Spherical geometry and Euler's polyhedral formula

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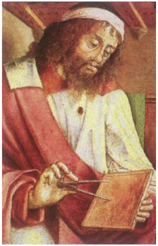
Euclid



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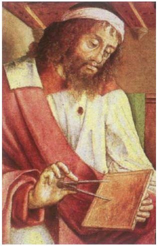

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- Euclid's Elements is referred to as the most influential work in mathematics.

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- Book 13 of the Elements constructs the five regular Platonic solids i.e. the tetrahedron, cube etc.

Euclid's Postulates

1. A straight line segment can be drawn joining any two points.
2. Any straight line segment can be extended indefinitely in a straight line.
3. Given any straight line segment, a circle can be drawn having the segment as radius and one endpoint as center.
4. All right angles are congruent.
5. Parallel Postulate: In a plane, given a line and a point not on it, at most one line parallel to the given line can be drawn through the point.

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- Formally, a geometry is defined as a complete locally homogeneous Riemannian manifold (i.e. way to measure distances which is same everywhere).

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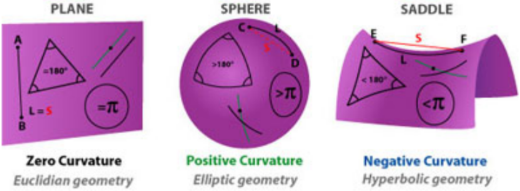

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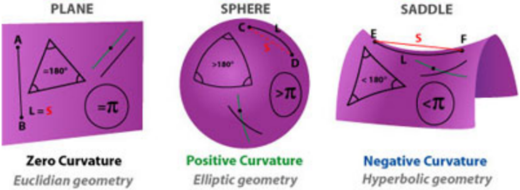

The possible 3-dimensional geometries include Euclidean, hyperbolic, and spherical, but also include five other types.

Spherical geometry

Set: The sphere S^2 is the unit sphere in $\mathbb{R}^3$ i.e. $S^2 = \{(x, y, z) \in \mathbb{R}^3 |\ x^2 + y^2 + z^2 = 1\ \}$. A point $P \in S^2$ can be thought of as the unit vector $\vec{OP}$.

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Lines: A line on the sphere is a great circle i.e. a circle which divides the sphere in half. In other words, a great circle is the interesection of S^2 with a plane passing through the origin.

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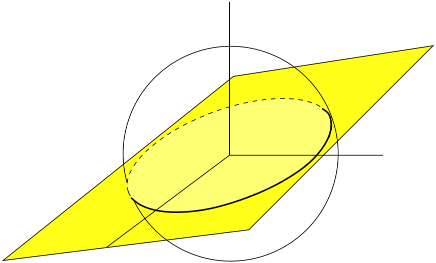
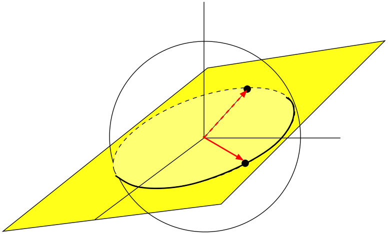


Euclid's first postulate for spherical geometry

Given two distinct points on S^2, there is a line passing through them.



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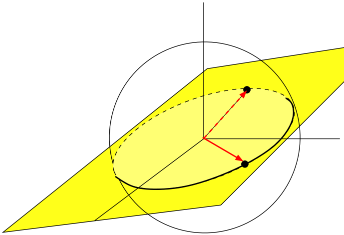

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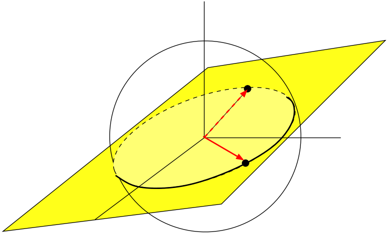

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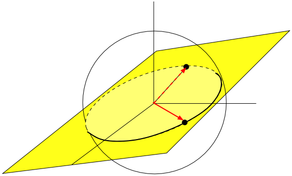
You can similarly verify the other three Euclid's posulates for geometry.

Lengths

Proposition

Let $P,\ Q \in S^2$ and let θ be the angle between the vectors $\vec{OP}$ and $\vec{OQ}$. The length of the shorter line segment PQ is θ.



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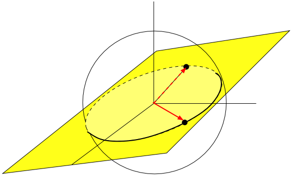

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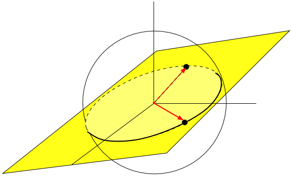

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Remark: In geometry, length of a line segment between two points is the shortest distance between the points.

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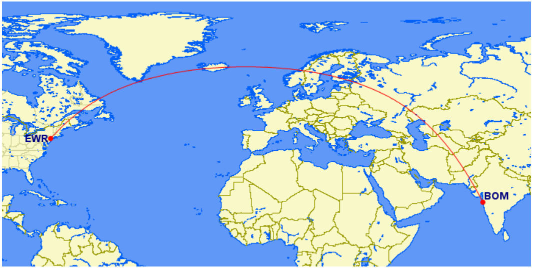

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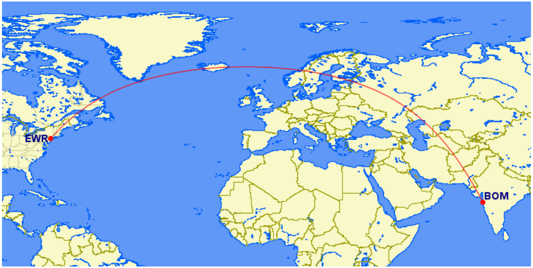
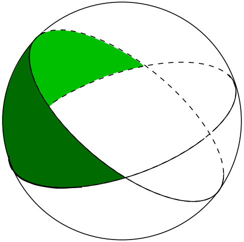


Diangles or lunes

Any two distinct lines intersect in two points which are negatives of each other. Thus this geometry does not satisfy the Parallel postulate.



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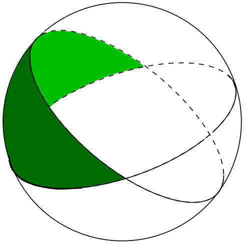

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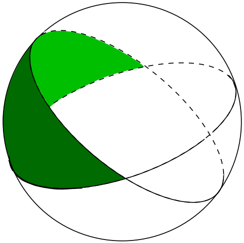

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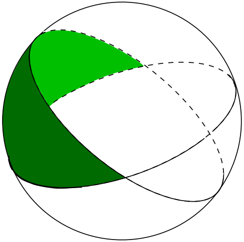

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Alternatively, one can compute this area directly as the area of a surface of revolution of the curve $z = \sqrt{1 - y^2}$ by an angle θ. This area is given by the integral $\int_{-1}^{1} \theta z \sqrt{1 + (z')^2}\, dy$.

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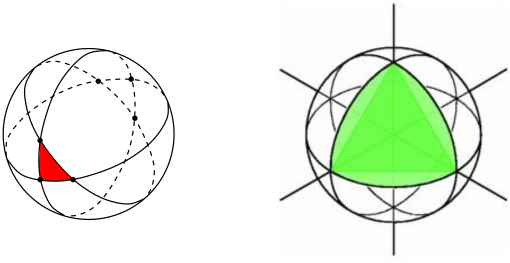
Examples: Spherical triangles



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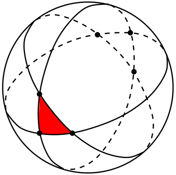

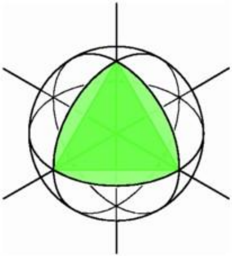

Question: What are the angles of the green triangle ?

Girard's Theorem: Area of a spherical triangle

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The area of a spherical triangle with angles α, β and γ is $\alpha+\beta+\gamma-\pi$.

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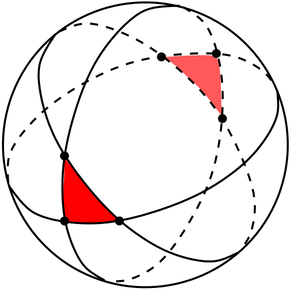

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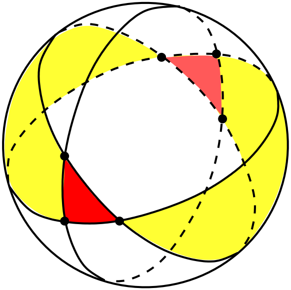

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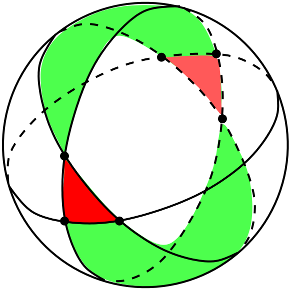

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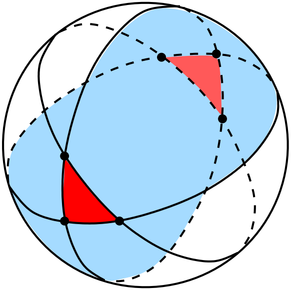

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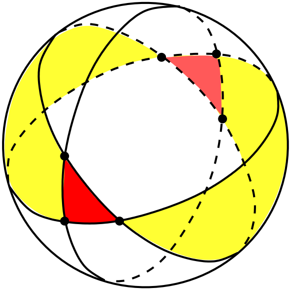

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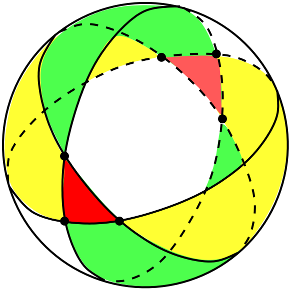

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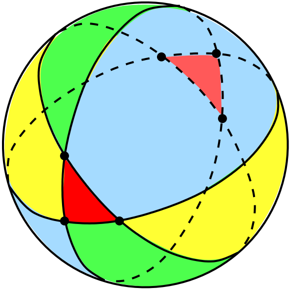

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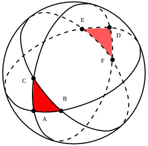
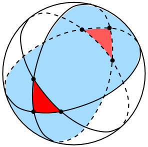
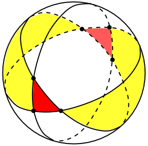


$\triangle ABC$



R_{AD}



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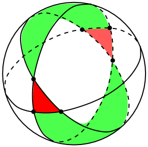
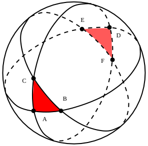


R_{CF}

Let R_{AD}, R_{BE} and R_{CF} denote pairs of diangles as shown. Then $\triangle ABC$ and $\triangle DEF$ each gets counted in every diangle.

Girard's Theorem: Area of a spherical triangle



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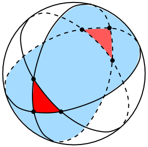

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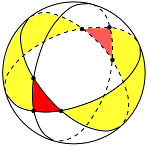

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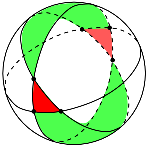

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$R_{AD} \cup R_{BE} \cup R_{CF} = S^2$, $\mathrm{Area}(\triangle ABC) = \mathrm{Area}(\triangle DEF) = X$.

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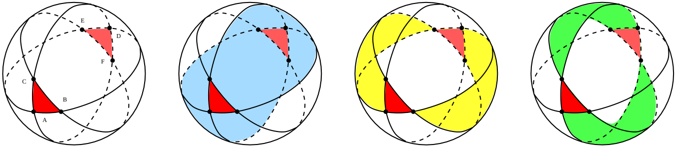

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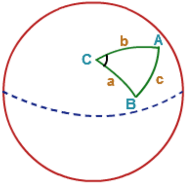
$$\begin{aligned} \mathrm{Area}(S^2) &= \mathrm{Area}(R_{AD}) + \mathrm{Area}(R_{BE}) + \mathrm{Area}(R_{CF}) - 4X \\ 4\pi &= 4\alpha + 4\beta + 4\gamma - 4X \\ X &= \alpha + \beta + \gamma - \pi \end{aligned}$$

■

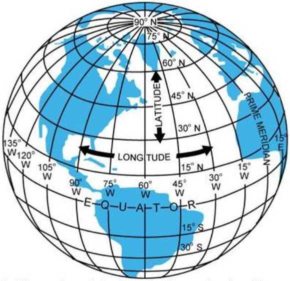
Spherical Pythagorean Theorem



Spherical Pythagorean Theorem In a spherical right angle triangle, let c denote the length of the side opposite to the right angle, and a, b denote the lengths of the other two sides, then

$$\cos a \ \cos b \ = \cos c.$$

Application 2: Navigation



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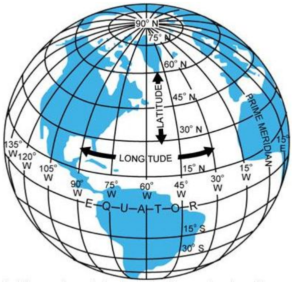

A prime meridian, based at the Royal Observatory, Greenwich, in London, was established in 1851. Greenwich Mean Time (GMT) is the mean solar time at the Royal Observatory in Greenwich, London.

By 1884, over two-thirds of all ships and tonnage used it as the reference meridian on their charts and maps.

Application 3: Map Projections

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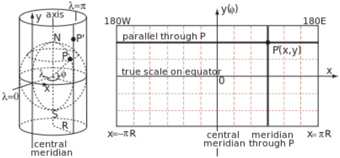

The Mercator projection is a cylindrical map projection presented by the cartographer Gerardus Mercator in 1569. It became the standard map projection for nautical purposes.

Application 3: Map Projections

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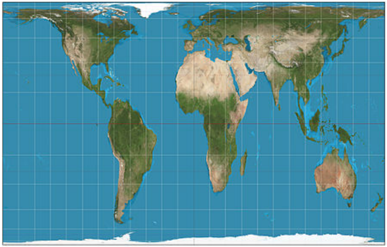
The Mercator projection portrays Greenland as larger than Australia; in actuality, Australia is more than three and a half times larger than Greenland.

Google Maps uses a close variant of the Mercator projection, and therefore cannot accurately show areas around the poles.

Application 3: Map Projections



The Gall-Peters projection, named after James Gall and Arno Peters, is a cylindrical equal-area projection.

It achieved considerable notoriety in the late 20th century as the centerpiece of a controversy surrounding the political implications of map design.

Euler

Leonhard Euler (1707-1783)

Leonhard Euler was a Swiss mathematician who made enormous contibutions to a wide range of fields in mathematics.

Convex Polyhedron

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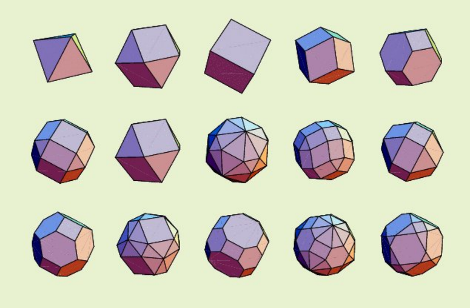
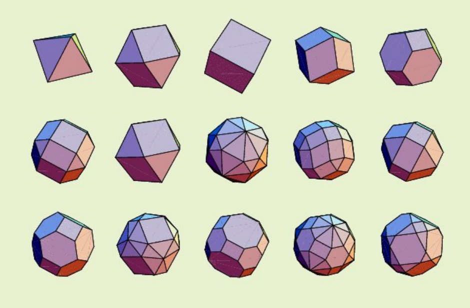
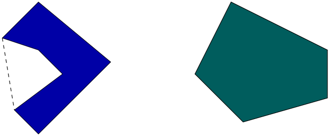


Convex Polyhedron

A polyhedron is a solid in $\mathbb{R}^3$ whose faces are polygons.



A polyhedron P is convex if the line segment joining any two points in P is entirely contained in P.



Euler's Polyhedral Formula

Euler's Formula

Let P be a convex polyhedron. Let v be the number of vertices, e be the number of edges and f be the number of faces of P. Then $v - e + f = 2$.

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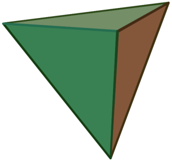
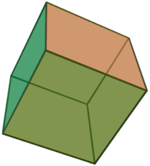
Examples



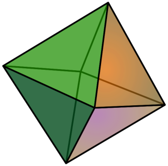
Tetrahedron

$v = 4,\ e = 6,\ f = 4$



Cube

$v = 8,\ e = 12,\ f = 6$



Octahedron

$v = 6,\ e = 12,\ f = 8$

Euler's Polyhedral Formula

Euler mentioned his result in a letter to Goldbach (of Goldbach's Conjecture fame) in 1750. However Euler did not give the first correct proof of his formula.

It appears to have been the French mathematician Adrian Marie Legendre (1752-1833) who gave the first proof using Spherical Geometry.

Adrien-Marie Legendre (1752-1833)

Area of a spherical polygon

Corollary

Let R be a spherical polygon with n vertices and n sides with interior angles $\alpha_1, \ldots, \alpha_n$. Then $\mathrm{Area}(R) = \alpha_1 + \ldots + \alpha_n - (n-2)\pi$.

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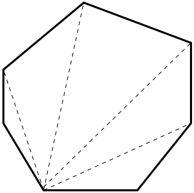
Proof: Any polygon with n sides for $n \geq 4$ can be divided into $n-2$ triangles.



The result follows as the angles of these triangles add up to the interior angles of the polygon. ∎

Application 4: Proof of Euler's Polyhedral Formula

Let P be a convex polyhedron in $\mathbb{R}^3$. We can "blow air" to make (boundary of) P spherical.

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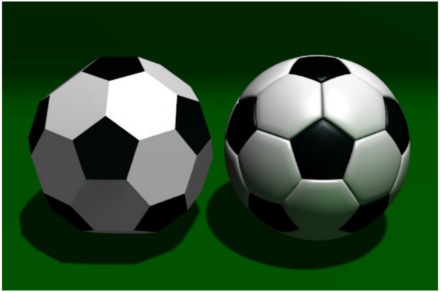

Application 4: Proof of Euler's Polyhedral Formula

Let v, e and f denote the number of vertices, edges and faces of P respectively. Let $R_1, \ldots, R_f$ be the spherical polygons on S^2.

Since their union is S^2, $\mathrm{Area}(R_1) + \ldots + \mathrm{Area}(R_f) = \mathrm{Area}(S^2)$.

Let n_i be the number of edges of R_i and α_{ij} for $j = 1, \ldots, n_i$ be its interior angles.

$$\begin{aligned} \sum_{i=1}^{f}\left(\sum_{j=1}^{n_i} \alpha_{ij} - n_i\pi + 2\pi\right) &= 4\pi. \\ \sum_{i=1}^{f}\sum_{j=1}^{n_i} \alpha_{ij} - \sum_{i=1}^{f} n_i\pi + \sum_{i=1}^{f} 2\pi &= 4\pi. \end{aligned}$$

Application 4: Proof of Euler's Polyhedral Formula

Since every edge is shared by two polygons

$$\sum_{i=1}^{f} n_i \pi = 2\pi e.$$

Since the sum of angles at every vertex is 2π

$$\sum_{i=1}^{f} \sum_{j=1}^{n_i} \alpha_{ij} = 2\pi v.$$

Hence $2\pi v - 2\pi e + 2\pi f = 4\pi$ that is $v - e + f = 2$. ■

Why Five ?

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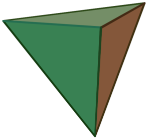
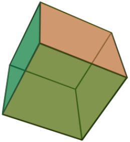
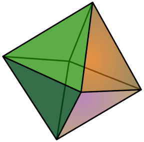
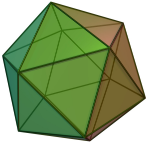
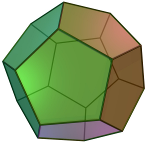
Chapter 13 in Euclid's Elements proved that there are only five platonic solids. Let us see why.

				
Tetrahedron	Cube	Octahedron	Icosahedron	Dodecahedron
$v = 4$ $e = 6$ $f = 4$	$v = 8$ $e = 12$ $f = 6$	$v = 6$ $e = 12$ $f = 8$	$v = 12$ $e = 30$ $f = 20$	$v = 20$ $e = 30$ $f = 12$

Why Five ?

Let P be a platonic solid and suppose the degree of each of its vertex is a and let each of its face be a regular polygon with b sides. Then $2e = af$ and $2e = bf$. Note that $a, b \geq 3$.

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By Euler's Theorem, $v - e + f = 2$, we have

$$\begin{aligned} \frac{2e}{a} - e + \frac{2e}{b} &= 2 \\ \frac{1}{a} + \frac{1}{b} &= \frac{1}{2} + \frac{1}{e} > \frac{1}{2} \end{aligned}$$

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If $a \geq 6$ or $b \geq 6$ then $\frac{1}{a} + \frac{1}{b} \leq \frac{1}{3} + \frac{1}{6} = \frac{1}{2}$. Hence $a < 6$ and $b < 6$ which gives us finitely many cases to check.

Why Five ?

a	b	e	v	Solid
3	3	6	4	Tetrahedron
3	4	12	6	Octahedron
3	5	30	12	Icosahedron
4	3	12	8	Cube
4	4			$\frac{1}{4}+\frac{1}{4}=\frac{1}{2}$!
4	5			$\frac{1}{4}+\frac{1}{5}=\frac{9}{20}<\frac{1}{2}$!
5	3	30	20	Dodecahedron
5	4			$\frac{1}{4}+\frac{1}{5}=\frac{9}{20}<\frac{1}{2}$!
5	5			$\frac{1}{5}+\frac{1}{5}=\frac{2}{5}<\frac{1}{2}$!

Thank You

Credits

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