A-POLYNOMIAL AND BLOCH INVARIANTS OF HYPERBOLIC 3-MANIFOLDS

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Abstract. Let N be a complete, orientable, finite-volume, one-cusped hyperbolic 3-manifold with an ideal triangulation. Using combinatorics of the ideal triangulation of N we construct a plane curve in $\mathbb{C}\times\mathbb{C}$ which contains the squares of eigenvalues of $\mathrm{PSL}(2,\mathbb{C})$ representations of the meridian and longitude. We show that the defining polynomial of this curve is related to the $\mathrm{PSL}(2,\mathbb{C})$ A-polynomial and has properties similar to the classical A-polynomial. We further show that a factor of this polynomial, $\overline{A}_0(l,m)$, associated to the discrete, faithful representation of $\pi_1(N)$ in $\mathrm{PSL}(2,\mathbb{C})$ is independent of the ideal triangulation. The Bloch invariant $\beta(N)$ of N is related to the volume and Chern-Simons invariant of N. The variation of Bloch invariant is defined to be the change of $\beta(N)$ under Dehn surgery on N. We relate $\overline{A}_0(l,m)$ to the variation of the Bloch invariant of N. We show that $\overline{A}_0(l,m)$ determines the variation of Bloch invariant in the case when $\overline{A}_0(l,m)$ is a defining equation of a rational curve. We also show that in this case the Bloch invariant reads the symmetry of $\overline{A}_0(l,m)$.

1. Introduction

This paper is divided into 2 parts. In the first part we study a two variable polynomial arising from the combinatorics of ideal triangulations of a cusped hyperbolic 3-manifold. Next we study the variation of Bloch invariants of hyperbolic 3-manifolds and relate it to this polynomial.

Essential surfaces are central to understanding the topology of 3-manifolds. Culler and Shalen [14] studied character varieties of $\mathrm{SL}(2,\mathbb{C})$ representations of 3-manifolds groups and used them to detect essential surfaces in 3-manifolds. The authors of [7] constructed a plane curve using representations of the meridian and longitude in $\mathrm{SL}(2,\mathbb{C})$ and using ideas developed in [14] showed that this plane curve carried topological information about essential surfaces in a 3-manifold with torus boundary. The A-polynomial is the defining equation of this plane curve. The A-polynomial provided a computable tool to study information obtained from $\mathrm{SL}(2,\mathbb{C})$ character varieties. It has many interesting properties and has emerged as an important tool in studying the topology of 3-manifolds. In an analogous manner the $\mathrm{PSL}(2,\mathbb{C})$ character

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variety was studied in [3]. For more details on $\mathrm{SL}(2,\mathbb{C})$ character varieties see [13], [14] and [35] and for A-polynomials see [7], [8], [9], [10].

Ideal triangulations are combinatorial triangulations with vertices at infinity. Ideal triangulations arise naturally in the study of of cusped hyperbolic 3-manifolds and play an important role in their study. Thurston [38] showed that any hyperbolic 3-manifold is obtainable from a cusped one by Dehn surgeries on some of the cusps. Epstein and Penner [20] showed that any cusped hyperbolic 3-manifold can be decomposed into ideal polyhedra. Ideal triangulations arise in practice, e.g. in the program SnapPea [40] written by Jeff Weeks for studying hyperbolic 3-manifolds. Ideal triangulations have been used to study geometric invariants like volume [31], Chern-Simons [26], arithmetic invariants like the invariant trace field [28], [34] and Bloch invariants [30], essential surfaces in hyperbolic 3-manifolds [39], [43] and Dehn surgeries on cusps of hyperbolic 3-manifolds [15].

We construct a two variable polynomial using ideal triangulations as follows. Let N be a one-cusped, complete, orientable, finite-volume hyperbolic 3-manifold having an ideal triangulation $\mathcal{T}$ with n tetrahedra. N is homeomorphic to the interior of a 3-manifold with torus boundary. Using the combinatorics of the triangulation developed in [31] one obtains n gluing equations and 2 completeness equations in the cross-ratio parameters $z_1, \ldots, z_n$ of the ideal tetrahedra, considered as hyperbolic ideal tetrahedra. The complete hyperbolic structure on N is given by parameters $z_1^0, \ldots, z_n^0$ which satisfy the gluing and completeness equations and lie in the upper-half plane. Let

$$\begin{aligned} P(N) &= \{(z_1, \ldots, z_n, t) \in \mathbb{C}^{n+1} : (z_1, \ldots, z_n) \text{ satisfy gluing equations and} \\ &\qquad t\prod_{i=1}^{n} z_i(1-z_i) = 1\} \end{aligned}$$

We call $P(N)$ the *Parameter Space* of N depending on the triangulation $\mathcal{T}$. The coordinate t ensures that the degenerate values of the parameters (i.e.$z_i = 0, 1$) do not occur in $P(N)$. The completeness equations give the square of eigenvalues of the meridian and longitude in the holonomy representation of $\pi_1(N)$ to $\mathrm{PSL}(2,\mathbb{C})$ as rational functions in z_i's. Using this we define a map Hol: $P(N) \rightarrow \mathbb{C} \times \mathbb{C}$ given by $\mathrm{Hol}(\mathbf{z}, t) = (l(\mathbf{z}), m(\mathbf{z}))$. The image of Hol is a curve and we call it the *Holonomy Variety* of N and denote it by $H(N)$. The defining polynomial of $H(N)$ is denoted by $H(l, m)$.

We define the $\mathrm{PSL}(2,\mathbb{C})$ analog of the classical A-polynomial as the defining polynomial of the curve containing the squares of eigenvalues of representations of the meridian and longitude in $\mathrm{PSL}(2,\mathbb{C})$ which extend to representations of $\pi_1(N)$. Let us denote this polynomial by $\overline{A}(l, m)$. We show:

Theorem *$H(l,m)$ divides the* $\mathrm{PSL}(2,\mathbb{C})$ *A-polynomial $\overline{A}(l,m)$.*

Let $P_0(N)$ denote the component of the *Parameter Space* containing the parameter $\mathbf{z} = (z_1^0, \dots z_n^0)$ for the complete structure. Let $H_0(N)$ be the image of $P_0(N)$ under the map Hol and let $H_0(l,m)$ denote the corresponding factor of $H(l,m)$. Since the complete hyperbolic structure can also be described as a discrete, faithful representation in $\mathrm{PSL}(2,\mathbb{C})$, we obtain a similar factor of $\overline{A}(l,m)$ which we denote by $\overline{A_0}(l,m)$. We show:

Theorem *$H_0(l,m) = \overline{A_0}(l,m)$ and hence the polynomial $H_0(l,m)$ is independent of the ideal triangulation of N.*

We will denote $H_0(l,m)$ by $\overline{A_0}(l,m)$ from now on. The Newton polygon of a polynomial $p(x,y) = \sum c_{mn}x^m y^n$ is the convex hull of the points $(m,n) \in \mathbb{Z} \times \mathbb{Z}$ where $c_{mn} \neq 0$. If S is an incompressible surface with non-empty boundary in N then ∂S is a family of simple closed curves on ∂N and hence determine a homology class in $H_1(\partial N;\mathbb{Z})$ given by $p\mathcal{M} + q\mathcal{L}$. The *boundary slope* of S is the rational number p/q. It is proved in [7] that the slopes of the sides of the Newton polygon of the A-polynomial are boundary slopes of essential surfaces in the 3-manifold.

Theorem *The slopes of the sides of the Newton Polygon of $H(l,m)$ are boundary slopes of incompressible surfaces in M which correspond to ideal points of $H(N)$.*

The terms of a two variable polynomial appearing along an edge of its Newton polygon may be viewed as a polynomial in a single variable called an edge polynomial. Another interesting property of the A-polynomial proved in [7] is that its edge polynomials are cyclotomic. Using a K- theoretic argument similar to the one given in [7] we show

Theorem *$H(l,m)$ has cyclotomic edge polynomials.*

It follows from the work of Thurston [38] that hyperbolic geometry is prevalent in 3-manifolds and understanding geometric invariants is an important tool in the study of 3-manifolds. Bloch invariants for hyperbolic 3-manifolds were introduced by Neumann and Yang in [30]. Bloch invariants capture geometric information such as volume, Chern-Simons invariant and scissors congruence information of the hyperbolic 3-manifold. $\mathbb{C} - \{0,1\}$ is the cross ratio parameter space for non-degenerate, ordered hyperbolic, ideal tetrahedra up to isometry. The pre-Bloch group $\mathcal{P}(\mathbb{C})$ is defined as

$$\mathcal{P}(\mathbb{C}) = \mathbb{Z}(\mathbb{C} - \{0,1\})/(\text{5-term relations})$$

where 5-term relations relate the parameters of tetrahedra when a hyperbolic polytope on five ideal vertices is decomposed as two ideal tetrahedra with a common face or as three ideal tetrahedra with a common edge. $\mathcal{P}(\mathbb{C})$ is the orientation sensitive analog of the scissors congruence group of $\mathbb{H}^3$. The analog of the Dehn invariant for scissors congruence is a map

$$\mu : \mathcal{P}(\mathbb{C}) \to \mathbb{C}^* \wedge \mathbb{C}^*$$

defined by $\mu([z]) = 2\ (z \wedge (1-z))$. The Bloch group of $\mathbb{C}$ is defined to be $\ker(\mu)$ and denoted as $\mathcal{B}(\mathbb{C})$. A well known conjecture is

Conjecture (*Bloch Rigidity Conjecture*) *The Bloch group $\mathcal{B}(\mathbb{C})$ is countable.*

Given an ideal triangulation of N with cross-ratio parameters $z_1^0, \ldots, z_n^0$, the Bloch invariant of N, $\beta(N) = \sum [z_i^0]$. In [30], Neumann and Yang proved that $\beta(N) \in \mathcal{B}(\mathbb{C})$ and is independent of the choice of ideal triangulation of N. The Bloch regulator map, $\rho : \mathcal{B}(\mathbb{C}) \to \mathbb{C}/\pi^2\mathbb{Q}$ relates $\beta(N)$ to the volume and Chern-Simons invariant of N.

For $(\mathbf{z}, t) \in P_0(N)$, let $N(\mathbf{z})$ denote the manifold obtained by gluing n ideal tetrahedra with parameters $z_1, \ldots, z_n$ and the same gluing pattern as N. Then the Bloch invariant of $N(\mathbf{z})$ is still defined but belongs to $\mathcal{P}(\mathbb{C})$ in general and belongs to $\mathcal{B}(\mathbb{C})$ when $\mathbf{z}$ corresponds to a (p,q) Dehn surgery. Let $\Delta\beta_N(\mathbf{z}) = \beta(N) - \beta(N(\mathbf{z}))$. We call $\Delta\beta_N$ the *variation* of the Bloch invariant. Δ_N is a function from $P_0(N)$ to $\mathcal{P}(\mathbb{C})$. *Variation* of the volume and Chern-Simons is similarly defined. It is shown implicitly in [31] that $\overline{A_0}(l,m)$ determines the variation of volume of N and implicitly in [42] that $\overline{A_0}(l,m)$ determines the variation of the Chern-Simons invariant of N. A well known conjecture from K-theory states that:

Conjecture (*Ramakrishnan Conjecture*) *The Bloch regulator map ρ is injective.*

In view of Ramakrishnan's Conjecture it is natural to make the following conjecture:

Conjecture $\overline{A}_0(l,m)$ *determines the variation of the Bloch invariant.*

We show:

Theorem *If two one-cusped hyperbolic 3-manifolds have the same* $\overline{A_0}(l,m)$ *then the difference of the variation of their Bloch invariant has image in* $\mathcal{B}(\mathbb{C})$.

Since $\Delta\beta_N$ is defined for values in $P_0(N)$, in view of the Bloch Rigidity Conjecture we expect that the difference in variation is constant and hence determined by $\overline{A_0}(l,m)$. We bypass the Bloch Rigidity Conjecture in the case when $\overline{A_0}(l,m)$ is an equation of a rational curve. We show:

Theorem $\overline{A_0}(l,m)$ *determines the variation of the Bloch invariant if* $\overline{A_0}(l,m)$ *is an equation of a rational curve.*

Let $N(p,q)$ denote the manifold obtained by Dehn filling a (p,q) curve on the torus boundary of N. Symmetries of $\overline{A_0}(l,m)$ translate to symmetries of the Bloch invariant. We show:

Theorem *If* $\overline{A_0}(l,m)$ *is an equation of a rational curve and* $\overline{A_0}(l,m) = \overline{A_0}(l^a m^b, l^c m^d)$ *then* $\beta(N(p,q)) = \beta(N(ap+bq, cp+dq))$.

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2. The PSL A-Polynomial

2.0.1. *Definitions in the* $\mathrm{SL}(2,\mathbb{C})$ *case.* Let N be a 3-manifold and let $G = \pi_1(N)$. Then the $\mathrm{SL}(2,\mathbb{C})$ *representation variety* of N is defined as $R(N) = \mathrm{Hom}(G, \mathrm{SL}(2,\mathbb{C}))$. Since N is a 3-manifold its fundamental group G is finitely presented. Let $G = \langle g_1, \ldots, g_n : r_1, \ldots r_m \rangle$ be a presentation of G. Using this presentation it is easy to see that $R(N) \subset \mathbb{C}^{4n}$ and is a solution of polynomial equations, one for every generator, which makes the determinant of its image equal 1, and four for each relator. Hence $R(N)$ is a complex affine algebraic set. We are interested in conjugacy classes of representations. $\mathrm{SL}(2,\mathbb{C})$ acts on $R(N)$ by conjugation: for any $A \in \mathrm{SL}(2,\mathbb{C})$ and for any representation $\rho \in R(N)$ we can define $A \cdot \rho = i_A \circ \rho$ where i_A is the inner automorphism $X \mapsto AXA^{-1}$. This action is algebraic and the *character variety* $X(N)$ is defined to be the algebro-geometric quotient of this action.

There is a direct way to see the character variety. For each $g \in G$, define $I_g : R(N) \to \mathbb{C}$ by setting $I_g(\rho) = trace(\rho(g))$ for every representation $\rho \in R(N)$. Then $I_g \in \mathbb{C}[R(N)]$, the coordinate ring of $R(N)$. Let the trace ring $T(G)$ be the sub-ring of $\mathbb{C}[R(N)]$ generated by all the functions I_g for $g \in G$. The following is shown in [14]] (see also [35]):

Proposition 2.1. *Suppose that the group G is generated by elements $g_1, \ldots, g_n$. Then the trace ring $T(G)$ is generated by the elements I_V, where V ranges over all elements of the form $g_{i_1} \ldots g_{i_k}$ with $1 \le k \le n$ and $1 \le i_1 < \ldots < i_k \le n$. (Note that this set of generators of $T(G)$ has $2^n - 1$ elements.)*

Using the above proposition we can describe the character variety explicitly. Let $N = 2^n - 1$ and let $V_1, \ldots, V_N$ be words of the above form in some order. Define a map $t : R(N) \to \mathbb{C}^N$ by $t(\rho) = (I_{V_1}(\rho), \ldots, I_{V_N}(\rho))$. Then $X(N)$ is parameterized by the image $t(R(N)) \subset \mathbb{C}^N$ and will be identified with the image $t(R(N))$. The map $t : R(N) \to X(N)$ is algebraic and surjective and it is shown in [14] that the image $t(R(N))$ is an algebraic set. A more elementary proof of the fact that $X(N)$ is an algebraic set is given in [21]. It is also shown in [21] that the trace ring $T(G)$ defined above is generated by the smaller set $\{I_V\}$ where $V \in \{g_i : 1 \le i \le n\} \cup \{g_i g_j : 1 \le i < j \le n\} \cup \{g_i g_j g_k : 1 \le i < j < k \le n\}$. This set contains $n(n^2 + 5)/6$ elements. The above definitions hold for any finitely generated group.

Character varieties are used to detect essential surfaces in 3-manifolds, as described in [14]. An ideal point of a curve C in $X(N)$ gives a valuation on the function field F of C. This valuation gives an action of $\mathrm{SL}(2, F)$ on a tree via the Bass-Serre-Tits theory of trees. This action can be pulled back to an action of $\pi_1(N)$ on a tree. The action of $\pi_1(N)$ on a tree can be used to construct essential surfaces in N using a construction by Stallings-Waldhausen.

Let N be a 3-manifold with torus boundary. Then we can study the representations of $\pi_1(\partial N)$ in $\mathrm{SL}(2, \mathbb{C})$ which extend to representations of $\pi_1(N)$ in $\mathrm{SL}(2, \mathbb{C})$. Let us fix a basis $B = \{L, M\}$ of $\pi_1(\partial N) = \mathbb{Z} \oplus \mathbb{Z}$. The inclusion map $\pi_1(\partial N)$ into $\pi_1(N)$ induces the restriction map $r : X(N) \to X(\partial N)$. Let $\Delta \subset R(\partial N)$ be the subvariety consisting of diagonal representations. Let $p_B : \Delta \to \mathbb{C}^* \times \mathbb{C}^*$ be defined as follows: if ρ is a representation defined by

$$\rho(L) = \begin{pmatrix} l & 0 \\ 0 & l^{-1} \end{pmatrix} \text{ and } \rho(M) = \begin{pmatrix} m & 0 \\ 0 & m^{-1} \end{pmatrix}$$

then $p_B(\rho) = (l, m)$. It follows that p_B is an isomorphism. Also the map $t : R(\partial N) \to X(\partial N)$ defined above restricts to a surjection $t_\Delta : \Delta \to X(\partial N)$ which is generically 2-to-1. We have:

$$\begin{tikzcd} X(N) \arrow[d, "r"'] & & \\ X(\partial N) & \Delta \arrow[l, "t_\Delta"', "2:1"] \arrow[r, "p_B", "1:1"'] & \mathbb{C}^* \times \mathbb{C}^* \end{tikzcd}$$

Let $X'(N)$ be the union of irreducible components Y' of $X(N)$ such that the closure of $r(Y')$ is 1-dimensional. For each component Z' of $X'(N)$ let Z be the curve $t_\Delta^{-1}(\overline{r(Y')}) \subset \Delta$.

Definition 2.1. *The curve D_N is the union of curves Z as Z' varies over all the components of $X'(N)$. The defining polynomial of the closure of the image of D_N in $\mathbb{C}^* \times \mathbb{C}^*$ is called the A-polynomial of N and denoted by $A_N(l,m)$.*

Remark 2.1.

1 *The curve D_N consists of the eigenvalues of the meridian and longitude of representations of $\pi_1(\partial N)$ in $\mathrm{SL}(2,\mathbb{C})$ which extend to $\pi_1(N)$ and is also referred to as the eigenvalue variety of the 3-manifold N.*

2 *A defining polynomial vanishes exactly on the curve and has no repeated irreducible factors. Such a polynomial is unique up to multiplication by non-zero constants and powers of l and m.*

3 *$A_N(l,m)$ depends upon the choice of the basis B of $\pi_1(\partial N)$. If $B_1 = \{L^aM^b, L^cM^d\}$, where det $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = 1$, is a different basis, then $A_N^{B_1}(l,m) \doteq A_N^B(l^am^b, l^cm^b)$ where $'' \doteq ''$ means equality up to factors of m and l.*

(4) *$A_N(l,m) \doteq A_N(1/l, 1/m)$ i.e. $\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ is always a symmetry of $A_N(l,m)$.*

2.0.2. *Definitions in the* $\mathrm{PSL}(2,\mathbb{C})$ *case.* Let us define the $\mathrm{PSL}(2,\mathbb{C})$ counterparts of the above. We will denote the $\mathrm{PSL}(2,\mathbb{C})$ counterparts by an over-line on their respective $\mathrm{SL}(2,\mathbb{C})$ notation. The $\mathrm{PSL}(2,\mathbb{C})$ *representation variety* of N is defined as $\overline{R}(N) = \mathrm{Hom}(G, \mathrm{PSL}(2,\mathbb{C}))$. Since $\mathrm{PSL}(2,\mathbb{C}) \simeq \mathrm{SO}(3,\mathbb{C}) \subset \mathrm{SL}(3,\mathbb{C})$, one can see as before that $\overline{R}(N) \subset \mathbb{C}^{9n}$ and is a complex affine algebraic set. Similar to the $\mathrm{SL}(2,\mathbb{C})$ case, there is a $\mathrm{PSL}(2,\mathbb{C})$ action on $\overline{R}(N)$ by conjugation and the $\mathrm{PSL}(2,\mathbb{C})$ *character variety* $\overline{X}(N)$ is defined to be the algebro-geometric quotient of this action. There is a surjective quotient map $\overline{t} : \overline{R}(N) \to \overline{X}(N)$ which is constant on conjugacy classes of representations. Let us denote $\overline{t}(\overline{\rho})$ by $\chi_{\overline{\rho}}$. The analogous result to Proposition 2.1 is proved in [3]. To describe it let $m = n(n^2+5)/6$ and let $\{y_1, \ldots, y_m\} = \{g_i : 1 \leq i \leq n\} \cup \{g_ig_j : 1 \leq i < j \leq n\} \cup \{g_ig_jg_k : 1 \leq i < j < k \leq n\}$. Let F_n denote the free group on the symbols $x_1, \ldots, x_n$.

Proposition 2.2. *Let $\overline{\rho}$, $\overline{\rho}' \in \overline{R}(N)$. Choose matrices $A_1, \ldots, A_n$, $B_1, \ldots, B_n \in \mathrm{SL}(2,\mathbb{C})$ satisfying $\overline{\rho}(g_i) = \pm A_i$ and $\overline{\rho}'(g_i) = \pm B_i$ for each i. Define $\rho, \rho' \in R(F_n)$ by requiring that $\rho(x_i) = A_i$ and $\rho'(x_i) = B_i$ for each i. Let $y_1, \ldots, y_m$ be the $n(n^2+5)/6$ elements of F_n associated to generators $x_1, \ldots, x_n$ as described above. Then $\chi_{\overline{\rho}} = \chi_{\overline{\rho}'}$ if and only if there is a homomorphism $\epsilon \in \mathrm{Hom}(F_n, \{\pm 1\})$ for which $trace(\rho'(y_j)) = \epsilon(y_j) trace(\rho(y_j))$ for each $j \in \{1, \ldots, m\}$.*

The quotient map $\Phi : \mathrm{SL}(2,\mathbb{C}) \to \mathrm{PSL}(2,\mathbb{C})$ induces an algebraic map $\Phi_* : R(N) \to \overline{R}(N)$. This map is not onto in general. The condition whether or not an element in $\overline{R}(N)$ lifts is well understood. See [3] for more details and examples of N such that Φ_* is not onto.

2.0.3. *Definition of the* $\mathrm{PSL}(2,\mathbb{C})$ *A-polynomial.* Let N be a 3-manifold with torus boundary. Let us define the $\mathrm{PSL}(2,\mathbb{C})$ analog of the A-polynomial. The inclusion of $\pi_1(\partial N)$ into $\pi_1(N)$ induces the restriction map $\overline{r} : \overline{X}(N) \to \overline{X}(\partial N)$. Let $\overline{\Delta} \subset \overline{R}(\partial N)$ be the subvariety consisting of diagonal representations. Let $\overline{p}_B : \overline{\Delta} \to \mathbb{C}^* \times \mathbb{C}^*$ be defined as follows: if $\overline{\rho}$ is a representation defined by

$$\overline{\rho}(L) = \pm \begin{pmatrix} l & 0 \\ 0 & l^{-1} \end{pmatrix} \text{ and } \overline{\rho}(M) = \pm \begin{pmatrix} m & 0 \\ 0 & m^{-1} \end{pmatrix}$$

then $\overline{p}_B(\overline{\rho}) = (l^2, m^2)$. It follows that $\overline{p}_B$ is an isomorphism. The map $\overline{t} : \overline{R}(\partial N) \to \overline{X}(\partial N)$ defined above restricts to a surjection $\overline{t}_\Delta : \overline{\Delta} \to \overline{X}(\partial N)$ which is generically 2-to-1. We have:

$$\begin{tikzcd} \overline{X}(N) \arrow[d, "\overline{r}"'] & & \\ \overline{X}(\partial N) & \overline{\Delta} \arrow[l, "\overline{t}_\Delta", "2:1"'] \arrow[r, "\overline{p}_B", "1:1"'] & \mathbb{C}^* \times \mathbb{C}^* \end{tikzcd}$$

Let $\overline{X}'(N)$ be the union of irreducible components $\overline{Y}'$ of $\overline{X}(N)$ such that the closure of $\overline{r}(\overline{Y}')$ is 1-dimensional. For each component $\overline{Z}'$ of $\overline{X}'(N)$ let $\overline{Z}$ be the curve $\overline{t}_\Delta^{-1}(\overline{r}(\overline{Y}')) \subset \overline{\Delta}$.

Definition 2.2. *The curve $\overline{D}_N$ is the union of curves $\overline{Z}$ as $\overline{Z}'$ varies over all the components of $\overline{X}'(N)$. The defining polynomial of the closure of the image of $\overline{D}_N$ in $\mathbb{C}^* \times \mathbb{C}^*$ is called the* $\mathrm{PSL}(2,\mathbb{C})$ *A-polynomial of N and denoted by $\overline{A}_N(l,m)$.*

Remark 2.2. *$\overline{A}(l,m)$ defines the curve given by the squares of the eigenvalues of the meridian and longitude of representations in* $\mathrm{PSL}(2,\mathbb{C})$ *of $\pi_1(\partial N)$ which extend to $\pi_1(N)$.*

We will be interested in the case when N is a complete, orientable, finite volume one-cusped hyperbolic 3-manifold. The cusp on N is homeomorphic to $T^2 \times [0,\infty)$ and N is homeomorphic to the interior of a 3-manifold with torus boundary. Although N is non-compact by ∂N we will mean the torus $T^2 \times 0$. The universal cover of N is the hyperbolic 3-space $\mathbb{H}^3$. The covering translations are isometries of $\mathbb{H}^3$ and hence lie in the group $\mathrm{PSL}(2,\mathbb{C})$. This gives us a representation of $\pi_1(N)$ into $\mathrm{PSL}(2,\mathbb{C})$ which is faithful and discrete. It is often referred to as the representation associated to the hyperbolic structure of N and is denoted by $\overline{\rho}_0$. It follows from Mostow Rigidity that any other discrete faithful representation is conjugate to $\overline{\rho}_0$. It is a theorem of Thurston that $\overline{\rho}_0$ lifts to a representation ρ_0 into $\mathrm{SL}(2,\mathbb{C})$. We will denote the irreducible component of $X(N)$ containing the character of ρ_0 by $X_0(N)$. Similarly we will denote the irreducible component of $\overline{R}(N)$ containing $\overline{\rho_0}$ by $\overline{R}_0(N)$ and the image $\overline{t}(\overline{R}_0(N))$ in $\overline{X}(N)$ by $\overline{X}_0(N)$. The corresponding curves and factors of the $\mathrm{SL}(2,\mathbb{C})$ and $\mathrm{PSL}(2,\mathbb{C})$ A-polynomials are also denoted by the subscript "$0''$.

Let us make an observation about the relation between the $\mathrm{SL}(2,\mathbb{C})$ and $\mathrm{PSL}(2,\mathbb{C})$ A_0-polynomials. The projection $\pi : \mathrm{SL}(2,\mathbb{C}) \to \mathrm{PSL}(2,\mathbb{C})$ induces the map $\pi : X_0(N) \to \overline{X}_0(N)$. Culler [12] showed that this map is surjective. Let $h : \mathbb{C}\times\mathbb{C} \to \mathbb{C}\times\mathbb{C}$ be defined by $h(x,y) = (x^2,y^2)$. Then

$$\begin{tikzcd}
X_0(N) \arrow[r, "r"] \arrow[d, "\pi"'] & X(\partial N) \arrow[d, "\pi"'] & \mathbb{C}^*\times\mathbb{C}^* \arrow[l, "p_B^{-1}\circ t_\Delta"', "2:1"] \arrow[d, "h"'] \\
\overline{X}_0(N) \arrow[r, "\overline{r}"] & \overline{X}(\partial N) & \mathbb{C}^*\times\mathbb{C}^* \arrow[l, "\overline{p_B}^{-1}\circ\overline{t}_\Delta"', "2:1"]
\end{tikzcd}$$

Let D_0 and $\overline{D_0}$ denote the curves defined by $A_0(l,m)$ and $\overline{A}_0(l,m)$ respectively. From the above diagram it follows that $h(D_0) = \overline{D}_0$. So we get

$$h^{-1}(\overline{D}_0) = D_0^{+,+} \cup D_0^{+,-} \cup D_0^{-,+} \cup D_0^{-,-}$$

where $D_0^{\pm,\pm} =$ *curve given by* $A_0(\pm l, \pm m)$. Hence

$$\overline{A}_0(l^2,m^2) | A_0(l,m)A_0(l,-m)A_0(-l,m)A_0(-l,-m)$$

It is shown in [7] that a homomorphism $\rho : \pi_1(N) \to \mathbb{Z}_2 = \{\pm 1\} \in \mathbb{C}$ which restricts non trivially to $\pi_1(\partial N)$ induces an involution on the $\mathrm{SL}(2,\mathbb{C})$ *A*-polynomial. For example in knot complements the Hurewicz homomorphism composed with the quotient map from $\mathbb{Z}$ to $\mathbb{Z}_2$ maps the standard meridian to -1 and standard longitude to 1 giving $A_0(l,m) = A_0(l,-m)$. In this case $\overline{A}_0(l^2,m^2) = A_0(l,m)A_0(-l,m)$. This relation or a similar one (e.g. $\overline{A}_0(l^2,m^2) = A_0(l,m)A_0(-l,-m)$ for m11) holds for many manifolds in the SnapPea's [40] census of cusped hyperbolic 3-manifolds. For the manifold $m208$ we get all the 4 factors. The curve $\overline{X}_0(N)$ is also studied in [24].

3. Combinatorics of ideal triangulations

3.0.4. *Ideal Tetrahedron.* An ideal tetrahedron is a geodesic hyperbolic tetrahedron with all its vertices on the sphere at infinity of the hyperbolic 3-space. Let $\mathbb{H}^3 = \{(x,y,z) \in \mathbb{R}^3 : z > 0\}$ denote the upper-half space model of the hyperbolic 3-space. The group of orientation preserving isometries of $\mathbb{H}^3$ is identified with the group $\mathrm{PSL}(2,\mathbb{C})$. The sphere (or boundary) at infinity of

$\mathbb{H}^3$ denoted by $\partial\mathbb{H}^3$ is the one point compactification of the x-y plane and is identified with $\mathbb{C} \cup \infty = \mathbb{CP}^1$. A horosphere centered at a point $p \in \partial\mathbb{H}^3$ is a hypersurface in $\mathbb{H}^3$ such that all geodesics with one endpoint at p are orthogonal to it. The horosphere inherits an Euclidean metric from $\mathbb{H}^3$. For instance, the horospheres at infinity in the upper half space model are the planes $\{(x, y, z) : z = \text{constant} > 0\}$. The Euclidean metrics on the horosphere obtained by different values of z are constant multiples of each other. See [2] or [33] for more details.

Given an oriented ideal tetrahedron T in $\mathbb{H}^3$, the horosphere centered at any of its vertices cuts out an Euclidean triangle which is well defined up to similarity. Similarity classes of Euclidean triangles are parameterized by the complex upper-half plane by arranging any Euclidean triangle to have vertices $0,\ 1$ and z, where z is in the upper-half plane, using Euclidean similarities (i.e. rotations, translations and dilations). The numbers $z,\ 1 - (1/z)$ and $1/(1 - z)$ give the same triangle depending on which vertices go to 0 and 1. Moreover using the $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ symmetry of the ideal tetrahedron we can see that opposite edges have the same dihedral angles and hence the Euclidean triangles cut by horospheres centered at each vertex are similar.

Hence an oriented ideal tetrahedron is described completely up to (oriented) hyperbolic isometry by a single complex number z in the upper half plane. The complex number $1/z$ describes the same tetrahedron with opposite orientation. To specify z uniquely we must pick an edge of T, the dihedral angle at this edge will be $\arg(z)$. To each edge of T is associated one of the numbers $z,\ 1 - (1/z)$ and $1/(1 - z)$, called the modulus of the edge, with opposite edges having the same modulus (see Figure 1). Once we fix an edge of the tetrahedron we will write $T = T(z)$. Observe that the modulus of the edges are of the form $\pm z^{\epsilon_1}(1 - z)^{\epsilon_2}$ where $\epsilon_i \in \{-1, 0, 1\}$.

In the upper-half space model of $\mathbb{H}^3$, the vertices of an ideal tetrahedron are extended complex numbers, say $z_1,\ z_2,\ z_3,\ z_4$. The modulus of the edge with ideal vertices z_1 and z_2 can also be obtained as the cross-ratio $[z_1 : z_2 : z_3 : z_4]$ of the vertices, where the cross-ratio is defined as:

$$[z_1 : z_2 : z_3 : z_4] = \frac{(z_3 - z_2)(z_4 - z_1)}{(z_3 - z_1)(z_4 - z_2)}$$

Hence the edge moduli are also referred to as cross-ratio parameters. An ideal tetrahedron is said to be *flat* if the cross-ratio has imaginary part 0 and *degenerate* if two or more of its vertices are identified. In the later case the cross-ratio parameters equal $0,\ 1$ or ∞. In all other cases the ideal tetrahedra is said to be *non-degenerate.*

The volume of an ideal tetrahedron is a function of its cross-ratio parameter.

$$\text{vol}(T(z)) = D_2(z)$$

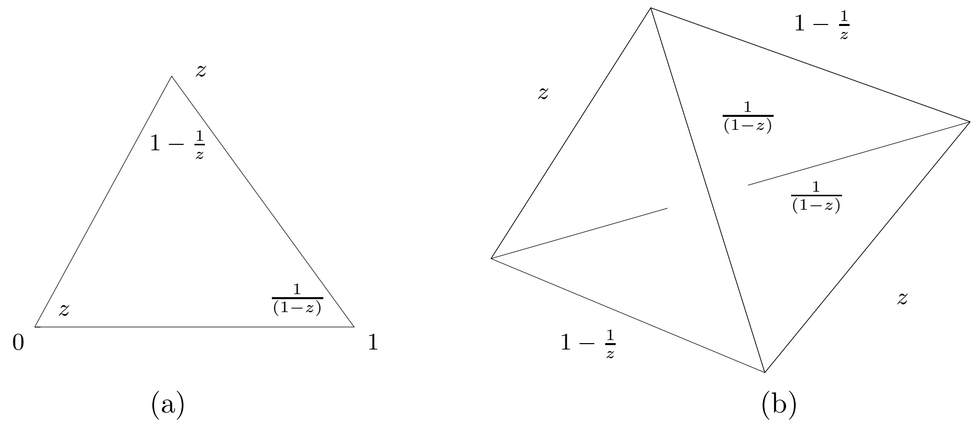

FIGURE 1. (a) Euclidean triangle cut out by horosphere. (b) Ideal tetrahedron with edge moduli

where $D_2(z)$ is the Bloch-Wigner dilogarithm defined as:

$$D_2(z) = \mathrm{Im}\ln_2(z) + \log|z|\arg(z),\ z \in \mathbb{C} - \{0,1\}$$

where $\ln_2(z)$ is the classical dilogarithm function defined as:

$$\ln_2(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2} \quad (|z| \leq 1) \tag{3.1}$$

3.0.5. *Ideal Triangulations.* An ideal triangulation of a manifold N is a cell complex X formed by gluing tetrahedra along faces in which the link of every vertex is a torus and N is homeomorphic to $X - X^{(0)}$. The vertices of the ideal triangulation can be visualized to be at infinity with torus cusps. Let N be a cusped hyperbolic 3-manifold with an ideal triangulation $\mathcal{T}$ with n ideal tetrahedra $\Delta_1, \ldots, \Delta_n$. N is homeomorphic to the interior of a compact 3-manifold with torus boundary. Since N has Euler characteristic zero the number of edges of N is equal to the number of tetrahedra. We number the tetrahedra by an index i and the edges by an index j where both i and j run from 1 to n. Assume that each Δ_i is an hyperbolic ideal tetrahedron and let $\Delta_i = \Delta(z_i)$.

At every edge of N the tetrahedra abutting the edge close up as one goes around the edge. Since the edge moduli at any edge look like $\pm z^{\epsilon_1}(1-z)^{\epsilon_2}$ for $\epsilon_i \in \{-1, 0, 1\}$, the edge moduli of these tetrahedra satisfy a gluing condition of the form

$$\prod_{i=1}^{n} z^{r'_{ji}} (1-z)^{r''_{ji}} \;= \pm 1 \quad (j = 1, 2, \ldots n) \tag{3.2}$$

These are called the *gluing equations*. The exponents r'_{ij} and r''_{ij} can be integers other than ± 1 as more than one edge of a single tetrahedron can be identified. Any solution to the above equations ensures that the hyperbolic metric is well defined around an edge and hence gives a hyperbolic metric on N which is in general incomplete.

Each cusp of N is homeomorphic to $T^2 \times [0, \infty)$. The cusp torus inherits a triangulation by triangles cut out by the horoshperes centered at vertices of the ideal tetrahedra. Since the Euclidean metric on the horospheres is defined only up to scale, these triangles are well defined only up to Euclidean similarities. Hence the cusp torus gets a similarity structure from its triangulation. This similarity structure can be realized as a holonomy representation of $\pi_1(\partial N)$ into the group of Euclidean similarities which is isomorphic to $\mathrm{Aff}(\mathbb{C}) = \{az + b : a \neq 0,\ b \in \mathbb{C}\}$. If we fix a basis $\{L_k,\ M_k\}$ of the k-th cusp torus then the derivative of the image of the basis in the holonomy representation, which we denote by l_k and m_k, can be written in terms of the cross-ratio parameters in the following way:

$$\begin{aligned} l_k = &\quad \pm \prod_{i=1}^n z^{l'_{ki}} (1-z)^{l''_{ki}} \\ m_k = &\quad \pm \prod_{i=1}^n z^{m'_{ki}} (1-z)^{m''_{ki}} \quad k = 1, \dots h \end{aligned} \tag{3.3}$$

For N to be complete the similarity structure on every cusp should be Euclidean. This means that $m_k = l_k = 1$ at every cusp. This condition gives two more equations at every cusp called *the completeness equations.* A solution $\mathbf{z}^0 = (z_1^0, \dots, z_n^0)$ to Equations 3.2 to 3.3 with the condition that each z_i^0 is in the upper-half plane or each z_i^0 is in the lower half plane, gives a complete, finite volume hyperbolic structure on N. We will refer to *the parameter for the complete hyperbolic structure on* N to mean the parameter $\mathbf{z}^0$ with each z_i in the upper-half plane. We call an ideal triangulation *geometric* if the gluing and completeness equations have a solution with each z_i in the upper half plane. A geometric ideal triangulation gives a hyperbolic structure on N. In this paper by an ideal triangulation we mean a geometric ideal triangulation. It follows from Mostow rigidity that the set of solutions to the above equations which give the parameters for the complete structure is a discrete set.

Let

$$R = \begin{pmatrix} r'_{11} & \dots & r'_{1n} & r''_{11} & \dots & r''_{1n} \\ \vdots & & \vdots & \vdots & & \vdots \\ r'_{n1} & \dots & r'_{nn} & r''_{n1} & \dots & r''_{nn} \end{pmatrix} \tag{3.4}$$

be the $n \times 2n$ matrix consisting of the powers $r'_{ji},\ r''_{ji}$. This encodes the gluing conditions. Similarly we have a $h \times 2n$ matrix L consisting of the powers $l'_{ki},\ l''_{ki}$ and $h \times 2n$ matrix M consisting of the powers $m'_{ki},\ m''_{ki}$ which encode the completeness equations. The matrix

$$U = \begin{pmatrix} L \\ M \\ R \end{pmatrix}_{(n+2h, 2n)} \tag{3.5}$$

is called the *gluing matrix* for the manifold N. See [31] for more details on the combinatorics of triangulation and properties of the gluing matrix U.

3.0.6. *Parameter space and Holonomies.* Let N be a hyperbolic 3-manifold with an ideal triangulation $\mathcal{T}$. Let $\mathbf{z}^0 = (z_i^0, \ldots, z_n^0)$ be the parameter for the complete hyperbolic structure on N.

Definition 3.1. *Let*

$P^{\mathcal{T}}(N) = \{(\mathbf{z}, t) \in \mathbb{C}^n \times \mathbb{C} : \mathbf{z}$ satisfies Equation 3.2 and $t\Pi_{i=1}^n z_i(1-z_i) = 1\}$

We call $P^{\mathcal{T}}(N)$ the parameter space of N with respect to the triangulation $\mathcal{T}$. Let $P_0^{\mathcal{T}}(N)$ denote the component of $P(N)$ which contains $\mathbf{z}^0$.

Remark 3.1.

(1) *$P^{\mathcal{T}}(N)$ can be seen as the space of all hyperbolic structures on N induced by the triangulation $\mathcal{T}$.*

(2) *We drop the superscript $\mathcal{T}$ when the context is clear. We will show in Section 3 that $P_0^{\mathcal{T}}(N)$ is independent of the ideal triangulation.*

(3) *The coordinate t ensures that the parameters do not degenerate to* 0 *or* 1. *We will drop the coordinate t in later discussions. $\mathbf{z} \in P(N)$ will mean $(\mathbf{z}, t)$ with t defined with the above equation.*

For $\mathbf{z} \in P(N)$, let $N(\mathbf{z})$ denote the manifold obtained from gluing n tetrahedra with parameters $z_1, \ldots, z_n$ and with the same gluing pattern as N. N has a hyperbolic metric which is in general incomplete and is complete when $\mathbf{z} = \mathbf{z}^0$. The volume of N is well defined and is the sum of the signed volumes of the individual tetrahedra, the sign being positive if z_i is in the upper-half space and negative if z_i is in the lower-half space. It is shown in [31] and [38] that $P_0(N)$ is smooth at the point $\mathbf{z}^0$ and has dimension h. Let $Def(N)$ be a small neighborhood of z^0 contained in $P_0(N)$ and call it the deformation space. Also let

$$\Phi : Def(N) \to \mathbb{C}^h, \text{ be defined by } \Phi(\mathbf{z}) = (\log(m_1(\mathbf{z})), \ldots, \log(m_h(\mathbf{z})))$$

Φ maps $Def(N) \subset P_0(N)$ biholomorphically to a neighborhood of $0 \in \mathbb{C}^h$. Let $\mathcal{D}$ be a neighborhood of $0 \in \mathbb{C}^h$ onto which $Def(N)$ is mapped by Φ. For $\mathbf{z} \in Def(N)$, let $\overline{N}(\mathbf{z})$ denote the completion of $N(\mathbf{z})$ with the hyperbolic metric given by the parameter value $\mathbf{z}$. $\overline{N}(\mathbf{z})$ differs from N topologically by the addition of a set γ_k of limit points at the k-th cusp. γ_k can be empty, a point or a circle. When γ_k is empty the cusp is left unsurgered, when γ_k is a point $N(\mathbf{z})$ is not a manifold and when γ_k is a circle $\overline{N}(\mathbf{z})$ is homeomorphic to a manifold obtained by doing a topological Dehn surgery on the k-th cusp. In the last case either the hyperbolic metric is singular along γ_k or $\overline{N}(\mathbf{z})$ is a complete hyperbolic 3-manifold.

In case $\overline{N}(\mathbf{z})$ is a complete hyperbolic 3-manifold then there exist co-prime integers p_k and q_k such that $p_k \log(m_k(\mathbf{z})) + q_k \log(l_k(\mathbf{z})) = 2\pi i$ and $\overline{N}(\mathbf{z})$ is obtained by (p_k, q_k)-Dehn surgery on the k-th cusp of N. We let $(p_k, q_k) = \infty$ if the cusp is left unsurgered. $m(\mathbf{z}) = l(\mathbf{z}) = 1$ if the cusp is left unsurgered and $m(\mathbf{z})^p l(\mathbf{z})^q = 1$ if the cusp is surgered along a (p, q) curve on the k-th cusp torus on N.

Let us define a *degree one ideal triangulation* for a hyperbolic 3-manifold. Gluing finitely many tetrahedra by identifying all the 2-faces in pairs gives a cellular complex X which is a manifold except possibly at the vertices. If the complement of the "bad" points is oriented then X is called a *geometric 3-cycle*. In this case the complement $X - X^{(0)}$ of the vertices is an oriented manifold. Let N be a hyperbolic 3-manifold. A *degree one ideal triangulation* of N consists of a geometric 3-cycle X and a map $f : X - X^{(0)} \to N$ such that:

(1) f is degree one almost everywhere in N.
(2) For each tetrahedron S of X there is a map f_S to an ideal tetrahedron in $\mathbb{H}^3 \cup \mathbb{CP}^1$, bijective on vertices, such that $f|_{S-S^{(0)}} : S - S^{(0)} \to N$ is the composition $\pi \circ f_S|_{S-S^{(0)}}$, where π is the projection $\pi : \mathbb{H}^3 \to N$.

Degree one ideal triangulations are much more general than what we will need. If M is a hyperbolic 3-manifold obtained from Dehn surgery on some of the cusps of a cusped manifold N then the ideal triangulation of N deforms to a degree one ideal triangulation of M.

Let N be a one-cusped hyperbolic 3-manifold ideally triangulated with n tetrahedra. Let $l = l_1$ and $m = m_1$. Define

$$\mathrm{Hol} : P(N) \to \mathbb{C} \times \mathbb{C}, \text{ by } \mathrm{Hol}(\mathbf{z}) = (l(\mathbf{z}), m(\mathbf{z}))$$

Hol (for holonomy) is a rational function on $P(N)$. Let $Z = \cup Y_i$ where Y_i is a component of $P(N)$ whose closure of the image under Hol is a curve in $\mathbb{C} \times \mathbb{C}$. We define

Definition 3.2. *The image* $\mathrm{Hol}(Z)$ *is called the Holonomy variety with respect to the triangulation* $\mathcal{T}$ *and denoted by* $H^{\mathcal{T}}(N)$. *The defining polynomial of the closure of* $H^{\mathcal{T}}(N)$ *is denoted by* $H^{\mathcal{T}}(l, m)$. *Let* $H_0^{\mathcal{T}}(N)$ *(respectively* $H_0^{\mathcal{T}}(l, m)$*) denote the image* $\mathrm{Hol}(P_0^{\mathcal{T}}(N))$ *(respectively factor of* $H^{\mathcal{T}}(l, m)$*).*

Remark 3.2.

(1) *We drop the superscript* $\mathcal{T}$ *when the context is clear. We will show in Section 3 that* $H_0^{\mathcal{T}}(N)$ *is independent of the ideal triangulation.*
(2) $H(l, m) \doteq H(1/l, 1/m)$ *i.e.* $\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ *is always a symmetry of* $H(l, m)$.

Let us describe a generalization of $H(l, m)$ to more than one cusps. Let N be a hyperbolic 3-manifold with $h > 1$ cusps. We can define the holonomy map $\mathrm{Hol} : P(N) \to \mathbb{C}^{2h}$ by $\mathrm{Hol}(\mathbf{z}) = (l_1(\mathbf{z}), m_1(\mathbf{z}), \ldots, l_h(\mathbf{z}), m_h(\mathbf{z}))$. We know that $P_0(N)$ has dimension h. Let $Q_k \subset P_0(N)$ be the subset consisting of parameters corresponding to leaving all but the k-th cusp unsurgered. Q_k is obtained by adding $h-1$ completeness equations, one for each unsurgered cusp, to the gluing equations. Hence Q_k is a curve in $P_0(N)$. Let $p_k : \mathbb{C}^{2h} \to \mathbb{C}^2$ be the map $p_k(z_1, w_1, \ldots, z_h, w_h) = (z_k, w_k)$. Then $p_k \circ \mathrm{Hol}(Q_k)$ is a curve in $\mathbb{C}^2$. We call this curve the *k-th Holonomy variety of* N and denote it by $H_0^k(N)$. The defining polynomial of the closure of $H_0^k(N)$ is denoted by $H_0^k(l, m)$.

4. The developing map

In this section we define a map $D : P(N) \to \overline{X}(N)$. Using this map we show that the polynomial $H(l, m)$ defined in Section 2 divides the $\mathrm{PSL}(2, \mathbb{C})$ A-polynomial. Furthermore we show that the the curves $P_0(N)$, $H_0(N)$ and the polynomial $H_0(l, m)$ is independent of the ideal triangulation.

4.0.7. *Construction of the developing map D.* Let N be a hyperbolic 3-manifold with an ideal triangulation $\mathcal{T}$ with n tetrahedra $\sigma_1, \ldots, \sigma_n$. Let $\mathbf{z}^0 = (z_1^0, \ldots, z_n^0)$ be the parameter for the complete hyperbolic structure on N. Fix a base point $x \in N$ such that $x \in$ interior (σ_1). Let $\tilde{N}$ be the universal cover of N and let $p : \tilde{N} \to N$ be the covering map. The triangulation $\mathcal{T}$ lifts to a triangulation $\tilde{\mathcal{T}}$ on $\tilde{N}$. Fix a base point $\tilde{x} \in \tilde{N}$ such that $p(\tilde{x}) = x$ and fix a lift $\tilde{\sigma_1}$ of σ_1 which contains $\tilde{x}$. The triangulation $\tilde{\mathcal{T}}$ is $\pi_1(N)$-equivariant i.e. it is invariant under the $\pi_1(N)$ action on $\tilde{N}$ and if $\tilde{y} = g \cdot \tilde{x}$ for some $g \in \pi_1(N)$ then $\tilde{y} \in$ interior $g(\tilde{\sigma_1})$.

For any parameter $\mathbf{z} = (z_1, \ldots, z_n) \in P(N)$ we construct a map $\phi : \tilde{N} \to \mathbb{H}^3$ inductively as follows: Send $\tilde{\sigma_1}$ to the ideal tetrahedra with vertices z_1, 1, ∞ and 0. Observe that $[z_1 : 1 : \infty : 0] = z_1$ and hence the cross-ratio parameter of $\phi(\tilde{\sigma_1})$ is z_1. Assume ϕ is defined for a triangulated subset S of $\tilde{N}$. Let $\tilde{\sigma}$ be a lift of σ_i for some i for which $\tilde{\sigma}$ has a face in common with S. Let a, b, c, d be the vertices of $\tilde{\sigma}$ and $\tilde{\sigma} \cap S$ equal the face of $\tilde{\sigma}$ with vertices a, b and c. Define $\phi(\tilde{\sigma})$ to be the unique hyperbolic tetrahedra with vertices $\phi|_S(a), \phi|_S(b), \phi|_S(c)$ and w such that the cross-ratio $[\phi|_S(a) : \phi|_S(b) : \phi|_S(c) : w] = z_i$. Observe that $\phi(\tilde{\sigma})$ is well defined even if all the four vertices are in S as the z_i's satisfy the gluing equations. The map ϕ is well-defined and continuous and is called the *developing map* for the hyperbolic structure (in general incomplete) induced on N by the parameter $\mathbf{z} = (z_1, \ldots, z_n)$.

Let us see how the map ϕ changes if we change the image of $\tilde{\sigma_1}$. Let ϕ' be the developing map defined using another choice of the image of $\tilde{\sigma_1}$. Let $\phi'(\tilde{\sigma_1}) = \Delta'$ and let a, b, c and d be the vertices of Δ' such that $[a : b : c : d] = z_1$. Since Δ' and Δ have the same cross-ratio there is a unique hyperbolic isometry α such that $\alpha(\Delta) = \Delta'$. Hence $\phi'(\tilde{\sigma_1}) = \alpha(\phi(\tilde{\sigma_1}))$. Since

ϕ and ϕ' are defined inductively we get:

$$\phi'(\tilde{\sigma}) = \alpha(\phi(\tilde{\sigma})) \tag{4.1}$$

for any $\tilde{\sigma} \in \tilde{\mathcal{T}}$ and hence $\phi'(x) = \alpha(\phi(x))$ for all $x \in \tilde{N}$.

The map ϕ gives rise to the representation $\overline{\rho} : \pi_1(N) \to \mathrm{PSL}(2,\mathbb{C})$ as follows: For any $g \in \pi_1(N)$ let $\tilde{\sigma_g} = g \cdot \tilde{\sigma_1}$. By definition of ϕ, the cross-ratio parameters of $\phi(\tilde{\sigma_1})$ and $\phi(\tilde{\sigma_g})$ are the same and hence there is a unique orientation preserving hyperbolic isometry τ which takes $\phi(\tilde{\sigma_1})$ to $\phi(\tilde{\sigma_g})$. Define $\overline{\rho}(g) = \tau$. The relation between ϕ and $\overline{\rho}$ is:

$$\phi(g \cdot \tilde{\sigma_1}) = \overline{\rho}(g)(\phi(\tilde{\sigma_1})) \tag{4.2}$$

Using 4.1 it is clear that this gives a representation of $\pi_1(N)$ into $\mathrm{PSL}(2,\mathbb{C})$. If $\overline{\rho}'$ is the representation obtained from ϕ' by changing the image of $\tilde{\sigma_1}$ then using 4.1 and 4.2 we can see that:

$$\overline{\rho}(g) = \alpha^{-1}\overline{\rho}'(g)\alpha \text{ for all } g \in \pi_1(N) \tag{4.3}$$

Let ϕ' be the developing map defined using a different base point, say $\tilde{x}' \in \tilde{\sigma_i}$, where $\tilde{\sigma_i}$ is a lift of σ_i and $\phi'(\tilde{\sigma_i})$ is the ideal tetrahedron with vertices $z_i,\ 1,\ \infty$ and 0. Since $\phi(\tilde{\sigma_1})$ and $\phi'(\tilde{\sigma_i})$ have 3-vertices in common, the induced representations $\overline{\rho}$ and $\overline{\rho}'$ are the same.

Hence a parameter $\mathbf{z} \in P(N)$ gives a conjugacy class of representations in $\mathrm{PSL}(2,\mathbb{C})$ and hence a well defined element of $\overline{X}(N)$. This representation is called the *holonomy representation* associated to the hyperbolic structure on N induced the parameter $\mathbf{z}$. Let ϕ_z be the developing map obtained as above starting with the image of $\tilde{\sigma_1}$ to be the tetrahedra with vertices $z_1,\ 1,\ \infty$ and 0 and let $\overline{\rho}_{\mathbf{z}}$ denote the induced holonomy representation. Let $D : P(N) \to \overline{X}(N)$ be defined by $D(\mathbf{z}) = \chi_{\overline{\rho}_{\mathbf{z}}}$.

4.0.8. *Properties of D.* D **is Algebraic:** We will show that D is an algebraic map. Let $D' : P(N) \to \overline{R}(N)$ be the map defined by $D(\mathbf{z}) = \overline{\rho}_{\mathbf{z}}$. Since $D = D' \circ \overline{t}$ and $\overline{t}$ is algebraic it is enough to show that the D' is algebraic. The fundamental group $\pi_1(N)$ of N is generated by face pairings of some fundamental domain $R \subset \tilde{N}$ which is triangulated in $\tilde{\mathcal{T}}$. Let $\tilde{\sigma_1}, \ldots, \tilde{\sigma_n}$ be the tetrahedra which make up R. Any face pairing of R is a composition of a face pairings of $\tilde{\sigma_i}$ and their inverses. Note that the face pairing of each $\tilde{\sigma_i}$ may or may not belong to $\pi_1(N)$. The image $\overline{\rho}_{\mathbf{z}}(\pi_1(N))$ is generated by the face pairings of the tetrahedra Δ_i with vertices $0,\ 1,\ \infty$ and z_i for $1 \le i \le n$ and their inverses. All the face pairings of Δ_i are generated by the following elements of $\mathrm{PSL}(2,\mathbb{C})$:

$$\pm\begin{pmatrix}\sqrt{z_i} & 0\\ 0 & 1/\sqrt{z_i}\end{pmatrix},\quad \pm\begin{pmatrix}\sqrt{z_i-1} & 1/\sqrt{z_i-1}\\ 0 & 1/\sqrt{z_i-1}\end{pmatrix},\quad \pm\begin{pmatrix}\sqrt{\frac{z_i}{1-z_i}} & 0\\ \sqrt{\frac{z_i}{1-z_i}} & \sqrt{\frac{1-z_i}{z_i}}\end{pmatrix},$$

$$\pm\begin{pmatrix}0 & 1\\ -1 & 1\end{pmatrix},\qquad \pm\begin{pmatrix}-i & i\\ 0 & i\end{pmatrix} \tag{4.4}$$

Hence the image of the representation $\overline{\rho}_{\mathbf{z}}$ is in the subgroup of $\mathrm{PSL}(2,\mathbb{C})$ generated by products of the above matrices and their inverses. Since $\mathrm{PSL}(2,\mathbb{C})$ acts on $\mathbb{C}^3$ via the action on 2×2 traceless matrices by conjugation, we get a faithful representation $\Theta : \mathrm{PSL}(2,\mathbb{C}) \to \mathrm{SL}(3,\mathbb{C})$ defined as:

$$\Theta(\pm\begin{pmatrix}a & b\\ c & d\end{pmatrix}) = \begin{pmatrix}ad+bc & -ac & bd\\ -2ab & a^2 & -b^2\\ 2cd & -c^2 & d^2\end{pmatrix} \tag{4.5}$$

It is easy to see that $\det(\Theta(\pm A)) = 1$ for all $\pm A \in \mathrm{PSL}(2,\mathbb{C})$ and the image of matrices in 4.4 are matrices with entries as rational functions of z_i's. Hence the map D' is algebraic and hence D is algebraic.

D **is 2:1:** In general D is not onto. We show that D is generically $2:1$ onto its image. Lifting representations to give geometric triangulations has been studied in [7] (Section 4.5) and also in [30] (Section 8). Consider the triangulated fundamental domain $R \subset \tilde{N}$ considered above and let $v_1, \dots v_s$ denote its vertices. Since N has one cusp, $\pi_1(N)$ acts transitively on the vertices of the triangulation $\tilde{\mathcal{T}}$ of $\tilde{N}$. Moreover since the triangulation $\mathcal{T}$ of N is an ideal triangulation, the only vertex in N is at infinity and hence $\pi_1(\partial N)$ has a fixes a vertex at infinity in $\tilde{N}$. Let v_1 be a vertex of $\tilde{\sigma}_1$ which always maps to ∞ under ϕ_z for all $z \in P(N)$. Let $g_1, \dots, g_s \in \pi_1(N)$ be such that $g_j(v_1) = v_j$ for all $1 \le j \le s$. To each tetrahedron $\tilde{\sigma}_i$ of R we can associate 4 group elements such that $g_{i_j}(v_1) = v_{i_j}$ for $j = 1, \dots 4$ where $v_{i_1}, \dots, v_{i_4}$ are the vertices of $\tilde{\sigma}_i$. For any $\mathbf{z} \in P(N)$ the cross ratio parameter of $\phi_z(\tilde{\sigma}_i)$ is

$$z_i = [\overline{\rho}_z(g_{i_1})(\infty) : \overline{\rho}_z(g_{i_2})(\infty) : \overline{\rho}(g_{i_3})(\infty) : \overline{\rho}(g_{i_4})(\infty)] \tag{4.6}$$

Let $\chi \in \overline{X}(N)$ be a character of a representation $\overline{\rho} \in \overline{R}(N)$. If the image $\overline{\rho}(\pi_1(\partial N)) \neq \mathbb{Z}_2 \oplus \mathbb{Z}_2$ then $\overline{\rho}(\pi_1(\partial N))$ has a fixed point on the sphere at infinity i.e. as a subgroup of $\mathrm{PSL}(2,\mathbb{C})$, $\overline{\rho}(\pi_1(\partial N))$ has at least one common eigenvector. When $\overline{\rho} = \overline{\rho_0}$, the representation associated to the hyperbolic structure on N, $\overline{\rho}(\pi_1(\partial N))$ consists of parabolic isometries and has a single fixed point on the sphere at infinity. In general $\overline{\rho}(\pi_1(\partial N))$ consists of hyperbolic or elliptic isometries and has two fixed points on the sphere at infinity. We can always conjugate a representation so that one of the fixed points of $\overline{\rho}(\pi_1(\partial N))$ is the point ∞. Starting with a character $\chi \in \overline{X}(N)$ let $\overline{\rho}$ be a representation such that $\chi_{\overline{\rho}} = \chi$ and $\overline{\rho}(\pi_1(\partial N))$ fixes ∞. The fundamental domain can be reconstructed using the vertices $\overline{\rho}(g_1)(\infty), \ldots \overline{\rho}(g_s)(\infty)$. Similarly the cross ratio parameters of each σ_i are obtained from Equation 4.6 by substituting $\overline{\rho}$ in place of $\overline{\rho}_z$. In case two or more of the vertices coincide the parameters degenerate, i.e. become 0, 1 or ∞, the pull back is not defined. Since the g_i's and $\pi_1(\partial N)$ generate $\pi_1(N)$, the image under D of the pull back of $\overline{\chi}$ goes to itself. Since generically $\overline{\rho}(\pi_1(\partial N))$ has two fixed points the map D is generically $2:1$. Observe that the pull back is defined for every element in the image of D.

Since the condition that two or more vertices coincide gives an additional equation, the characters on the component $\overline{X}_0(N)$ for which the pull back is not defined is 0-dimensional. Hence the image $D(P_0(N))$ consists of almost all the characters on $\overline{X}_0(N)$.

Holonomies of cusp torus: Let T be the cusp torus of N. Then T inherits a triangulation by similarity classes of Euclidean triangles cut off by the horospheres centered at the vertices of the ideal tetrahedra. This gives a similarity structure on T which is Euclidean if the ideal triangulation gives the complete hyperbolic structure on N. The similarity structure gives a holonomy representation $\xi : \pi_1(T) \to \mathrm{Sim}(\mathbb{E}^2) = \mathrm{Aff}(\mathbb{C}) = \{az + b : a, b \in \mathbb{C},\ a \neq 0\}$. Euclidean similarities can be identified with a subset of $\mathrm{PSL}(2, \mathbb{C})$ via the identification:

$$az + b \longmapsto \pm \begin{pmatrix} \sqrt{a} & b/\sqrt{a} \\ 0 & 1/\sqrt{a} \end{pmatrix}$$

Fix a basis $B = \{L, M\}$ of $\pi_1(T) \subset \pi_1(N)$. Let $\xi(L) = lz + b_1$ and $\xi(M) = mz + b_2$. Then the holonomy representations of L and M are:

$$\overline{\rho}(L) = \pm \begin{pmatrix} \sqrt{l} & b_1/\sqrt{l} \\ 0 & 1/\sqrt{l} \end{pmatrix} \text{ and } \overline{\rho}(M) = \pm \begin{pmatrix} \sqrt{m} & b_2/\sqrt{m} \\ 0 & 1/\sqrt{m} \end{pmatrix}$$

Hence the square of the eigenvalues of the meridian and longitude of the holonomy representation $\overline{\rho}_z$ induced by the developing maps to the derivatives of the representation of the meridian and longitude in $\mathrm{Sim}(\mathbb{E}^2) = \mathrm{Aff}(\mathbb{C})$. It is proved in [38] (see also [31] Lemma 2.1) that this derivative is the number l and m from Equations 3.3. Hence we have shown:

Theorem 4.1. *The map $D : P(N) \to \overline{X}(N)$ defined by $D(\mathbf{z}) = \overline{\chi}_{\overline{\rho}_{\mathbf{z}}}$ is algebraic, generically $2:1$ onto its image and the following diagram commutes:*

$$\begin{array}{ccc} \overline{X}(N) & \xleftarrow[2:1]{D} & P(N) \\ \overline{r}\big\downarrow & & Hol\big\downarrow \\ \overline{X}(\partial N) & \xleftarrow[2:1]{\overline{p}_B^{-1}\circ \overline{t}_\Delta} & \mathbb{C}^* \times \mathbb{C}^* \end{array}$$

Hence $H(l,m)$ *divides the* $\mathrm{PSL}(2,\mathbb{C})$ *A-polynomial* $\overline{A}(l,m)$*.*

The above diagram restricted to $P_0(N)$ gives that the curve $H_0(N) = \overline{p}_B(\overline{D}_0)$. Hence we have:

Theorem 4.2. $H_0(l,m) = \overline{A}(l,m)$ *and hence the curve* $H_0(N)$ *and its polynomial are independent of the ideal triangulation of* N*.*

In [16], Dunfield has shown that the map $\overline{r} : \overline{X}_0(N) \to \overline{X}(\partial N)$ is a birational isomorphism onto its image. Hence from the above diagram we have

Theorem 4.3. $\mathrm{Hol} : P_0(N) \to \mathbb{C} \times \mathbb{C}$ *is a birational isomorphism onto its image and hence the curve* $P_0(N)$ *is birational to the curve* $H_0(N)$ *and is independent of the triangulation of* N*.*

5. Boundary slopes

One of the striking properties of the A-polynomial is that the slopes of the sides of the Newton polygon of the A-polynomial are boundary slopes of essential surfaces in the manifold. We give a similar relationship for the $\mathrm{PSL}(2,\mathbb{C})$ A-polynomial along the same lines using the $\mathrm{PSL}(2,\mathbb{C})$ character variety theory developed in [3]. Most of the proofs in Section 3 of [7] go through with minor modifications.

5.0.9. *Ideal points of* $\overline{D}_N$. Let N be a one-cusped hyperbolic 3-manifold. Fix a basis $B = \{L, M\}$ of $\pi_1(\partial N) = H_1(\partial N; \mathbb{Z})$. If S is an incompressible surface with non-empty boundary in N then ∂S is a family of homologous simple closed curves on ∂N and hence determines a homology class in $H_1(\partial N; \mathbb{Z})$ given by $pM + qL$. The *boundary slope* of S is the extended rational number p/q. Fix a curve $C \subset \overline{X}(N)$. Then it is shown in [3] (Section 4) that there exists a finite extension F of the function field $\mathbb{C}(C)$ of C and a *tautological representation*

$$P : \pi_1(N) \to \mathrm{PSL}(2, F)$$

.

Proposition 5.1. *Let $\overline{Y}$ be an irreducible component of $\overline{D}_N$. There exists a finite extension F of the function field $\mathbb{C}(\overline{Y})$ of $\overline{Y}$ and a representation $\overline{P} : \pi_1 N \to \mathrm{PSL}(2, F)$ such that*

$$\overline{P}(L) = \pm \begin{pmatrix} l & 0 \\ 0 & l^{-1} \end{pmatrix} \text{ and } \overline{P}(M) = \pm \begin{pmatrix} m & 0 \\ 0 & m^{-1} \end{pmatrix}$$

where l and m are regarded as elements of $\mathbb{C}[\overline{Y}]$.

Proof: By the construction of the curve $\overline{D}_N$ there is a curve $\overline{Y}_{\partial N} \subset \overline{X}(\partial N)$ and an irreducible component $\overline{Y}' \subset \overline{X}(N)$ such that $\overline{t}_\Delta(\overline{Y}) = \overline{r}(\overline{Y}') = \overline{Y}_{\partial N}$. By Section 4 of [3] there exists a finite extension F_1 of $\mathbb{C}(\overline{Y}')$ and a tautological representation $P_1 : \pi_1(N) \to \mathrm{PSL}(2, F_1)$. Since the maps $\overline{r}$ and $\overline{t}_\Delta$ are surjective, the function fields $\mathbb{C}(\overline{Y}')$ and $\mathbb{C}(\overline{Y})$ are finite extensions of the function field $\mathbb{C}(\overline{Y}_{\partial N})$. Hence we can find a common finite extension, call it F, of F_1 and $\mathbb{C}(\overline{Y})$. We may regard P_1 as a representation of $\pi_1(N)$ in $\mathrm{PSL}(2, F)$. Since F contains $\mathbb{C}(\overline{Y})$, it contains l and m. Since l and m are eigenvalues of commuting matrices $P_1(L)$ and $P_1(M)$, the representation P_1 is conjugate in $\mathrm{GL}(2, F)$ to a representation P satisfying the conclusion of the proposition. ■

Proposition 5.2. *To each ideal point x of $\overline{D}_N$ there corresponds an incompressible surface with non-empty boundary in N. If ν is the valuation on $\mathbb{C}(\overline{D}_N)$ associated to x then the boundary slope of this incompressible surface is $-\nu(l)/\nu(m)$.*

Proof: Let $\overline{Y}$ be an irreducible component of $\overline{D}_N$ and ν be the discrete valuation induced on $\mathbb{C}(\overline{Y})$ by x. Let F be the finite extention of $\mathbb{C}(\overline{Y})$ and P be the representation obtained using Proposition 5.1. Since F is a finite extention of $\mathbb{C}(\overline{Y})$, the discrete valuation ν extends to a discrete valuation ν' of F with the property that $\nu'(f) = N\nu(f)$ for some N and for all $f \in \mathbb{C}(\overline{Y})$. As shown in [14] and [3] we obtain an action of $\pi_1(N)$ on the tree of $\mathrm{PSL}(2, F)$ determined by the representation P. Since l and m are the coordinates on $\overline{D}_N$ and x is an ideal point, $\nu(l)$ or $\nu(m)$ is non-zero. This means that this action is non-trivial and Theorem 4.3 of [3] implies that N has a splitting along an essential surface with nonempty boundary. Morever Proposition 4.7 of [3] implies that the boundary of this essential surface is the unique boundary slope r such that $\nu'(r) = 0$ i.e. if $r = p/q$ then the element $\gamma = l^q m^p \in \pi_1(\partial N) \subset \pi_1(N)$ satisfies $\nu'(\gamma) = 0$. Since ν' is a discrete valuation

$$\nu'(\gamma) = \nu'(l^q m^p) = q\nu'(l) + p\nu'(m) = 0$$

One solution to this is $p = \nu'(l)$ and $q = -\nu'(m)$ and hence the slope $r = -\nu'(l)/\nu'(m) = -\nu(l)/\nu(m)$, since $\nu'(f) = N\nu(f)$ for $f \in \mathbb{C}(\overline{Y})$. Proposition 4.7 of [3] says that this is the unique slope with this property. This proves our proposition. ■

5.0.10. *Newton polygon and Puiseaux parametrizations.* The Newton polygon of a polynomial $F(x,y) = \sum a_{mn} x^m y^n$ is the convex hull of the points $(m,n) \in \mathbb{Z} \times \mathbb{Z}$ where $a_{mn} \neq 0$ and is denoted by $\mathcal{N}_F$. To relate the boundary slopes of essential surfaces to slopes of sides of the Newton Polygon of the $\overline{A}(l,m)$ we find valuations which arise from the slopes of the sides of Newton polygons. This is done using Puiseaux parametrization.

Let $F(x,y) = \sum_{m,n} a_{mn} x^m y^n = 0$ be the defining polynomial of an irreducible plane curve $C \subset \mathbb{C} \times \mathbb{C}$. Let S be a side of $\mathcal{N}_F$ having slope $-p/q$ with $p,\ q > 0$ and lying below $\mathcal{N}_F$. Let S lie on the line with equation $px + qy = d$. Assume that F has no factors of x or y. Consider the following part of F:

$$G_1(x,y) = \sum_{pm+qn=d} a_{mn} x^m y^n \tag{5.1}$$

Substituting $x = x_1^p$ and $y = t x_1^q$ in G_1 we get:

$$\sum_{pm+qn=d} a_{mn} x_1^{pm} t^n x_1^{nq} = \sum_{pm+qn=d} a_{mn} x_1^{pm+nq} t^n = x_1^d \sum_{pm+qn=d} a_{mn} t^n$$

If $F(x,y) = 0$ then $G_1(x,y) = 0$ and this happens when t is substituted by a root $a_0 \neq 0$ of the polynomial $\sum_{pm+qn=d} a_{mn} t^n$. Hence $y_0 = a_0 x_1^q$ can be seen as a first approximation to solving y in terms of x for the implicit equation $F = 0$. One can iterate this process by substituting $x = x_1^p$ and $y = x_1^q(a_0 + y_1)$ in F to get

$$F(x_1^p, x_1^q(a_0 + y_1)) = \sum_{m,n} a_{mn} x_1^{pm+qn} (a_0 + y_1)^n = x_1^d F_1(x_1, y_1)$$

since S is below $\mathcal{N}_F$ and $pm + qn = d$ is lowest among all m and n with $a_{mn} \neq 0$. Now we can repeat this recursively with F substituted by F_1. It is a classical result that this gives a power series expansion of y which converges (see [4] or [22]). Hence we get a parametrization of the curve C of the form:

$$x(t) = t^p \quad \text{and} \quad y(t) = t^q \sum_{n=0}^{\infty} b_n t^n$$

with $b_0 \neq 0$. Such a parametrization is called a *Puiseaux parametrization.*

A Puiseaux parametrization gives us a valuation on the function field $\mathbb{C}(C)$ of the curve C as follows. Given a rational function $k(x,y)$ which is non-zero in $\mathbb{C}(C)$, one defines the order of k to be the integer n such that $k((x(t), y(t)) = t^n z(t)$ where $z(t)$ is a power series with a non-zero constant term. It is shown in [22] that the order is well defined and the induced function is a discrete valuation on $\mathbb{C}(C)$. If this valuation is ν then $\nu(x) = p$ and $\nu(y) = q$ where x and y are seen as elements of $\mathbb{C}(C)$.

In order to handle the case of the sides of $\mathcal{N}_F$ which have non-negative slope or a side which lies above $\mathcal{N}_F$ we change coordinates appropriately. See the proof of Proposition 3.3 of [7] for more details. From the above discussion we have

Proposition 5.3. *Let C be a plane curve with defining polynomial $F(x, y)$. Assume F is not divisible by x or y. If the Newton polygon of F has a side of slope p/q then there is a valuation v on the function field of some irreducible component of C such that $p/q = -\nu(x)/\nu(y)$.*

Combining Propositions 5.2 and 5.3 we have

Theorem 5.1. *The slopes of the sides of the Newton polygon of $\overline{A}(l, m)$ are boundary slopes of incompressible surfaces in N which correspond to the ideal points of $\overline{D_N}$.*

Using Theorem 4.1 we immediately get:

Theorem 5.2. *The slopes of the sides of the Newton polygon of $H(l, m)$ are boundary slopes of incompressible surfaces in N which correspond to the ideal points of $H(N)$.*

6. Computations and examples

In this section we will discuss the computational aspect of $H(l, m)$ and give examples.

6.0.11. *Computation of $H(l, m)$.* The computation of $H(l, m)$ is reduced to classical elimination theory once the gluing and completeness equations are set up. Suppose we are given an ideal triangulation of a cusped hyperbolic 3-manifold N. Then we can compute the gluing and completeness equations using the triangulation data. Jeff Weeks' program SnapPea [40] reads the triangulation data of a manifold, computes the gluing and completeness equations and computes the hyperbolic structure simply by solving these equations numerically. Using the numerical solution SnapPea computes hyperbolic invariants like volume, Chern-Simons invariant, length of shortest geodesic and topological invariants like the fundamental group, first homology group, cyclic covers. Although experimental SnapPea is very accurate and experimentally reliable. SnapPea has been used extensively to study examples of hyperbolic 3-manifolds and test conjectures. A manifold can be entered into SnapPea in many ways. A knot or a link complement can be entered by drawing a knot or link projection. SnapPea includes a census of cusped hyperbolic 3-manifolds which can be triangulated by 7 or less ideal tetrahedra [6] which is usually referred to as SnapPea's cusped census. A manifold from the census can be loaded into SnapPea. Punctured torus bundles can be loaded into SnapPea using inbuilt functions. Once loaded a manifold can be saved as a data file. The program Snap [11] uses SnapPea data to compute the hyperbolic structure to very high precision and uses a number theory package called PARI-GP [1] to compute the the hyperbolic structure using exact arithmetic. Snap also computes arithmetic invariants such as the trace field, invariant trace field, quaternion algebras and Bloch invariants. Snap reads manifold data files from SnapPea and hence can be used to study all the manifolds which SnapPea can study. Snap includes SnapPea's cusped census and the knot census up to 16 crossings. Once a hyperbolic 3-manifold is loaded into Snap the gluing matrix can be obtained using the command "print gluing equations". Once we have the gluing matrix it is easy to set up the gluing equations. Once the gluing equations are set up we can use Groebner bases or resultant theory to obtain the polynomial. In our computations we have used various programs which compute Groebner bases like Macaulay and Magma.

Remark 6.1. *The census manifolds have the following notation: "m" denotes a manifold with 5 or less tetrahedra, "s" denotes a manifold with exactly 6 tetrahedra and "v" denotes a manifold with exactly 7 tetrahedra.*

6.0.12. *Examples.* **Example 1- m004 also known as the figure-8 knot complement:** The gluing matrix obtained from Snap is:

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & -2 & 0 & 4 & 2 \\ 2 & -1 & -1 & 2 & 0 \\ -2 & 1 & 1 & -2 & 0 \end{pmatrix}$$

Each row gives first the powers of $z_1, \ldots, z_n$, then the power of $1-z_1, \ldots, 1-z_n$, and finally the power of $e^{\pi i}$. The first row is the gluing equation for the meridian, the second row for the longitude and the remaining n rows are the for edges. Neumann and Zagier [31] proved that you only need $n-1$ of the n edge equations. In the above case we have 2 tetrahedra. Let the parameters be z_1 and z_2. The equations look like:

$$\begin{aligned} z_1(1-z_2) &= m \\ \frac{(1-z_2)^4}{z_2^2} &= l \\ \frac{z_1^2(1-z_2)^2}{(1-z_1)z_2} &= 1 \\ \frac{z_2(1-z_1)}{z_1^2(1-z_2)^2} &= 1 \end{aligned}$$

We first clear denominators in the above equations. In order to eliminate z_1 and z_2 from the above equations we can proceed in two ways. Either (1) take resultants or (2) we can form the ideal I generated by the above 4 equations (with cleared denominators and equated to 0) in the ring $\mathbb{Z}[z_1, z_2, l, m]$ and compute a Grobner basis with respect to the ordering which eliminates the first 2 variables. It is often convenient to add in another variable t and the extra equation $tz_1z_2(1-z_1)(1-z_2)=1$. This ensures that none of the parameters degenerate (i.e. equal 0 or 1). We give the $p \times q$ coefficient matrix Q from which we can retrieve the polynomial as:

$$\begin{pmatrix} 1 & m & m^2 & \dots & m^{p-1} \end{pmatrix} Q \begin{pmatrix} 1 & l & l^2 & \dots & l^{q-1} \end{pmatrix}^T$$

The coefficient matrix also gives the shape of the Newton polygon from which we can read the boundary slopes of essential surfaces. For the figure-8 knot complement we get only one factor:

$$\begin{pmatrix} & 1 & \\ & -2 & \\ & -3 & \\ & 2 & \\ -1 & 6 & -1 \\ & 2 & \\ & -3 & \\ & -2 & \\ & 1 & \end{pmatrix}$$

Example 2- m009:

$$\begin{pmatrix} & -1 & & \\ & 4 & & \\ & & & \\ 1 & -8 & 2 & \\ & & -4 & \\ & 4 & & \\ & -2 & 8 & -1 \\ & & & \\ & & -4 & \\ & & 1 & \end{pmatrix}$$

Example 3- m129 or the Whitehead link complement: We can compute the polynomial at each cusp as indicated in Section 2. For the Whitehead link complement we get the same polynomial at both the cusps.

$$\begin{pmatrix} & & -1 & \\ & & 8 & \\ & 2 & -16 & 1 \\ & -8 & 8 & \\ -1 & 16 & -2 & \\ & -8 & & \\ & 1 & & \end{pmatrix}$$

Example 4- m130:

$$\begin{pmatrix}
 & & & -1 & & & & & & \\
 & & & 9 & -3 & 3 & & & & \\
 & & 1 & -40 & 33 & -37 & 10 & -3 & & \\
 & & -2 & 87 & -98 & 132 & -61 & 32 & -7 & 1 \\
 & 1 & -13 & -44 & & -40 & -22 & -7 & -1 & \\
 & 1 & 7 & 22 & 40 & & 44 & 13 & -1 & \\
-1 & 7 & -32 & 61 & -132 & 98 & -87 & 2 & & \\
 & & 3 & -10 & 37 & -33 & 40 & -1 & & \\
 & & & & -3 & 3 & -9 & & & \\
 & & & & & & 1 & & &
\end{pmatrix}$$

Example 5- m137: We get only one factor. This manifold is studied in [17].

$$\begin{pmatrix}
 & & 1 & & \\
 & & -5 & & \\
 & -2 & -6 & -2 & \\
 & 2 & 13 & -10 & -1 \\
1 & 4 & 30 & -2 & -2 \\
2 & -10 & 12 & 24 & -3 \\
3 & -24 & -29 & 30 & -1 \\
1 & -2 & -48 & -2 & 1 \\
-1 & 30 & -29 & -24 & 3 \\
-3 & 24 & 12 & -10 & 2 \\
-2 & -2 & 30 & 4 & 1 \\
-1 & -10 & 13 & 2 & \\
 & -2 & -6 & -2 & \\
 & & -5 & & \\
 & & 1 & &
\end{pmatrix}$$

Example 6- s773:

$$\begin{pmatrix}
 & & & & & & -1 \\
 & & & 2 & 8 & & -3 \\
-1 & -8 & & 6 & -40 & 24 & -2 \\
-3 & 40 & -24 & 4 & 8 & 8 & 2 \\
-2 & -8 & -8 & -4 & 24 & -40 & 3 \\
2 & -24 & 40 & -6 & & 8 & 1 \\
3 & & -8 & -2 & & & \\
1 & & & & & &
\end{pmatrix}$$

7. Fundamental Identity

We will prove the identity which will be our fundamental tool in working with the variation of Bloch invariant.

7.0.13. *Prerequisites.* We recall some results from [31] and prove an identities which follows easily. Let N be a hyperbolic 3-manifold with h cusps. Let

$$U = \begin{pmatrix} L \\ M \\ R \end{pmatrix}_{(n+2h,2n)}$$

be the gluing matrix for a triangulation $\mathcal{T}$ of N. Let C denote the matrix $C = \left(\begin{smallmatrix} L \\ M \end{smallmatrix}\right)$ so $U = \left(\begin{smallmatrix} C \\ R \end{smallmatrix}\right)$. Let J_{2m} denote the symplectic matrix

$$J_{2m} = \begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix}$$

On $\mathbb{R}^{2n}$ we have the symplectic form $< \mathbf{x}, \mathbf{y} >= \frac{1}{2}\mathbf{x}J_{2n}\mathbf{y}^t$. The following theorem is proved in [31] (Theorem 2.2):

Theorem 7.1. $UJ_{2n}U^t = 2\begin{pmatrix} J_{2h} & 0 \\ 0 & 0 \end{pmatrix}$.

If we denote the row space of a matrix A by $[A]$, the above theorem implies that $[U]$ is orthogonal w.r.t $<,>$ to $[R]$ and the rows of C form a symplectic basis of $[C]$. It is also shown in [31] that

Proposition 7.1. *Rank $R = n - h$ and rank $U = n + h$. Moreover if $\perp$ denotes the orthogonal complement with respect to $<,>$ then $[U]^\perp = [R]$.*

Corollary 7.1. *Suppose $\mathbf{x}, \mathbf{y} \in \mathbb{Z}^{2n}$ satisfy $R\mathbf{x}^t = R\mathbf{y}^t = 0$. Then $2\mathbf{x}J_{2n}\mathbf{y}^t = \mathbf{x}C^tJ_{2h}C\mathbf{y}^t$.*

Proof: Since $0 = R\mathbf{x}^t = RJ_{2n}(\mathbf{x}J_{2n})^t$, it follows from Proposition 7.1 that the vector $\mathbf{x}J_{2n} \in [R]^\perp = [U]$. Let $\mathbf{x}J_{2n} = \mathbf{z}U$ and $\mathbf{y}J_{2n} = \mathbf{w}U$, where $\mathbf{z} = ({z_1}', \ldots, z_h', z_1'', \ldots, z_h'', z_1, \ldots, z_n)$ and $\mathbf{w} = ({w_1}', \ldots, w_h', w_1'', \ldots, w_h'', w_1, \ldots, w_n)$. This means that

$$\mathbf{x}J_{2n} = \sum_{i=1}^{h} z_i' L_i + \sum_{i=1}^{h} z_i'' M_i + \sum_{i=1}^{h} z_j R_j$$

$$\mathbf{y}J_{2n} = \sum_{i=1}^{h} w_i' L_i + \sum_{i=1}^{h} w_i'' M_i + \sum_{i=1}^{h} w_j R_j$$

By Theorem 7.1 we have

$$< \mathbf{x}J_{2n}, L_k >=< \sum_{i=1}^{h} z_i' L_i + \sum_{i=1}^{h} z_i'' M_i + \sum_{i=1}^{h} z_j R_j, L_k >=< z_k'' M_k, L_k >= -z_k''$$

$$< \mathbf{x}J_{2n}, M_k > = < \sum_{i=1}^{h} z_i' L_i + \sum_{i=1}^{h} z_i'' M_i + \sum_{i=1}^{h} z_j R_j, M_k > = < z_k' L_k, M_k > = z_k'$$

and similarly $< \mathbf{y}J_{2n}, L_k > = -w_k''$ and $< \mathbf{y}J_{2n}, M_k > = w_k'$. For $\mathbf{a}$,$\mathbf{b}$ ϵ $\mathbb{R}^{2n}$, $\mathbf{a}\mathbf{b}^t = -\mathbf{a}J_{2n}^2\mathbf{b}^t = -2 < \mathbf{a}J_{2n}, \mathbf{b} >$. So

$$\begin{aligned}
\mathbf{x}C^t &= \mathbf{x}(L_1^t, \ldots, L_h^t, M_1^t, \ldots, M_h^t) \\
&= -2(< \mathbf{x}J_{2n}, L_1 >, \ldots, < \mathbf{x}J_{2n}, L_n >, < \mathbf{x}J_{2n}, M_1 >, \ldots, < \mathbf{x}J_{2n}, M_n >) \\
&= -2(-z_1'', \ldots, -z_h'', z_1', \ldots, z_h') \\
&= -2\mathbf{z}\left(\begin{smallmatrix} J_{2h} \\ 0 \end{smallmatrix}\right)
\end{aligned}$$

and similarly $\mathbf{y}C^t = -2\mathbf{w}\left(\begin{smallmatrix} J_{2h} \\ 0 \end{smallmatrix}\right)$. By Theorem 7.1 we have

$$\begin{aligned}
2\mathbf{x}J_{2n}\mathbf{y}^t &= 2(\mathbf{x}J_{2n})J_{2n}(\mathbf{y}J_{2n})^t \\
&= 2(\sum_{i=1}^{h} z_i' L_i + \sum_{i=1}^{h} z_i'' M_i + \sum_{i=1}^{h} z_j R_j) J_{2n} (\sum_{i=1}^{h} w_i' L_i + \sum_{i=1}^{h} w_i'' M_i + \sum_{i=1}^{h} w_j R_j)^t \\
&= 4\sum_{i=1}^{h} z_i{}' w_i'' - z_i'' w_i' \\
&= 4(z_1', \ldots, z_h', z_1'', \ldots, z_h'') J_{2h} (w_1', \ldots, w_h', w_1'', \ldots, w_h'')^t \\
&= 4\mathbf{z}\left(\begin{smallmatrix} J_{2h} \\ 0 \end{smallmatrix}\right) J_{2h} (\mathbf{w}\left(\begin{smallmatrix} J_{2h} \\ 0 \end{smallmatrix}\right))^t \\
&= \mathbf{x}C^t J_{2n} (\mathbf{y}C^t)^t \\
&= \mathbf{x}C^t J_{2h} C \mathbf{y}^t
\end{aligned}$$

as desired. ■

7.0.14. *Identity.* Let $B_{2n} : \mathbb{Z}^{2n} \times \mathbb{Z}^{2n} \to \mathbb{Z}$ be an anti-symmetric bilinear form defined by $B_{2n}(\mathbf{x}, \mathbf{y}) = \mathbf{x}J_{2n}\mathbf{y}^t$, where J_{2n} is the symplectic matrix defined above. It is easy to see that B_{2n} descends to a homomorphism $B_{2n} : \mathbb{Z}^{2n} \wedge \mathbb{Z}^{2n} \to \mathbb{Z}$. Let $C : \mathbb{Z}^{2n} \to \mathbb{Z}^{2h}$ be the map defined by $C(\mathbf{x}) = \mathbf{x}C^t$. Since R has rank $n-h$, the row space of R is generated by $n-h$ vectors. Let R' be a $n-h \times 2n$ matrix whose rows generate the row space for R and let $R : \mathbb{Z}^{2n} \to \mathbb{Z}^{n-h}$ be the map defined by $R(\mathbf{x}) = \mathbf{x}(R')^t$. Let

$$0 \longrightarrow K \xrightarrow{\ i\ } \mathbb{Z}^{2n} \xrightarrow{\ R\ } \mathbb{Z}^{n-h} \tag{7.1}$$

be an exact sequence where K =ker(R) and i is the inclusion. Let $C \wedge C : \mathbb{Z}^{2n} \wedge \mathbb{Z}^{2n} \to \mathbb{Z}^{2h} \wedge \mathbb{Z}^{2h}$ be defined by $(C \wedge C)(\mathbf{x}, \mathbf{y}) = \mathbf{x}C^t \wedge \mathbf{y}C^t$. Let $m2$ denote the homomorphism from $\mathbb{Z}$ to $\mathbb{Z}$ defined by multiplication by 2. Then the conclusion of Corollary 7.1 can be seen as the following commutative diagram:

$$
\begin{array}{ccc}
K \wedge K & \xrightarrow{B_{2n}} & \mathbb{Z} \\
{\scriptstyle C\wedge C}\big\downarrow & & {\scriptstyle m2}\big\downarrow \\
\mathbb{Z}^{2h} \wedge \mathbb{Z}^{2h} & \xrightarrow{B_{2h}} & \mathbb{Z}
\end{array}
\tag{7.2}
$$

The above diagram implies that if $R(\mathbf{x}) = R(\mathbf{y}) = 0$ then 2 $B_{2n}(\mathbf{x}, \mathbf{y}) = B_{2h}(\mathbf{x}C^t, \mathbf{y}C^t)$.

Let G be an abelian group. Let $G \wedge G$ denote the second exterior product i.e., if A_2 is the subgroup of $G \otimes G$ generated by all the elements of the type $g \otimes g$ then $G \wedge G = G \otimes G/A_2$. For $\mathbf{g} = (g'_1, \dots, g'_n, g''_1, \dots, g''_n)$ and $\mathbf{h} = (h'_1, \dots, h'_n, h''_1, \dots, h''_n) \in G^{2n}$. Define

$$
\mathbf{g} \bigwedge \mathbf{h} = \sum_{i=1}^{n} g'_i \wedge h''_i - g''_i \wedge h'_i \ \ \epsilon \ \ G \wedge_{\mathbb{Z}} G
$$

Using the identification $G^{2n} \simeq \mathbb{Z}^{2n} \otimes G$ all the maps defined earlier can be extended to G^{2n} as follows: Define $R \otimes id : \mathbb{Z}^{2n} \otimes G \ \ \to \mathbb{Z}^{n-h} \otimes G$ by $(R \otimes id)(\mathbf{a} \otimes g) = R(\mathbf{a}) \otimes g$. Similarly the map C can be extended to a map $C \otimes id : \mathbb{Z}^{2n} \otimes G \to \mathbb{Z}^{n-h} \otimes G$. The map $C \wedge C$ extends to the map $C \wedge C : (\mathbb{Z}^{2n} \otimes G) \wedge (\mathbb{Z}^{2n} \otimes G) \to (\mathbb{Z}^{2h} \otimes G) \wedge (\mathbb{Z}^{2h} \otimes G)$ defined by $(C \wedge C)((\mathbf{a} \otimes g) \wedge (\mathbf{b} \otimes h)) = (\mathbf{a}C^t \otimes g) \wedge (\mathbf{b}C^t \otimes h)$. The map B_{2n} can be extended to the map $B_{2n} : (\mathbb{Z}^{2n} \otimes G) \times (\mathbb{Z}^{2n} \otimes G) \to G \wedge G$ defined by $B_{2n}((\mathbf{a} \otimes g) \wedge (\mathbf{b} \otimes h)) = B_{2n}(\mathbf{a}, \mathbf{b})(g \wedge h)$. Observe that using the identification $G^{2n} \simeq \mathbb{Z}^{2n} \otimes G$, $B_{2n}(\mathbf{g}, \mathbf{h}) = \mathbf{g} \bigwedge \mathbf{h}$ and B_{2n} descends to a map from $G^{2n} \wedge G^{2n}$ to $G \wedge G$.

From the diagram (7.2) we immediately obtain the following diagram:

$$
\begin{array}{ccc}
(K \otimes G) \wedge (K \otimes G) & \xrightarrow{B_{2n}} & G \wedge G \\
{\scriptstyle C\wedge C}\big\downarrow & & {\scriptstyle m2}\big\downarrow \\
(\mathbb{Z}^{2h} \otimes G) \wedge (\mathbb{Z}^{2h} \otimes G) & \xrightarrow{B_{2h}} & G \wedge G
\end{array}
\tag{7.3}
$$

Let $\mathbf{x}, \mathbf{y} \in K$ and $g, h \in G$. Let $\mathbf{g}$ and $\mathbf{h}$ correspond to $\mathbf{x} \otimes g$ and $\mathbf{y} \otimes h$ respectively Observe that $\mathbf{g}$ and $\mathbf{h} \in \ker(R \otimes id)$. The diagram (7.3) implies that 2 $\mathbf{g} \bigwedge \mathbf{h} = (C \otimes id)(\mathbf{g}) \wedge (C \otimes id)(\mathbf{h})$.

Suppose the map R in the sequence (7.1) is surjective i.e. we have the following short exact sequence :

$$
0 \longrightarrow K \xrightarrow{i} \mathbb{Z}^{2n} \xrightarrow{R} \mathbb{Z}^{n-h} \longrightarrow 0 \tag{7.4}
$$

Then tensoring the above sequence with G we get:

$$
0 \longrightarrow K \otimes G \xrightarrow{i \otimes id} \mathbb{Z}^{2n} \otimes G \xrightarrow{R \otimes id} \mathbb{Z}^{n-h} \otimes G \longrightarrow 0 \tag{7.5}
$$

This sequence is exact because $\mathrm{Tor}(\mathbb{Z}^{n-h}, G) = 0$ for any abelian group G. In this case $\ker(R \otimes id) = K \otimes G$

In general R is not surjective. We would like to show that a diagram (and the corresponding identity) similar to the diagram (7.3) holds for $\ker(R \otimes id)$. Let us explain the general situation.

For a cupsed hyperbolic 3-manifold N, let N^* denote the complex obtained by identitying each torus cusp to a point. If N has an ideal triangulation given by a complex X such that $N = X - X^{(0)}$ then $N^* = X$. Observe that $H_1(N^*; \mathbb{Z}/2) = H_1(N, \partial N; \mathbb{Z}/2)$. In is proved in [26] (Theorem 4.2) that the following sequence is exact:

$$0 \longrightarrow K \xrightarrow{i} \mathbb{Z}^{2n} \xrightarrow{R} \mathbb{Z}^{n-h} \longrightarrow H_1(N^*; \mathbb{Z}/2) \longrightarrow 0 \tag{7.6}$$

Let us denote the group $H_1(N^*; \mathbb{Z}/2)$ by A. The sequence (7.6) is a free resolution of $H_1(N^*; \mathbb{Z}/2)$. Tensoring it with G we get the following chain complex:

$$\mathcal{C} : 0 \longrightarrow K \otimes G \xrightarrow{i \otimes id} \mathbb{Z}^{2n} \otimes G \xrightarrow{R \otimes id} \mathbb{Z}^{n-h} \otimes G \longrightarrow 0 \tag{7.7}$$

The homology of this complex is as follows: $H_0(\mathcal{C}) = G \otimes A$, $H_1(\mathcal{C}) = \mathrm{Tor}(G, A) = \ker(R \otimes id)/(K \otimes G)$ and $H_i(\mathcal{C}) = 0$ for $i \geq 2$.

Since $A = H_1(N^*; \mathbb{Z}/2)$, the group $\mathrm{Tor}(G, A)$ contains only 2-torsion. Observe that in the following cases $\mathrm{Tor}(G, A) = 0$ and hence $\ker(R \otimes id) = K \otimes G$ which gives us the commutative diagram (7.3).

(1) G is a torsion-free abelian group.
(2) Torsion in G is other than 2-torsion.
(3) $A = H_1(N^*; \mathbb{Z}/2) = 0$ which holds for instance when N is a knot complement or more generally when the map $i_* : H_1(\partial N; \mathbb{Z}/2) \to H_1(N; \mathbb{Z}/2)$ is surjective.

Since $\mathrm{Tor}(G, A) = \ker(R \otimes id)/(K \otimes G)$ we have the following short exact sequence:

$$0 \longrightarrow K \otimes G \xrightarrow{i \otimes id} \ker(R \otimes id) \longrightarrow \mathrm{Tor}(G, A) \longrightarrow 0 \tag{7.8}$$

We will show that this sequence splits. Let $\mathcal{C}'$ be the following chain complex:

$$\mathcal{C}' : 0 \longrightarrow \mathbb{Z}^{2n} \xrightarrow{R} \mathbb{Z}^{n-h} \longrightarrow 0 \tag{7.9}$$

Then by the Universal Coefficient Theorem the following sequence splits:

$$0 \longrightarrow H_1(\mathcal{C}') \otimes G \longrightarrow H_1(\mathcal{C}' \otimes G) \longrightarrow \mathrm{Tor}(H_0(\mathcal{C}', G)) \longrightarrow 0 \tag{7.10}$$

Now $H_1(\mathcal{C}') = \ker(R)$, $H_1(\mathcal{C}' \otimes G) = \ker(R \otimes id)$ and $H_0(\mathcal{C}') = A$. The splitting of the sequence (7.10) gives us a splitting of the short exact sequence (7.8). Hence $\ker(R \otimes id) = (K \otimes G) \oplus \mathrm{Tor}(G, A)$. Since $A = H_1(N^*; \mathbb{Z}/2)$,

$\ker(R\otimes id)$ differs from $K\otimes G$ in just 2-torsion. So the commutative diagram (7.3) holds for $\ker(R\otimes id)$ in place of $K\otimes G$ after multiplication by 2 i.e.

$$\begin{array}{ccc} \ker(R\otimes id)\wedge\ker(R\otimes id) & \xrightarrow{2B_{2n}} & G\wedge G \\ {\scriptstyle C\wedge C}\Big\downarrow & & {\scriptstyle m2}\Big\downarrow \\ (\mathbb{Z}^{2h}\otimes G)\wedge(\mathbb{Z}^{2h}\otimes G) & \xrightarrow{2B_{2h}} & G\wedge G \end{array} \tag{7.11}$$

Let $\mathbf{g}\in G^{2n}$ then using the identification $G^{2n}\simeq\mathbb{Z}^{2n}\otimes G$ we can see $\mathbf{g}=\sum \mathbf{x}_i\otimes g_i$. Let us denote $R(\mathbf{g})=(R\otimes id)(\sum\mathbf{x}_i\otimes g_i)$ and $C(\mathbf{g})=(C\otimes id)(\sum\mathbf{x}_i\otimes g_i)$. We proved:

Lemma 7.1. *Let $\mathbf{g},\mathbf{h}\in G^{2n}$, then $R(\mathbf{g})=R(\mathbf{h})=0\Rightarrow 4\ \mathbf{g}\bigwedge\mathbf{h}=2\ C(\mathbf{g})\bigwedge C(\mathbf{h})$. In case G is torsion-free or $H_1(N^*;\mathbb{Z}/2)=0$ then $R(\mathbf{g})=R(\mathbf{h})=0\Rightarrow 2\ \mathbf{g}\bigwedge\mathbf{h}=C(\mathbf{g})\bigwedge C(\mathbf{h})$.*

8. Bloch Invariants

Bloch invariants of hyperbolic 3-manifolds were introduced in [30]. Using a degree one ideal triangulation of N, Neumann and Yang defined an invariant $\beta(N)$ which is determined by the cross-ratio parameters of the ideal triangulation. Using group homology they showed that $\beta(N)$ is independent of the triangulation and lies in the Bloch group $\mathcal{B}(\mathbb{C})$. In this section we will give basic definitions and show that $\beta(N)$ lies in $\mathcal{B}(\mathbb{C})$ using the identity proved earlier.

8.0.15. *Bloch group.* As seen in Section 2.2, $\mathbb{C}-\{0,1\}$ is the parameter space of hyperbolic ideal tetrahedra. We can study the free abelian group generated by $\mathbb{C}-\{0,1\}$ modulo the relations induced by the geometry.

Definition 8.1. *The pre-Bloch group $\mathcal{P}(\mathbb{C})$ is the quotient of the free $\mathbb{Z}$-module $\mathbb{Z}(\mathbb{C}-\{0,1\})$ by all instances of the relations:*

$$[x]-[y]+[\frac{y}{x}]-[\frac{1-x^{-1}}{1-y^{-1}}]+[\frac{1-x}{1-y}]=0 \tag{8.1}$$

$$[x]=[1-\frac{1}{x}]=[\frac{1}{1-x}]=-[\frac{1}{x}]=-[\frac{x-1}{x}]=-[1-x] \tag{8.2}$$

The relations (8.1) are called the 5-term relations and are induced as follows: An ideal polytope in $\mathbb{H}^3$ on 5 vertices can be decomposed into 2 tetrahedra with a face in common or into 3 tetrahedra with an edge in common. See Figure 2. This is often referred to as the 2-3 move. It is proved in [32] (see also [23]) that any two (topological) ideal triangulations containing at least 2 tetrahedra of a 3-manifold are related by 2-3 moves. This gives

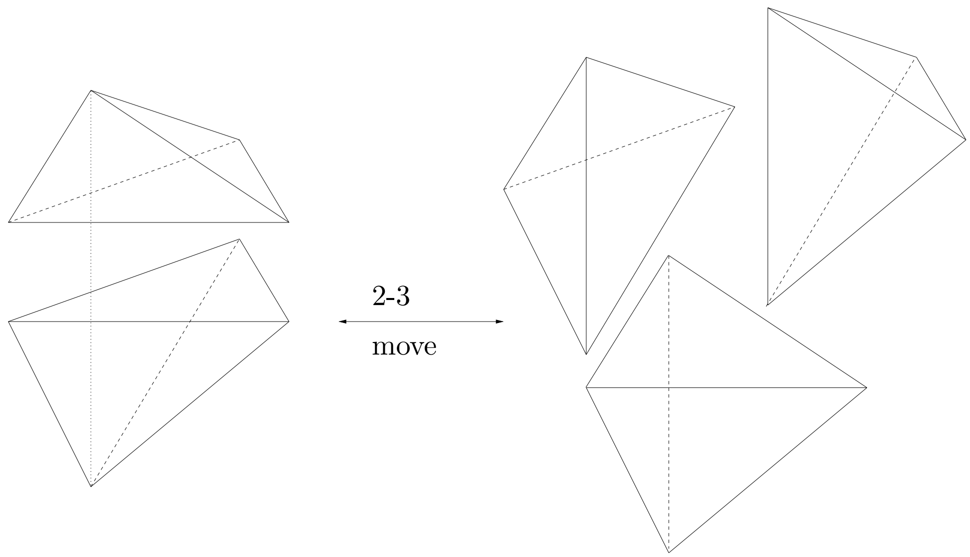

FIGURE 2. The 5 tetrahedra obtained from the 2-3 move in the 5-term relations

an immediate relation on the "sums" of tetrahedra which involves 5 cross-ratios. If $z_0, \ldots, z_4$ are the 5 ideal vertices then the 5-term relation for the cross-ratios looks like:

$$\sum_{i=0}^{4}(-1)^i[z_0 : \ldots : \hat{z}_i : \ldots : z_4] = 0$$

This equation takes the form of Equation (8.1) for $z_0 = y,\ z_1 = x,\ z_2 = 1,\ z_3 = \infty$ and $z_4 = 0$. The relations 8.2 are induced by the fact that $z,\ 1/(1-z),\ 1-1/z$ give the same tetrahedron and $1/z,\ 1-z,\ z/(z-1)$ give the same tetrahedron with opposite orientation.

The group $\mathcal{P}(\mathbb{C})$ comes up in the study of scissors congruence in $\mathbb{H}^3$. See [27] for more details. The volume of an ideal tetrahedra $\mathrm{vol}(\Delta(z)) = D_2(z)$ where $D_2(z)$ is the "Bloch-Wigner dilogarithm function" defined earlier in Section 2.2. It follows that $D_2(z)$ satisfies a functional equation corresponding to the 5-term relation and hence gives a map vol: $\mathcal{P}(\mathbb{C}) \to \mathbb{R}$. The analog of the Dehn invariant is a map $\mu : \mathcal{P}(\mathbb{C}) \to \mathbb{C}^* \wedge_{\mathbb{Z}} \mathbb{C}^*$ defined as $\mu([z]) = 2(z \wedge (1-z))$. We define:

Definition 8.2. *The Bloch group $\mathcal{B}(\mathbb{C})$ is the kernel of the map μ.*

Remark 8.1.

(1) *The pre-Bloch and the Bloch groups can be defined for any field k.*

(2) *There are several definitions of the Bloch group in the literature. They differ from each other by at most torsion and agree for algebraically closed fields. See [30] for more details on the differences and other definitions of the Bloch and the pre-Bloch groups.*

One of the main conjectures about the Bloch group states that:

Conjecture 8.1. (**Bloch Rigidity Conjecture**) *The Bloch group $\mathcal{B}(\mathbb{C})$ is countable.*

8.0.16. *The Bloch invariant.* Let N be a hyperbolic 3-manifold with a degree-one ideal triangulation with cross-ratio parameters $z_1, \ldots, z_n$.

Definition 8.3. *The Bloch invariant $\beta(N)$ is defined as the element $\sum_{i=1}^{n}[z_i] \in \mathcal{P}(\mathbb{C})$.*

In order to show that the Bloch invariant $\beta(N) \in \mathcal{B}(\mathbb{C})$ we need some algebraic preliminaries. Let $(\mathbb{C}^*, \cdot)$ denote the multiplicative group of complex numbers and let μ denote the subgroup of roots of unity in $\mathbb{C}^*$. It is easy to see that μ is isomorphic to $\mathbb{Q}/\mathbb{Z}$. Let z_μ denote the equivalence class of z in $\mathbb{C}^*/\mu$.

Lemma 8.1. *$\mathbb{C}^*/\mu$ is a $\mathbb{Q}$-vector space with addition defined by complex multiplication and scalar multiplication defined by $(\frac{p}{q})z_\mu = z_\mu^{p/q}$.*

Proof: For any non-zero complex number z and a positive integer q, $z^{1/q}$ is well defined modulo roots of unity. Hence $z_\mu^{1/q}$ is well defined in $\mathbb{C}^*/\mu$. This makes the scalar multiplication well defined and gives $\mathbb{C}^*/\mu$ a $\mathbb{Q}$-vector space structure. Moreover the exponential map $\exp : \mathbb{C} \to \mathbb{C}^*$ defined by $\exp(z) = e^z$ is a group homomorphism and $\exp^{-1}(\mu) = 2\pi i\mathbb{Q}$. Hence $\exp : \mathbb{C}/2\pi i\mathbb{Q} \to \mathbb{C}^*/\mu$ is a vector space isomorphism. ■

Lemma 8.2. $\mathbb{C}^* \wedge_{\mathbb{Z}} \mathbb{C}^* \simeq (\mathbb{C}^*/\mu) \wedge_{\mathbb{Z}} (\mathbb{C}^*/\mu)$.

Proof: The projection map $p : \mathbb{C}^* \to \mathbb{C}^*/\mu$ defined by $p(z) = z_\mu$ induces a surjective homomorphism $q : \mathbb{C}^* \wedge_{\mathbb{Z}} \mathbb{C}^* \to (\mathbb{C}^*/\mu) \wedge_{\mathbb{Z}} (\mathbb{C}^*/\mu)$. The kernel $\ker(q) = (\mu \wedge_{\mathbb{Z}} \mu) \oplus (\mu \otimes \mathbb{C}^*/\mu)$. We will show that if α is a root of unity then $\alpha \otimes z = 0$ for all $z \in \mathbb{C}^*$. Let $\alpha^n = 1$. Since we can take n-th roots in $\mathbb{C}$, there is a $w \in \mathbb{C}^*$ such that $w^n = z$. So we have

$$\alpha \otimes z = \alpha \otimes w^n = n\ \alpha \otimes w = \alpha^n \otimes w = 1 \otimes w = 0$$

Hence the map q is injective and hence an isomorphism. ■

Since $\mathbb{C}/\mu$ is a $\mathbb{Q}$-vector space, $(\mathbb{C}^*/\mu) \wedge_{\mathbb{Z}} (\mathbb{C}^*/\mu)$ is also a $\mathbb{Q}$-vector space and hence $\mathbb{C}^* \wedge_{\mathbb{Z}} \mathbb{C}^*$ is a $\mathbb{Q}$- vector space.

Lemma 8.3. *If* $(z_1, \ldots, z_n) \in P_0(N)$ *then* $2\sum_{i=1}^n z_i \wedge (1 - z_i) = \sum_{k=1}^h l_k \wedge m_k \in \mathbb{C}^* \wedge_{\mathbb{Z}} \mathbb{C}^*$.

Proof: Since $z_1, \ldots z_n$ satisfy the gluing equations 3.2 the tuple $\mathbf{x} = ((z_1)_\mu, \ldots (z_n)_\mu) \in (\mathbb{C}^*/\mu)^n$ satisfies the condition $R\mathbf{x}^t = 0$, where R is the gluing matrix. Since $\mathbb{C}^*/\mu$ is a $\mathbb{Q}$-vector space by above we can use Lemma 7.1 to get

$$2\sum_{i=1}^{n}(z_i)_\mu \wedge (1 - (z_i)_\mu) = \mathbf{x} \bigwedge \mathbf{x} = \frac{1}{2}C\mathbf{x}^t \bigwedge C\mathbf{x}^t = \sum_{k=1}^{h}(l_k)_\mu \wedge (m_k)_\mu$$

This identity holds in $\mathbb{C}^*/\mu \wedge_{\mathbb{Z}} \mathbb{C}^*/\mu$. Since $\mathbb{C}^* \wedge_{\mathbb{Z}} \mathbb{C}^* \simeq (\mathbb{C}^*/\mu) \wedge_{\mathbb{Z}} (\mathbb{C}^*/\mu)$ from above we have that the corresponding identity $2\sum_{i=1}^n z_i \wedge (1 - z_i) = \sum_{k=1}^h l_k \wedge m_k$ also holds in $\mathbb{C}^* \wedge_{\mathbb{Z}} \mathbb{C}^*$. ■

Let $\mathcal{T}$ be a degree 1 ideal triangulation of N with parameters $z_1, \ldots z_n$. This means that either $\mathcal{T}$ is a genuine ideal triangulation in which case the holonimies $m_k = l_k = 1$ or N is obtained by Dehn surgery on some (or all) of the cusps in which case $m_k^{p_k} l_k^{q_k} = 1$ for some integers p_k and q_k.

Proposition 8.1. $\beta(N) \in \mathcal{B}(\mathbb{C})$.

Proof: $\beta(N) = \displaystyle\sum_{i=1}^{n}[z_i] \in \mathcal{P}(\mathbb{C})$. By the Lemma 8.3

$$\mu(\sum_{i=1}^{n}[z_i]) = 2\sum_{i=1}^{n}z_i \wedge (1 - z_i) = \sum_{k=1}^{h}l_k \wedge m_k$$

Now either $m_k = l_k = 1$ or $(m_k)^p(l_k)^q = 1$ and in either case $l_k \wedge m_k = 0$ and hence $\beta(N) \in \mathcal{B}(\mathbb{C})$. ■

The key point in the proof is the identity $2\sum_{i=1}^n z_i \wedge (1 - z_i) = \sum_{k=1}^h l_k \wedge m_k$. This means that the holonomies at the cusps determine the image $\mu(\beta(N))$. In particular we are interested in manifolds with the same holonomies at the cusp.

9. Variation of the Bloch invariant

9.0.17. *Bloch invariant, volume and Chern-Simons invariant.* $\overline{A}(l, m)$ has appeared in literature in disguises. For example in [31] Neumann and Zagier implicitly showed that $\overline{A}(l, m)$ determines the variation of volume of N and in [42] Yoshida implicitly showed that $\overline{A}(l,m)$ determines the variation of the Chern-Simons invariant of N. The Bloch invariant is related to the volume and the Chern-Simons invariant via the Bloch regulator map. Define

$$\rho : \mathcal{P}(\mathbb{C}) \to \mathbb{C} \wedge_{\mathbb{Z}} \mathbb{C}$$

by

$$\rho(z) = \frac{\log z}{2\pi i} \wedge \frac{\log(1-z)}{2\pi i} + 1 \wedge \frac{\mathcal{R}(z)}{2\pi^2}$$

where $\mathcal{R}(z)$ is the Rogers dilogarithm function defined by

$$\mathcal{R}(z) = \frac{1}{2}\log z \log(1-z) - \int_o^z \frac{\log(1-t)}{t} dt$$

This map restricts to $\mathcal{B}(\mathbb{C})$ to give a map $\rho : \mathcal{B}(\mathbb{C}) \to \mathbb{C}/\mathbb{Q}$. In [30] Neumann and Yang showed that

$$\frac{2\pi^2}{i}\rho(\beta(N)) = vol(N) + iCS(N) \in \mathbb{C}/(i\pi^2\mathbb{Q}).$$

So it is natural to expect that $\overline{A}(l,m)$ is related to the variation of the Bloch invariant of N. There is further evidence given by Ramakrishnan's conjecture from K-theory.

Conjecture 9.1. (**Ramakrishnan's Conjecture**) *$\rho : \mathcal{B}(\mathbb{C}) \to \mathbb{C}/\pi^2\mathbb{Q}$ is injective.*

9.0.18. *Bloch invariant and $\overline{A}_0(l,m)$.* Let us begin by defining the variation of the Bloch invariant.

Definition 9.1. *For $\mathbf{z} \in P_0(N)$ define $\Delta\beta_N(z) = \beta(N) - \beta(N(\mathbf{z}))$ where $\beta(N(\mathbf{z})) = \sum_{i=1}^n [z_i]$. $\Delta\beta(N)$ is called the variation of the Bloch invariant. $\beta(N(\mathbf{z})) \in \mathcal{P}(\mathbb{C})$ in general and $\beta(N(\mathbf{z})) \in \mathcal{B}(\mathbb{C})$ when $\mathbf{z}$ corresponds to a hyperbolic Dehn surgery.*

In view of Ramakrishnan's Conjecture it is natural to make the following conjecture:

Conjecture 9.2. *For a hyperbolic 3-manifold N, $\overline{A}_0(l,m)$ determines the variation of the Bloch invariant $\Delta\beta_N$.*

Let N_1 and N_2 be one-cusped hyperbolic 3-manifolds such that $\overline{A}_{0,N_1}(l,m) = \overline{A}_{0,N_2}(l,m)$. Hence $H_0(N_1) = H_0(N_2)$. We can define a curve $P = \{(\mathbf{z},\mathbf{w}) \in P_0(N_1) \times P_0(N_2) : Hol_{N_1}(z) = Hol_{N_2}(w)\}$. We have the following commutative diagram:

$$\begin{array}{ccc} P & \xrightarrow{p_1} & P_0(N_1) \\ {\scriptstyle p_2}\downarrow & & {\scriptstyle Hol_{N_1}}\downarrow \\ P_0(N_2) & \xrightarrow{Hol_{N_2}} & \mathbb{C}\times\mathbb{C} \end{array} \tag{9.1}$$

where $p_i : P \to P_0(N_i)$ are the projection maps on the i-th coordinate. We can now define the variation of the Bloch invariant for elements of P. For $(\mathbf{z},\mathbf{w}) \in P$ define *variation of Bloch invariant on P* to be $\Delta\beta(\mathbf{z},\mathbf{w}) = \Delta\beta_{N_1}(\mathbf{z}) - \Delta\beta_{N_2}(\mathbf{w})$. It follows from Theorem 4.2 that the maps p_1 and p_2 are birational isomorphisms and that the curve P parametrizes both $P_0(N_1)$ and $P_0(N_2)$. From the results of the previous sections we get:

Theorem 9.1. *Let N_1, N_2 be one-cusped hyperbolic 3-manifolds. If $\overline{A}_{0,N_1}(l,m) = \overline{A}_{0,N_2}(l,m)$ then $\Delta\beta(\mathbf{z},\mathbf{w}) \in \mathcal{B}(\mathbb{C})$ for all $(z,w) \in P$.*

Proof:

$$\begin{aligned}
\mu(\Delta\beta(\mathbf{z},\mathbf{w})) &= \mu(\Delta\beta_{N_1}(\mathbf{z}) - \Delta\beta_{N_2}(\mathbf{w})) \\
&= \mu(\beta(N_1)) - \mu\beta(N_2)) + 2\sum_{i=1}^{n} z_i \wedge (1-z_i) - 2\sum_{j=1}^{m} w_j \wedge (1-w_j) \\
&= l(\mathbf{z}) \wedge m(\mathbf{z}) - l(\mathbf{w}) \wedge m(\mathbf{w}) \text{ by Lemma 8.3} \\
&= 0 \text{ by the diagram 9.1}
\end{aligned}$$

which shows that $\Delta\beta(\mathbf{z},\mathbf{w}) \in \mathcal{B}(\mathbb{C})$ for all $(z,w) \in P$. ∎

From the above Theorem we get a family of Bloch invariants parametrized by a complex curve in the Bloch group. The Bloch Rigidity Conjecture would imply that this family is constant. This implies:

Theorem 9.2. *Bloch Rigidity Conjecture implies Conjecture 9.2.*

The symmetries of $\overline{A_0}(l,m)$ give symmetries on the Bloch invariant. To describe the precise result let us set up some notation. Let N be a one cusped hyperbolic 3-manifold and let $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with $ad-bc=1$ be a symmetry of $\overline{A_0}(l,m)$. This means that $\overline{A_0}(l,m) = \overline{A_0}(l^a m^b, l^c m^d)$ up to multiplication by powers of l and m and that the curve defined by $\overline{A_0}(l,m)$ is invariant under the map $s : \mathbb{C}^2 \to \mathbb{C}^2$ given by $s(l,m) = (l^a m^b, l^c m^d)$. We have a map $S : P_0(N) \to P_0(N)$ such that the following diagram commutes

$$\begin{tikzcd}
P_0(N) \arrow[r, "S"] \arrow[d, "Hol"'] & P_0(N) \arrow[d, "Hol"'] \\
\mathbb{C}^2 \arrow[r, "s"] & \mathbb{C}^2
\end{tikzcd} \tag{9.2}$$

Then we have

Theorem 9.3. *If $\overline{A_0}(l,m) \doteq \overline{A_0}(l^a m^b, l^c m^d)$ then $\beta(N(\mathbf{z})) - \beta(N(S(\mathbf{z}))) \in \mathcal{B}(\mathbb{C})$.*

Proof: As in the proof of proposition 5 we have

$$\begin{aligned}
\mu(\beta(N(\mathbf{z})) - \beta(N(s(\mathbf{z})))) &= 2\sum_{i=1}^{n} z_i \wedge (1-z_i) - 2\sum_{i=1}^{n} S(\mathbf{z})_i \wedge (1 - S(\mathbf{z})_i) \\
&\quad \text{by Lemma 8.3} \\
&= l(\mathbf{z}) \wedge m(\mathbf{z}) - l(S(\mathbf{z})) \wedge m(S(\mathbf{z})) \\
&= l \wedge m - l^a m^b \wedge l^c m^d \text{ by diagram(9.2)} \\
&= l \wedge m - (ad - bc) l \wedge m \\
&= 0
\end{aligned}$$

and hence $\beta(N(\mathbf{z})) - \beta(N(S(\mathbf{z}))) \in \mathcal{B}(\mathbb{C})$. ■

We conjecture that this should be a constant. We show this in the case when $\overline{A}_0(l,m)$ is an equation of a rational curve in the next section.

We have a similar situation for manifolds with more than one cusp. Let us set up some notation before we describe it. Let N by a hyperbolic 3-manifold with h cusps. As described in section 2, $P_0(N)$ has dimension h. For $1 \leq k \leq h$ let $Q_k \subset P_0(N)$ be the subset consisting of parameters corresponding to leaving all but the kth cusp unsurgered and surgering the kth cusp. Q_k is obtained by adding $h-1$ completeness equations, one for each cusp, to the gluing equations and hence $\dim Q_K = 1$ and Q_k is a curve in $P_0(N)$. For $\mathbf{z} \in Q_k$, $Hol(z) = (1, \ldots, l_k, m_k, \ldots, 1)$. Let $q_k : \mathbb{C}^{2h} \to \mathbb{C}^2$ be defined by $q_k(l_1, m_1, \ldots, l_h, m_h) = (l_k, m_k)$, then $q_k \circ Hol(Q_k)$ is a curve in $\mathbb{C}^2$. Let us denote the defining polynomial of this curve by $\overline{A}_0^k(l,m)$. This is the A-polynomial associated to the kth cusp of the manifold N.

Let N_1, N_2 be two hyperbolic 3-manifolds such that $\overline{A}_{0,N_1}^{k_1}(l,m) = \overline{A}_{0,N_2}^{k_2}(l,m)$ As before we can define $Q = \{(\mathbf{z},\mathbf{w}) \in Q_{k_1}(N_1) \times Q_{k_2}(N_2) : q_{k_1} \circ Hol_{N_1}(\mathbf{z}) = q_{k_2} \circ Hol_{N_2}(\mathbf{w})\}$. We can define similarly a variation of the Bloch invariant on Q to be $\Delta\beta(\mathbf{z},\mathbf{w}) = \Delta\beta_{N_1}(\mathbf{z}) - \Delta\beta_{N_2}(\mathbf{w})$. Similar to Theorem 9.1 we have

Theorem 9.4. *Let N_1, N_2 be two hyperbolic 3-manifolds. If $\overline{A}_{0,N_1}^{k_1}(l,m) = \overline{A}_{0,N_2}^{k_2}(l,m)$ then $\Delta\beta(\mathbf{z},\mathbf{w}) \in \mathcal{B}(\mathbb{C})$ for all $(\mathbf{z},\mathbf{w}) \in Q$.*

10. The case for rational curves

In the case that $\overline{A_0}(l,m)$ is a defining equation of a rational curve, we can prove Conjecture 9.2. The key idea in the proof is that if C is a rational curve then the function field $\mathcal{M}(C) = \mathbb{C}(t)$ i.e the field of rational functions over $\mathbb{C}$ and in this case it is a classical result in K-theory that the $\mathcal{B}(\mathbb{C}(t)) = \mathcal{B}(\mathbb{C})$. We will also give an elementary proof of this fact using the 5-term relations following [18] and [41].

10.0.19. *Identity in the function field of any curve.* Let N be a one-cusped hyperbolic 3-manifold and let $k = \mathbb{C}(P_0(N))$. k^* is an abelian group under multiplication. We will denote the coordinate functions z_i and the holonomy functions m and l by the same letters while identifying them as elements of k.

Lemma 10.1. *The coordinate functions $z_1, \ldots, z_n$ and the holonomy functions l and m as elements of k^* satisfy the identity*

$$8 \sum_{i=1}^{n} z_i \wedge (1 - z_i) = 4 \; l \wedge m$$

in $k^* \wedge_{\mathbb{Z}} k^*$

Proof: Let $\mathbf{z} = (\pm z_1, \ldots, \pm z_n, 1 - z_1, \ldots, 1 - z_n) \in (k^*)^{2n}$ be such that the signs on the z_i's are chosen so that the gluing equation always equals 1. Since the gluing equations are defining equations of $P_0(N)$ and m and l are functions on $P_0(N)$, as an 2n-tuple $\mathbf{z}$ satisfies $R(\mathbf{z}) = 0$ for R defined in Section (3.1) and hence by Lemma 7.1 $4 \; \mathbf{z} \bigwedge \mathbf{z} = 2 \; C(\mathbf{z}) \bigwedge C(\mathbf{z})$. Since $C(\mathbf{z}) = (l, m)$ we get $8 \sum_{i=1}^{n} (\pm z_i) \wedge (1 - z_i) = 4 \; (\pm l) \wedge (\pm m)$. Since $2 \; (\pm 1) \wedge f = 1 \wedge f = 0$ the above identity simplifies to $8 \; \sum_{i=1}^{n} z_i \wedge (1 - z_i) = 4 \; (l \wedge m)$. ■

10.0.20. *Bloch group of* $\mathbb{C}(t)$. It follows from [36] that $\mathcal{B}(\mathbb{C}(t)) = \mathcal{B}(\mathbb{C})$. We will give a direct proof following [18] and [41]. The following proposition is proved in [18] and [41]. Let us give the proof for completeness.

Proposition 10.1. *Any rational function in $\mathbb{C}(t)$ is equal to a linear combination of constants and linear polynomials modulo relations (8.1) and (8.2) i.e $\mathcal{P}(\mathbb{C}(t))$ is generated by constants and polynomials of the form $p(z) = a(z + b)$ where $a, b \in \mathbb{C}$ and $a \neq 0$.*

Proof: Observe that in the pre-Bloch group of any field the relations $[x] = -[1/x]$ and $[x] = -[1 - x]$ generate all the other relations in (8.2). For $f(t) = p(t)/q(t)$, where $p(t)$ and $q(t)$ are polynomials with no common factors define $d(f) = \max(\deg(p(t)), \deg(q(t)))$. We say that a rational function $f(t)$ is linear is $f(t) = az + b$. We can write

$$f = \alpha \frac{\prod_{i=1}^{n} (t - a_i)}{\prod_{j=1}^{m} (t - b_j)} \quad \text{and} \quad (1 - f) = \alpha' \frac{\prod_{k=1}^{r} (t - c_k)}{\prod_{j=1}^{m} (t - b_j)}$$

where $\alpha, \alpha' \in \mathbb{C}^*$ and the numerator and denominator of $f(t)$ has no common factors. Using the relation $[x] = -[1/x]$ we can assume that $n \geq m$. Let

$$x = \frac{(c_1 - a_1)(t - b_1)}{(c_1 - b_1)(t - a_1)} \quad \text{and} \quad y = \alpha \frac{(c_1 - a_1)\prod_{i=2}^{n}(t - a_i)}{(c_1 - b_1)\prod_{j=2}^{m}(t - b_j)}$$

Then we get

$$\frac{y}{x} = f(t), \quad (1 - x) = \frac{(a_1 - b_1)(t - c_1)}{(c_1 - b_1)(t - a_1)} \quad \text{and}$$

$$(1-y) = \frac{(c_1 - b_1)\prod_{j=2}^{m}(t - b_j) - \alpha(c_1 - a_1)\prod_{i=2}^{n}(t - a_i)}{(c_1 - b_1)\prod_{j=2}^{m}(t - b_j)} = \frac{(t - c_1)g(t)}{(c_1 - b_1)\prod_{j=2}^{m}(t - c_1))}$$

where $\deg(g(t)) \leq n - 2$. Hence

$$\frac{1 - x}{1 - y} = \frac{(a_1 - b_1)\prod_{j=2}^{m}(t - b_1)}{(t - a_1)g(t)} \quad \text{and} \quad \frac{1 - x^{-1}}{1 - y^{-1}} = \frac{y(1 - x)}{x(1 - y)} = \alpha \frac{(a_1 - b_1)\prod_{i=2}^{n}(t - a_1)}{(t - b_1)g(t)}$$

If $n \geq 2$ we get $d(x), d(y), d(\frac{1-x}{1-y}), d(\frac{1-x^{-1}}{1-y^{-1}}) \leq (n-1)$. Since $\frac{y}{x} = f(t)$ we can express $f(t)$ as a linear combination of rational functions g_i with $d(g_i) < d(f)$ using the 5-term relations (8.1). Proceeding inductively in this way we get $f(t)$ as a linear combination of rational functions g_i with $d(g_i) = 1$. In case $f(t)$ is a polynomial start with $x = \dfrac{(c_1 - a_1)}{t - a_1)}$ and proceed similarly.

If $f(t) = \alpha \dfrac{t - a}{t - b}$ then let $x = 1 - \alpha$ and $y = \dfrac{b - a}{t - a}$. We get

$$\frac{y}{x} = \frac{(b - a)}{(1 - \alpha)(t - a)}, \quad \frac{1 - x^{-1}}{1 - y^{-1}} = \alpha \frac{(b - a)}{(1 - \alpha)(z - b)} \text{ and } \frac{1 - x}{1 - y} = \alpha \frac{t - a}{t - b} = f(t)$$

If $\alpha = 1$ i.e. $f(t) = \dfrac{t - a}{t - b}$ then using relations $[x] = -[1 - x] = -[1/x]$ we get

$$[f] = -[1 - f] = -[\frac{a - b}{t - b}] = [\frac{t - b}{a - b}]$$

Hence using the relation (8.1) and (8.2) any rational function is a linear combination of linear polynomials. ■

From relations (8.2) we can see that $[a(z+b)] = -[1-a(z+b)] = -[-a(z+b-1/a)]$ are the only relations among the generators of $\mathcal{P}(\mathbb{C}(t))$ of the form $[a(z+b)]$, $a \neq 0$. Proposition 10.1 tells us that if $[f] \in \mathcal{P}(\mathbb{C}(t))$ then

$$[f] = \sum_{j=1}^{m} m_j [x_j] + \sum_{i=1}^{n} n_i [a_i(z+b_i)]$$

where $[x_j] \in \mathcal{P}(\mathbb{C})$ and only one of $[a(z+b)]$ or $[-a(z+b-1/a)]$ is included. Let $\mu : \mathcal{P}(\mathbb{C}(t)) \to \mathbb{C}(t)^* \wedge_{\mathbb{Z}} \mathbb{C}(t)^*$ be the map $\mu([f]) = 2\ f \wedge (1-f)$ as defined in Section 3.2. Then by definition $\ker(\mu) = \mathcal{B}(\mathbb{C}(t))$.

Lemma 10.2. *For $[f]$ as above, if $\mu([f]) = 0$ then $n_i = 0$ for all $1 \leq i \leq n$.*

Proof: $\mathbb{C}(t)^* = \mathbb{C}^* \oplus H$ where $H = \oplus_{\alpha \in \mathbb{C}} \mathbb{Z}(z-\alpha)$. So

$$\begin{aligned}
\mathbb{C}(t)^* \wedge_{\mathbb{Z}} \mathbb{C}(t)^* &= (\mathbb{C}^* \oplus H) \wedge_{\mathbb{Z}} (\mathbb{C}^* \oplus H) \\
&= (\mathbb{C}^* \wedge_{\mathbb{Z}} \mathbb{C}^*) \ \oplus\ (\mathbb{C}^* \otimes H) \ \oplus\ (H \wedge_{\mathbb{Z}} H)
\end{aligned}$$

Let $[a(z+b)] \in \mathcal{P}(\mathbb{C}(t))$, then

$$\begin{aligned}
\mu([a(z+b)]) &= 2\ (a(z+b) \wedge (1 - a(z+b))) \\
&= 2\ (a(z+b) \wedge (1 - az - ab)) \\
&= 2\ (a(z+b) \wedge -a(z+b-1/a)) \\
&= 2\ ((a \wedge (-a)) \ \oplus\ (a \otimes (z+b-1/a) \ -\ a \otimes (z+b)) \\
&\quad \oplus ((z+b) \wedge (z+b-1/a))) \\
&= 2\ ((a \otimes (z+b-1/a) -\ a \otimes (z+b)) \oplus ((z+b) \wedge (z+b-1/a))) \\
&\quad \text{since } 2\ (x \wedge (-y)) = 2\ (x \wedge y)
\end{aligned}$$

Since $H = \oplus_{\alpha \in \mathbb{C}} \mathbb{Z}(z-\alpha)$, $H \wedge_{\mathbb{Z}} H$ is a free abelian group with basis $I = \{z - \alpha \wedge z - \beta : \alpha \neq \beta,\ \alpha < \beta \text{ in the lexicographic ordering on } \mathbb{C}\}$. Let $p_H : \mathbb{C}(t)^* \wedge_{\mathbb{Z}} \mathbb{C}(t)^* \to H \wedge_{\mathbb{Z}} H$ be the projection map on $H \wedge_{\mathbb{Z}} H$. Then $p_H(\mu([a(z+b)])) = 2\ (z+b) \wedge (z+b-1/a)$ such that $\pm(z+b) \wedge (z+b-1/a) \in I$. Observe that

$$\begin{aligned}
p_H(\mu(-[-a(z+b-1/a)])) &= -2\ (z+b-1/a) \wedge (z+b) \\
&= 2\ (z+b) \wedge (z+b-1/a) \\
&= p_H(\mu([a(z+b)]))
\end{aligned}$$

Since $\mu([f]) = 0$ we get

$$\begin{aligned} p_H(\mu([f])) &= p_H(\mu(\sum_{i=1}^{n} n_i\ [a_i(z+b_i)])) \\ &= \sum_{i=1}^{n} n_i\ p_H(\mu([a_i(z+b_i)])) \\ &= 2\ \sum_{i=1}^{k} (\pm n_i)\ (z+b_i) \wedge (z+b_i-1/a_i) \\ &= 0 \end{aligned}$$

Since $\pm(z+b_i) \wedge (z+b_i-1/a_i) \in I$, the basis of $H \wedge_{\mathbb{Z}} H$ we get $\pm n_i = 0$ for all i and hence $[f] \in \mathcal{P}(\mathbb{C})$ and since $\mu([f]) = 0$, $[f] \in \mathcal{B}(\mathbb{C})$. ∎

Lemma 10.3. *$\mathcal{P}(\mathbb{C}(t))$ has no 2-torsion and hence no 4-torsion.*

Proof: Let $[f] \in \mathcal{P}(\mathbb{C}(t))$ be such that $2[f] = 0$. Let $[f] = \sum_{j=1}^{m} m_j[x_j] + \sum_{i=1}^{n} n_i[a_i(z+b_i)]$. Since $2[f] = 0$, $2\mu([f]) = 0$ and from the proof of Lemma 10.2 we can see that $n_i = 0$ for all i. Hence $[f] \in \mathcal{P}(\mathbb{C})$. It is proved in [36] and [37] that $\mathcal{P}(\mathbb{C})$ and $\mathcal{B}(\mathbb{C})$ are $\mathbb{Q}$-vector spaces. Hence $2[f] = 0 \Rightarrow [f] = 0$. This shows that $\mathcal{P}(\mathbb{C}(t))$ has no 2-torsion hence no 4-torsion. ∎

We have

Theorem 10.1. $\mathcal{B}(\mathbb{C}(t)) = \mathcal{B}(\mathbb{C})$

Proof: By definition $\mathcal{B}(\mathbb{C}(t)) = \ker(\mu)$. Since $\mathbb{C} \subset \mathbb{C}(t)$, $\mathcal{B}(\mathbb{C}) \subset \mathcal{B}(\mathbb{C}(t))$. By Lemma 10.2 if $[p] \in \mathcal{B}(\mathbb{C}(t))$ then $[p] = \sum_{j=1}^{n} m_j[x_j]$ with $[x_j] \in \mathcal{P}(\mathbb{C})$ for all j. Hence $[p] \in \mathcal{B}(\mathbb{C})$ which implies $\mathcal{B}(\mathbb{C}(t)) \subset \mathcal{B}(\mathbb{C})$. ∎

10.0.21. *Variation of Bloch invariant for rational curves.* Let N_1 and N_2 be one-cusped hyperbolic 3-manifolds with $\overline{A}_{0,N_1}(l,m) = \overline{A}_{0,N_2}(l,m)$. Let $P = \{(\mathbf{z},\mathbf{w}) \in P_0(N_1) \times P_0(N_2) : Hol_{N_1}(\mathbf{z}) = Hol_{N_2}(\mathbf{w})\}$ be the curve defined in Section 3.3 Theorem 9.1. For $(\mathbf{z},\mathbf{w}) \in P$ let $\Delta\beta(\mathbf{z},\mathbf{w}) = \Delta\beta_{N_1}(\mathbf{z}) - \Delta\beta_{N_2}(\mathbf{w})$

Theorem 10.2. *Conjecture 2 is true if $\overline{A}(l,m)$ is a defining equation of a rational curve i.e. if N_1 and N_2 are one-cusped hyperbolic 3-manifolds with $\overline{A}_{0,N_1}(l,m) = \overline{A}_{0,N_2}(l,m)$ and $\overline{A}_{0,N_1}(l,m) = \overline{A}_{0,N_2}(l,m)$ is a defining equation of a rational curve then $\Delta\beta(\mathbf{z},\mathbf{w}) = 0$ which implies that $\Delta\beta_{N_1} = \Delta\beta_{N_2}$.*

Proof: Since $\overline{A}_{0,N_1}(l,m) = \overline{A}_{0,N_2}(l,m)$ is an equation of a rational curve, P is rational and hence the function field $\mathcal{M}(P) \simeq \mathbb{C}(t)$. Now the functions

$z_i(p)$, $w_j(p)$ and $l(p), m(p)$ can be seen as elements of $\mathbb{C}(t)$. Moreover the Bloch variation $\Delta\beta(\mathbf{z}, \mathbf{w})$ can be seen as an element of $\mathcal{P}(\mathbb{C}(t))$. We have

$$\begin{aligned}
\mu(4(\Delta\beta)) &= \mu(4(\Delta\beta_{N_1}) - 4(\Delta\beta N_2)) \\
&= 4\ \mu(\beta(N_1)) - 4\ \mu(\beta(N_2)) + 8 \sum_{i=1}^{n} z_i \wedge (1 - z_i) - 8 \sum_{j=1}^{m} w_j \wedge (1 - w_j) \\
&= 4\ l(\mathbf{z}) \wedge m(\mathbf{z}) - 4\ l(\mathbf{w}) \wedge m(\mathbf{w}) \text{ by Lemma 10.1} \\
&= 0 \text{ by the definition of } P
\end{aligned}$$

Hence we have that $4\ (\Delta\beta) \in \mathcal{B}(\mathbb{C}(t))$. By Theorem 10.1 we have that $4(\Delta\beta) \in \mathcal{B}(\mathbb{C})$. By Lemma 10.3, $\mathcal{P}(\mathbb{C}(t))$ and hence $\mathcal{B}(\mathbb{C})$ has no 4-torsion. Morover it is proved in [36] and [37] that $\mathcal{B}(\mathbb{C})$ is a $\mathbb{Q}$-vector space. Hence $\Delta\beta \in \mathcal{B}(\mathbb{C})$ and equals a constant. This constant can be found out by evaluating at the point in P which corresponds to the complete parameter in $P_0(N_1)$ and $P_0(N_2)$. So $\Delta\beta(\mathbf{z}^0, \mathbf{w}^0) = \Delta\beta_{N_1}(\mathbf{z}^0) - \Delta\beta_{N_2}(\mathbf{w}^0) = (\beta(N_1)) - \beta(N_1)) - (\beta(N_2)) - \beta(N_2)) = 0$. Hence $\Delta\beta_{N_1} = \Delta\beta_{N_2}$. ∎

Corollary 10.1. *For N_1, N_2 satisfying the above hypothesis we have $\beta(N_1(p, q)) = \beta(N_2(p, q))$*

Theorem 10.3. *For N satisfying that $\overline{A}_0(l, m)$ is the equation of a rational curve and $\overline{A}(l, m) = \overline{A}_0(l^a m^b, l^c m^d)$ then $\beta(N(p, q)) = \beta(N(ap+bq, cp+dq))$ and in particular $(N(p, q)) = \mathrm{vol}(N(ap + bq, cp + dq))$.*

10.0.22. *Examples.* The polynomial $\overline{A}_0(l, m)$ of the one-cupsed census manifold m208 has many interesting symmetries. The coefficient matrix of $\overline{A}_0(l, m)$ is:

$$\begin{pmatrix}
 & & & & & & 1 & & \\
 & & & & -2 & -2 & -2 & -2 & \\
 & & 1 & -2 & 29 & -28 & 29 & -2 & 1 \\
 & & -2 & -28 & 2 & 2 & -28 & -2 & \\
 & -2 & 29 & 2 & 12 & 2 & 29 & -2 & \\
 & -2 & -28 & 2 & 2 & -28 & -2 & & \\
1 & -2 & 29 & -28 & 29 & -2 & 1 & & \\
 & -2 & -2 & -2 & -2 & & & & \\
 & & 1 & & & & & &
\end{pmatrix}$$

Assuming the coefficient 12 to be the origin we can see the symmetries of the above matrix as the following matrices in $\mathrm{SL}(2, \mathbb{Z})$:

$$a = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix},\ b = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

a has order 6 and b has order 2. Note that all A-polynomials have the symmetry $\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ which implies $\overline{A}_0(1/l, 1/m) \doteq \overline{A}_0(l, m)$.

Another source of examples is the rigidity of cusps phenomenon studied by Neumann and Reid in [29]. Neumann and Reid construct infinitely many examples of two cusped manifolds such that surgery on one cusp does not affect the other. In particular one obtains infinitely many one-cusped manifolds with the same A-polynomial. In [5] Calegari studies some of these examples in more detail.

11. Cyclotomic edge polynomials

Given a polynomial $G(x,y) = \sum a_{mn}x^m y^n$, the Newton polygon $\mathcal{N}_G$ was defined in Section 3. Every edge of $\mathcal{N}_G$ lies on some line with equation $px + qy = d$. Fix an edge e of $\mathcal{N}_G$. The part of the G corresponding to e is of the form

$$G_e' = \sum_{pm+qn=d} a_{mn}x^m y^n$$

As in Section 2 after the substitution $x = x_1^p$ and $y = tx_1^q$ we get:

$$G_e' = x_1^d \sum_{pm+qn=d} a_{mn}t^n$$

The polynomial $G_e(t)$ obtained from G_e' by dividing by x_1^d is called the *edge polynomial* of G corresponding to the edge e of the Newton polygon $\mathcal{N}_G$. Another striking property of the A-polynomial proved in [7] is that the A-polynomial has cyclotomic edge polynomials. In this section we will prove that the polynomial $H(l,m)$ obtained in Section 2 from the combinatorics of triangulations also has this property. In fact we only need the combinatorial properties defining this polynomial.

11.0.23. *K_2 and the tame symbol.* Let F be a field with a discrete valuation ν. Then the *valuation ring* $\mathcal{O}_\nu = \{x \in F : \nu(x) \geq 0\}$ is a principal ideal domain with the unique maximal ideal $\mathcal{M}_\nu = (\pi)$, where $\pi \in F$ is such that $\nu(\pi) = 1$. The *residue field* of ν is $k_\nu = \mathcal{O}_\nu/\mathcal{M}_\nu$. If $a,\ b \in F$ and $s = \gcd(\nu(a), \nu(b))$, then the element $a^{\nu(b)/s}b^{\nu(a)/s}$ determines a well-defined element of the residue field k_ν. This element is denoted by $\tau_\nu(a,b)$.

There is a homomorphism $d_\nu : K_2(F) \to k_\nu^*$ given by $d_\nu(\{a,b\}) = \tau_\nu(a,b)$. The homomorphism d_ν is called the *tame symbol.* See [25] for more details. Using results of Matsumoto (see [25]) and Bloch (see [19] and [36]) we have the following exact sequence modulo 2:

$$0 \longrightarrow \mathcal{B}(F) \xrightarrow{\ i\ } \mathcal{P}(F) \xrightarrow{\ \mu\ } F^* \wedge_{\mathbb{Z}} F^* \xrightarrow{\text{sym}} K_2(F) \longrightarrow 0 \tag{11.1}$$

where i is the inclusion of the Bloch group $\mathcal{B}(F)$ into the pre-Bloch group $\mathcal{P}(F)$, $\mu([x]) = 2\ (x \wedge (1-x))$ and $\text{sym}(x \wedge y) = \{x, y\}$.

As described in Section 3, given $G(x,y) = \sum a_{mn}x^m y^n$, an edge e of $\mathcal{N}_G$ with the equation $px + qy = d$ and a root α of G_e we obtain a Puiseaux parametrization of some irreducible component C of the curve given by $G(x,y) = 0$. This Puiseaux parametrization also gives us a valuation ν on the function field $F = \mathbb{C}(C)$ of C such that $\nu(x) = p$ and $\nu(y) = q$. The residue field for this valuation is $\mathbb{C}$ and the tame symbol $\tau_\nu(x,y) = \alpha^{\nu(x)}$ where x and y are seen as elements of F.

11.0.24. *Edge polynomials of $H(l,m)$.* Let N be a one-cusped hyperbolic 3-manifold and let $H(N)$ be the holonomy variety as defined in Section 2. Let $H(l,m)$ be the defining polynomial of $H(N)$.

Proposition 11.1. *Let l and m denote elements of F then the symbol $\{l,m\}^8 = 1$ in $K_2(F)$.*

Proof: From the exact sequence 11.1 we have that $\mathrm{sym}(l \wedge m) = \{l,m\}$. Now by the Lemma 10.1 we have $8 \sum z_i \wedge (1-z_i) = 4\ l \wedge m$ and since $\mu(4 \sum [z_i]) = 8 \sum z_i \wedge (1-z_i)$ we have that $(\mathrm{sym} \circ \mu)(\sum [z_i]) = \{l,m\}^4$. Hence by the exactness of the sequence we get $\{l,m\}^4 = 1$ modulo 2 and hence $\{l,m\}^8 = 1$ in $K_2(F)$. ∎

Theorem 11.1. *The edge polynomials of $H(l,m)$ are cyclotomic.*

Proof: Let e be an edge of the Newton polygon $\mathcal{N}_H$ of $H(l,m)$ and let C be the curve in $H(N)$ associated to e. Then given a root α of the edge polynomial H_e associated to e we obtain a valuation ν on the function field $F = \mathbb{C}(C)$. As described in the previous subection we get the tame symbol $d_\nu : K_2(F) \to k_\nu^*$ defined by $d_\nu(f,g) = \tau_\nu(f,g) = \alpha^{\nu(f)}$. Since $\{l,m\}^8 = 1$ by Proposition 11.1 we get $\alpha^{8\nu(l)} = 1$ which implies that α is a root of unity. Since α was arbitrary we get that every root of H_e is a root of unity implying that H_e is cyclotomic. ∎

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