorem, $v - e + f = 2$, we have

\[
\frac{2e}{a} - e + \frac{2e}{b} = 2
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\[
\frac{1}{a} + \frac{1}{b} = \frac{1}{2} + \frac{1}{e} > \frac{1}{2}
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If $a \geq 6$ or $b \geq 6$ then $\frac{1}{a} + \frac{1}{b} \leq \frac{1}{3} + \frac{1}{6} = \frac{1}{2}$. Hence $a < 6$ and $b < 6$ which gives us finitely many cases to check.
<table>
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<th>a</th>
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Thank You
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