Dessins d’enfant and Khovanov homology

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Abstract

Dessins d’enfant are graphs embedded in closed surfaces. A quasi-tree of a dessin is a spanning sub-dessin with one face, which we show is equivalent to an ordered chord diagram. We show that for any link diagram $L$, there is an associated dessin whose quasi-trees correspond bijectively to spanning trees of the graph obtained by checkerboard coloring $L$. This correspondence preserves the bigrading used for the spanning tree model of Khovanov homology, whose Euler characteristic is the Jones polynomial of $L$. Thus, Khovanov homology can be expressed entirely in terms of dessins, with generators given by ordered chord diagrams.

1 Introduction

Dessins d’enfant are graphs embedded in closed surfaces. The Jones polynomial of any link can be obtained as a specialization of the Bollobás-Riordan-Tutte polynomial of a dessin obtained from the link diagram [3]. The Jones polynomial also has an expansion in terms of spanning trees of the Tait graph, obtained by checkerboard coloring the link diagram. Moreover, with an appropriate bigrading, these spanning trees generate Khovanov homology, whose bigraded Euler characteristic is the Jones polynomial [2].
We show that there is a one-to-one correspondence between spanning trees of the Tait graph and quasi-trees, which are spanning sub-dessins with one face. We translate the data used to define the bigrading for spanning trees to the language of dessins using ordered chord diagrams. The Khovanov homology results in [2] are then expressed in terms of dessins and ordered chord diagrams. This leads to the question: Do any of the algebraic structures known for chord diagrams carry over to Khovanov homology?

This project was inspired by [4] and [6], and the correspondence we establish implies many of their results.

2 Quasi-trees and spanning trees

Let $D$ be a connected link diagram. A checkerboard coloring of $D$ determines the Tait graph $G$. An edge of $G$ is positive if the shaded regions of its endpoints are joined by $A$-smoothing the corresponding crossing of $D$. Otherwise, the edge is negative. We take either $G$ or its planar dual so that $E_+(G) \geq E_-(G)$.

Let $D$ be the all-$A$ dessin of $D$. Let $V(D)$ be the number of vertices of $D$, which is the number of components in the all-$A$ state of $D$. A sub-dessin $H \subset D$ is called a spanning sub-dessin if $V(H) = V(D)$. Let $F(H)$ be the number of faces of $H$, which is the number of complementary regions in the orientable surface of minimal genus on which $H$ embeds. A quasi-tree $Q$ is a spanning sub-dessin of $D$ with $F(Q) = 1$ (see Definition 3.1 of [4]).

**Theorem 1** Quasi-trees of $D$ are in one-one correspondence with spanning trees of $G$:

$$Q_j \leftrightarrow T_v \quad \text{where} \quad v + j = \frac{V(G) + E_+(G) - V(D)}{2}$$

$Q_j$ denotes a quasi-tree of genus $j$, and $T_v$ denotes a spanning tree with $v$ positive edges.

The proof will use the following lemma. A state $s$ of $D$ is given by $s : \text{Edges}(D) \rightarrow \{A, B\}$. Let $|s|$ denote the number of components in the corresponding smoothing of $D$. In Section 4 of [3], the dessin $D(s)$ was defined such that $V(D(s)) = |s|$. We now define a different correspondence between states of $D$ and dessins:

**Lemma 1** Spanning sub-dessins $H \subset D$ are in one-one correspondence with states $s$ of $D$, such that $s(H)(e) = B$ iff $e \in H$. Thus, $F(H) = |s|$ and $V(H) = V(D)$.

**Proof:** For any state $s$ of $D$, let $D_s$ denote the following link diagram:

$$D_s = \begin{cases} A - \text{smoothing at } e & \text{if } s(e) = A \\ D - \text{crossing at } e & \text{if } s(e) = B \end{cases}$$
Let \(\mathbb{D}_A(D_s)\) and \(\mathbb{D}_B(D_s)\) denote the all-\(A\) and all-\(B\) dessins of \(D_s\), respectively. For any state \(s\) of \(D\), define \(\mathbb{H}(s) = \mathbb{D}_A(D_s)\). It follows that \(s = s(\mathbb{H}(s))\) and \(\mathbb{H} = \mathbb{H}(s(\mathbb{H}))\).

The dessins \(\mathbb{H}(s) = \mathbb{D}_A(D_s)\) and \(\mathbb{D}_B(D_s)\) are dual in the sense of Lemma 4.1 of [3]. By this duality, \(F(\mathbb{H}(s)) = V(\mathbb{D}_B(D_s)) = |s|\). Also, \(V(\mathbb{H}(s)) = V(\mathbb{D}_A(D_s)) = V(\mathbb{D})\).  

The Jordan trail of a connected link diagram is a simple closed curve obtained by smoothing each crossing [5, p. 2]. There is a one-one correspondence between Jordan trails of \(D\) and spanning trees of the Tait graph \(G\) [5, p. 56]. In particular, the Jordan trail of a spanning tree \(T\) bounds a planar neighborhood of \(T\).

**Proof of Theorem 1:** In the table below, let \(\tau, t, \bar{\tau}, \bar{t}\) denote a positive edge in \(T\), a positive edge in \(G - T\), a negative edge in \(T\), and a negative edge in \(G - T\), respectively. The Jordan trail of \(T\) is then given by the smoothings of \(D\) shown in the second row.

<table>
<thead>
<tr>
<th>(\tau)</th>
<th>(t)</th>
<th>(\bar{\tau})</th>
<th>(\bar{t})</th>
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<tbody>
<tr>
<td>(A)</td>
<td>(B)</td>
<td>(B)</td>
<td>(A)</td>
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<tr>
<td>(q)</td>
<td>(Q)</td>
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To prove the numerical claim, for any \(T\) in \(G\), \(v(T) = \#\tau\) and \(E(Q) = \#t + \#\bar{\tau}\).

\[
j = \frac{1 - V(Q) + E(Q)}{2} = \frac{1 - V(\mathbb{D}) + \#t + \#\bar{\tau}}{2}
\]

\[
v + j = \frac{2(\#\tau) + 1 - V(\mathbb{D}) + \#t + \#\bar{\tau}}{2} = \frac{V(G) + E_+(G) - V(\mathbb{D})}{2}
\]

since \(\#\tau + \#\bar{\tau} = E(T) = V(G) - 1\) and \(\#\tau + \#\bar{t} = E_+(G)\).

**3 Quasi-tree complex for Khovanov homology**

To construct the spanning tree chain complex in [2], every spanning tree \(T\) of the Tait graph \(G\) was given a bigrading \((u(T), v(T))\). By Theorem 1, the \(v\)-grading, which is the number of positive edges in \(T\), is determined by the genus of the corresponding quasi-tree \(Q\). The \(u\)-grading, which was defined using activities in the sense of Tutte, also has a quasi-tree analogue in terms of the ordered chord diagram for \(Q\).

If \(D\) has \(n\) ordered crossings, let \(\mathbb{D}\) be given by permutations \((\sigma_0, \sigma_1, \sigma_2)\) of the set \(\{1, \ldots, 2n\}\), such that the \(i\)-th crossing corresponds to half-edges \(\{2i - 1, 2i\}\), which are marked on the components of the all-\(A\) state of \(D\). To be precise, suppose at the crossings of \(D\), the strands are parallel to \(y = x\) or \(y = -x\), then we require that the marks \(2i - 1\) and \(2i\) be in the half-planes \(y > x\) and \(y < x\), respectively. For an example, see Figure 2 in Section 4. We give the components of the all-\(A\) state of \(D\) the admissible
orientation for which outer ones are oriented counterclockwise (see [3]). In this way, every component has a well-defined positive direction.

The orbits of $\sigma_0$ form the vertex set. In particular, $\sigma_0$ is given by noting the half-edge marks when going in the positive direction around the components of the all-A state of $D$. The other permutations are given by $\sigma_1 = \prod_{i=1}^n (2i-1, 2i)$ and $\sigma_2 = \sigma_1 \circ \sigma_0^{-1}$.

Let an ordered chord diagram denote a circle marked with $\{1, \ldots, 2n\}$ in some order, and chords joining all pairs $\{2i-1, 2i\}$.

**Proposition 1** Every quasi-tree $Q$ corresponds to the ordered chord diagram $C_Q$ with consecutive markings in the positive direction given by the permutation:

$$\sigma(i) = \begin{cases} 
\sigma_0(i) & i \notin Q \\
\sigma_2^{-1}(i) & i \in Q 
\end{cases}$$

**Proof:** Since $Q$ is a quasi-tree, $\gamma_Q$ is one simple closed curve. If we choose an orientation on $S(D)$, we can traverse $\gamma_Q$ along successive boundaries of bands and vertex discs, such that we always travel around the boundary of each disc in a positive direction (i.e., the disc is on the left). If a half-edge is not in $Q$, $\gamma_Q$ will pass across it travelling along the boundary of a vertex disc to the next band. If a half-edge is in $Q$, $\gamma_Q$ traverses along one of the edges of its band. On $\gamma_Q$, we mark a half-edge not in $Q$ when $\gamma_Q$ passes across it along the boundary of the vertex disc and we mark a half-edge in $Q$ when we traverse an edge of a band in the direction of the half-edge. If the half-edge $i$ is not in $Q$, travelling along the boundary of a vertex disc, the next half-edge is given by $\sigma_0$. If the half-edge $i$ is in $Q$, traversing the edge of its band to the vertex disc and then along the boundary of that disc, the next half-edge is given by $\sigma_0 \sigma_1 = \sigma_2^{-1}$. For example, see Figure 1.

As $Q$ is a quasi-tree, each of its half-edges must be in the orbit of its single face, while the complementary set of half-edges are met along the boundaries of the vertex discs. Since we mark all half-edges traversing $\gamma_Q$, the chord diagram $C_Q$ parametrizes $\gamma_Q$.

Note that if $D$ is the all-A dessin of a connected link diagram $D$, by the proof of Theorem 1, following $\gamma_Q$ along an edge of $Q$ is given by the $B$-smoothing of that crossing of $D_{s(Q)}$.  

Figure 1: Dessin $D$, quasi-tree $Q = (12)(56)$ and the curve $\gamma_Q$
Therefore, the chord diagram $C_Q$ parametrizes both $\gamma_Q$ and the Jordan trail for $T$, which is the all-$B$ state of $D_{\sigma(Q)}$.

To compute the genus $g(Q)$ from $C_Q$, let $C$ be the sub-chord diagram of chords that correspond to edges in $Q$. Then $g(Q)$ is half the rank of the adjacency matrix of the intersection graph of $C$ [1].

**Definition 1** Using $\min(i, \sigma_1(i))$, there is an induced total order on the chords of $C_Q$. A chord is live if it does not intersect lower-ordered chords, and otherwise it is dead. For any quasi-tree $Q$, an edge $e$ is live or dead when the corresponding chord of $C_Q$ is live or dead.

**Lemma 2** If $T$ corresponds to $Q$, as in Theorem 1, then the $i$-th edge of $D$ is live with respect to $Q$ if and only if the $i$-th edge of $G$ is live with respect to $T$.

**Proof:** In $C_Q$, the $i$-th and $j$-th chords intersect if and only if going around the Jordan trail for $T$ in some direction, we see cyclic permutations of the marks $(2i-1, 2j-1, 2i, 2j)$ or $(2i-1, 2j, 2i, 2j-1)$. Now, $e_i \in \text{cut}(T, e_j)$ or $e_i \in \text{cyc}(T, e_j)$ if and only if the Jordan trail becomes disconnected when the $j$-th smoothing is changed, and is re-connected when the $i$-th smoothing is changed. Equivalently, $C_Q$ becomes disconnected when unzipped along the $j$-th chord, and becomes re-connected when unzipped along the $i$-th chord, which occurs if and only if the $i$-th and $j$-th chords intersect:

\[
\begin{align*}
\text{(a)} & \rightarrow \begin{array}{c}
\text{(b)} \\
\text{(c)}
\end{array} \\
\text{(a)} & \rightarrow \begin{array}{c}
\text{(b)} \\
\text{(c)}
\end{array}
\end{align*}
\]

Therefore, $e_i$ is live with respect to $T$ exactly when the $i$-th chord does not intersect lower-ordered chords. ■

**Definition 2** For any quasi-tree $Q$ of $D$, we define

\[
u(Q) = \# \{\text{live not in } Q\} - \# \{\text{live in } Q\} \quad \text{and} \quad \nu(Q) = -g(Q)
\]

Define $C(D) = \oplus_{u,v} C^{u,v}(D)$, where $C^{u,v}(D) = \mathbb{Z}\langle Q \subset D | u(Q) = u, v(Q) = v \rangle$

**Theorem 2** For a knot diagram $D$, there exists a quasi-tree complex $C(D) = \{C^{u,v}(D), \partial\}$ with $\partial : C^{u,v} \to C^{u-1,v}$ that is a deformation retract of the reduced Khovanov complex. In particular, the reduced Khovanov homology $\tilde{H}^{i,j}(D; \mathbb{Z})$ is given by

\[
\tilde{H}^{i,j}(D; \mathbb{Z}) \cong H^{u,v}_i(C(D); \mathbb{Z})
\]

with the indices related as follows:

\[
u = j - i - w(D) + 1 \quad \text{and} \quad \nu = j/2 - i + (V(D) - c_+(D))/2
\]

where $w(D)$ is the writhe, $c_+(D)$ is the number of positive crossings of $D$, and $V(D)$ is the number of components in the all-$A$ state of $D$, which is the number of vertices of $D$. 5
Proof: The result follows from Theorem 5 of [2] and Theorem 1, once we establish for a spanning tree $T$ corresponding to a quasi-tree $Q$ that their bigradings are related as claimed.

By Lemma 2, edges of $Q$ and $T$ are live exactly when they correspond. From [2], $u(T) = \#L - \#\ell - \#\overline{L} + \#\overline{\ell}$. By the proof of Theorem 1, $\{L, \overline{\ell}\}$-edges of $T$ correspond to live edges not in $Q$, and $\{\ell, \overline{L}\}$-edges of $T$ correspond to live edges in $Q$. Therefore,

$$u(Q) = \#\{\text{live not in } Q\} - \#\{\text{live in } Q\} = u(T) = j - i - w(D) + 1$$

By Theorem 5 of [2] and Theorem 1,

$$v(Q) = -g(Q) = v(T) - \frac{V(G) + E_+(G) - V(D)}{2}$$

$$= \left(\frac{j}{2} - i - \frac{w(D) - k(D) - 2}{4}\right) - \frac{V(G) + E_+(G) - V(D)}{2}$$

$$= \frac{j}{2} - i - \frac{w(D) + E(G)}{4} + \frac{V(D)}{2} = \frac{j}{2} - i + \frac{V(D) - c_+(D)}{2}$$

where we used that $w(D) + E(G) = w(D) + c(D) = 2c_+(D)$. ■

The following bound for the thickness of Khovanov homology in terms of dessin genus was obtained by Manturov [6]:

**Corollary 3** Let $g(D)$ denote the genus of the all-$A$ dessin of $D$. The thickness of the reduced Khovanov homology of $D$ is less than or equal to $g(D) + 1$.

**Proof:** For any quasi-tree $Q$ of $D$, $-g(D) \leq v(Q) \leq 0$. Therefore, the quasi-tree complex $C(D)$ has $g(D) + 1$ rows, so $H^\text{nu}_k(C(D); \mathbb{Z})$ has at most $g(D) + 1$ rows. ■

Corollary 3 is stronger than Theorem 13(ii) of [2] because, for instance, dessin-genus one links are a much richer class than 1-almost alternating links. (For example, see Lemma 4.3 [4].) However, by the correspondence in Theorem 1, the two proofs are the same: Since $0 \leq j \leq g(D)$,

$$g(D) = \max_{T \subseteq G} v(T) - \min_{T \subseteq G} v(T)$$

### 4 An example

As an example, we use a 4-crossing diagram of the trefoil. In Figure 2, we show the diagram $D$, the Tait graph $G$, and the all-$A$ dessin $D$, given by

$$\sigma_0 = (15724863)$$

$$\sigma_1 = (12)(34)(56)(78)$$

$$\sigma_2 = (14)(2835)(67)$$
The ordered chord diagram for each quasi-tree is given by Corollary 1. This order can be seen from the corresponding Jordan trail, which is shown for the quasi-tree $Q_1$ with edges $(12)$ and $(56)$:

Below we show the correspondence between the spanning trees of $G$, the quasi-trees of $D$, and their chord diagrams. The circled numbers on each chord diagram indicate edges in $D - Q$. The activities follow the convention: capital letters for edges in the spanning tree or quasi-tree, bar for negative edges, $L$ or $\ell$ for live, $D$ or $d$ for dead.

<table>
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<th>$Q_1$</th>
<th>$Q_2$</th>
<th>$Q_3$</th>
<th>$Q_4$</th>
<th>$Q_5$</th>
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<td>$LdDd$</td>
<td>$Ld\ell D$</td>
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<td>$LdD\ell\ell d$</td>
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References


