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# TWISTED ALEXANDER POLYNOMIAL FOR FINITELY PRESENTABLE GROUPS

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## INTRODUCTION

LET $\Gamma$ be a finitely presentable group of which a surjective homomorphism $\alpha$ to a (multiplicative) free abelian group with generators $t_1, \ldots, t_r$ is specified. To each linear representation

$$\rho: \Gamma \to GL_n(R)$$

of the group $\Gamma$ over a unique factorization domain $R$ we will assign a rational expression

$$\Delta_{\Gamma,\rho}(t_1, \ldots, t_r)$$

of the indeterminates $t_1, \ldots, t_r$ with coefficients in $R$ called the twisted Alexander polynomial of $\Gamma$ associated to $\rho$. The twisted Alexander polynomial is well-defined up to a factor of $\varepsilon t_1^{e_1} \cdots t_r^{e_r}$, where $\varepsilon \in R^\times$ is a unit of $R$ and $e_1, \ldots, e_r$ are integers.

The twisted Alexander polynomial is a generalization of the Alexander polynomial (cf. [3]) in the following sense. Let $\Gamma$ be a finitely presentable group whose abelianization $\alpha: \Gamma \to \langle t \rangle$ is of rank 1. Then the Alexander polynomial of $\Gamma$ is written as

$$\Delta_\Gamma(t) = (1 - t)\Delta_{\Gamma,\rho}(t),$$

where $\rho$ is the trivial, 1-dimensional representation of $\Gamma$.

We are mainly interested in the case where $\Gamma$ is the group of a knot or of a link and where $\alpha$ is the abelianization. As an invariant of a link we can refine the definition of the twisted Alexander polynomial so that it is well-defined up to a factor of $\varepsilon t_1^{ne_1} \cdots t_r^{ne_r}$ ($\varepsilon \in R^\times$, $e_1, \ldots, e_r \in \mathbb{Z}$), where $n$ is the dimension of the representation space of $\rho$. See Section 5 for the detail.

The twisted Alexander polynomial is not an invariant of a knot or of a link by itself, for it depends not only on the group but also on the representation. One way to get a link invariant out of the twisted Alexander polynomial is to consider representations over a finite field $\mathbb{F}_p$. These "discrete representations" have been studied extensively since [8]. The point here is that there are only finitely many homomorphisms of $\pi_1(S^3 - K)$ to $GL_n(\mathbb{F}_p)$. Therefore we may consider the collection of twisted Alexander polynomials as a link invariant. In Section 6, we show that Kinoshita-Terasaka and Conway's 11 crossing knots are distinguished by the twisted Alexander polynomial in this way.

In [7] X-S. Lin has defined a version of twisted Alexander polynomial for knots using regular Seifert surfaces. He defines the twisted Alexander polynomial as a generator of the

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order ideal of the twisted Alexander module. Thus his definition of the twisted Alexander polynomial corresponds to the numerator of ours. His approach may allow more insight into geometry of knots while ours is toward easy calculations, and generalization to links and to arbitrary finitely presentable groups.

## 1. TIETZE TRANSFORMATIONS

By saying that

$$(x_1, \ldots, x_s \mid r_1, \ldots, r_t) \tag{1.1}$$

is a presentation of a given group $\Gamma$, we mean that a specific surjective homomorphism

$$\phi : F_s \to \Gamma$$

of the free group $F_s = \langle x_1, \ldots, x_s \rangle$ to the group $\Gamma$ is given and that the kernel of the homomorphism $\phi$ is normally generated by the words $r_1, \ldots, r_t \in F_s$.

The Tietze transformation theorem [9] states that one presentation (1.1) of a given group $\Gamma$ can be transformed to any other presentation of $\Gamma$ by an application of a finite sequence of operations of the following types and their inverse operations, called Tietze transformations:

I. To add a consequence $r$ of the relators $r_1, \ldots, r_t$ to the set of relators. The resulting presentation is

$$(x_1, \ldots, x_s \mid r_1, \ldots, r_t, r).$$

II. To add a new generator $x$ and a new relator $xw^{-1}$, where $w$ is any word in $x_1, \ldots, x_s$. Thus the resulting presentation is

$$(x_1, \ldots, x_s, x \mid r_1, \ldots, r_t, xw^{-1}).$$

We will first define the twisted Alexander polynomial for presentations, then prove its invariance under Tietze transformations.

## 2. FREE DIFFERENTIAL CALCULUS

Here we review some basic notions about group derivations. For a systematic treatment of the subject, the reader is referred to [1].

Let $G$ be a group and $\mathbb{Z}G$ its integral group ring. A $\mathbb{Z}$-linear map

$$d : \mathbb{Z}G \to V$$

of $\mathbb{Z}G$ to a left $\mathbb{Z}G$-module $V$ is called a derivation if it satisfies the condition

$$d(uv) = du + u\,dv \quad (\forall u, v \in G). \tag{2.1}$$

From this we can easily obtain

$$d(u^{-1}) = -u^{-1}du \quad (\forall u \in G). \tag{2.2}$$

Let $DG$ denote the left ideal of $\mathbb{Z}G$ generated by the elements of the form $g - 1$ ($g \in G$). The $\mathbb{Z}$-linear map

$$d : \mathbb{Z}G \to DG$$

given by $dg = g - 1$ for $g \in G$ is a derivation of $\mathbb{Z}G$, and is called the universal derivation. It is universal in the following sense: For any derivation

$$f : \mathbb{Z}G \to V$$

of $\mathbb{Z}G$, there is a unique $\mathbb{Z}G$-module homomorphism

$$h : DG \to V$$

such that $h \circ d = f$. In fact $h$ is simply the restriction of $f$ to $DG$.

Let us consider the universal derivation of the free group $F_s$,

$$d : \mathbb{Z}F_s \to DF_s.$$

Since every element $w \in F_s$ is a product of $x_1^{\pm 1}, \ldots, x_s^{\pm 1}$, one can apply (2.1) and (2.2) repeatedly to $dw$ and express it as a $\mathbb{Z}F_s$-linear combination of $dx_1, \ldots, dx_s$ as

$$dw = \sum_{i=1}^{s} \frac{\partial w}{\partial x_i} dx_i. \tag{2.3}$$

The coefficient $\frac{\partial w}{\partial x_i} \in \mathbb{Z}F_s$ is called the free derivative of $w$ with respect to $x_i$. In fact the module $DF_s$ is freely generated by $dx_1, \ldots, dx_s$ over $\mathbb{Z}F_s$. According to the definition of the universal derivation the formula (2.3) means

$$w - 1 = \sum_{i=1}^{s} \frac{\partial w}{\partial x_i}(x_i - 1). \tag{2.4}$$

## 3. TWISTED ALEXANDER POLYNOMIAL FOR GROUPS

Suppose that we are given a finitely presentable group $\Gamma$ and a surjective homomorphism

$$\alpha : \Gamma \to T_r$$

of $\Gamma$ to the free abelian group $T_r = \langle t_1, \ldots, t_r \mid t_i t_j = t_j t_i (\forall i, j) \rangle$ of rank $r \geq 1$. The group ring of the free abelian group $T_r$ over a commutative ring $R$ is called the Laurent polynomial ring of $t_1, \ldots, t_r$, and is denoted by $R[t_1^{\pm 1}, \ldots, t_r^{\pm 1}]$. The homomorphism $\alpha$ induces a ring homomorphism of the integral group ring

$$\tilde{\alpha} : \mathbb{Z}\Gamma \to \mathbb{Z}[t_1^{\pm 1}, \ldots, t_r^{\pm 1}].$$

Let

$$P = (x_1, \ldots, x_s \mid r_1, \ldots, r_t) \tag{3.1}$$

be a presentation of $\Gamma$, and

$$\phi : F_s \to \Gamma$$

the associated homomorphism of the free group $F_s$ to $\Gamma$. Extending the homomorphism $\phi$ linearly to the integral group rings, we obtain a ring homomorphism

$$\tilde{\phi} : \mathbb{Z}F_s \to \mathbb{Z}\Gamma.$$

Let $\rho$ be a representation of $\Gamma$ on a finitely generated free module $V$ over some unique factorization domain $R$; for instance, a finite dimensional vector space $V$ over some field $R$. Choosing a basis for $V$, we may regard $\rho$ as a homomorphism

$$\rho : \Gamma \to GL_n(R),$$

where $n$ is the rank of the representation space $V$. The corresponding ring homomorphism of the integral group ring $\mathbb{Z}\Gamma$ to the matrix algebra $M_n(R)$ of degree $n$ over $R$ is denoted by

$$\tilde{\rho} : \mathbb{Z}\Gamma \to M_n(R).$$

The composition of the ring homomorphism $\tilde{\phi}$ and the tensor product homomorphism

$$\tilde{\rho} \otimes \tilde{\alpha} : \mathbb{Z}\Gamma \to M_n(R[t_1^{\pm 1}, \dots, t_r^{\pm 1}])$$

will be used so often that we introduce a new symbol

$$\Phi = (\tilde{\rho} \otimes \tilde{\alpha}) \circ \tilde{\phi} : \mathbb{Z}F_s \to M_n(R[t_1^{\pm 1}, \dots, t_r^{\pm 1}]). \tag{3.2}$$

We remark that this composition $\Phi$ is also a ring homomorphism.

Let us consider the "big" $t \times s$ matrix $M$ whose $(i, j)$ component is the $n \times n$ matrix

$$\Phi\left(\frac{\partial r_i}{\partial x_j}\right) \in M_n(R[t_1^{\pm 1}, \dots, t_r^{\pm 1}]).$$

This matrix $M$ is called the Alexander matrix of the presentation (3.1) associated to the representation $\rho$. The following proposition, though not needed in the proof of our theorem, illustrates the meaning of the Alexander matrix.

PROPOSITION 1. *Consider the matrix $M$ as a linear map of the module $(R[t_1^{\pm 1}, \dots, t_r^{\pm 1}])^{ns}$ to $(R[t_1^{\pm 1}, \dots, t_r^{\pm 1}])^{nt}$. Then there is a natural one-to-one correspondence between the kernel of $M$ and the set of derivations of $\Gamma$ with values in $(R[t_1^{\pm 1}, \dots, t_r^{\pm 1}])^n$, which is regarded as a $\mathbb{Z}\Gamma$-module via $\tilde{\rho} \otimes \tilde{\alpha}$.*

*Proof.* Every derivation

$$f : \mathbb{Z}\Gamma \to (R[t_1^{\pm 1}, \dots, t_r^{\pm 1}])^n$$

defines a derivation of $F_s$,

$$f \circ \tilde{\phi} : \mathbb{Z}F_s \to (R[t_1^{\pm 1}, \dots, t_r^{\pm 1}])^n.$$

By the universal property of the derivation

$$d : \mathbb{Z}F_s \to DF_s,$$

there is a $\mathbb{Z}F_s$-module homomorphism

$$h : DF_s \to (R[t_1^{\pm 1}, \dots, t_r^{\pm 1}])^n$$

such that $h \circ d = f \circ \tilde{\phi}$. Since $DF_s$ is freely generated by $dx_1, \dots, dx_s$ over $\mathbb{Z}F_s$, such a $\mathbb{Z}F_s$-module homomorphism $h$ is determined exactly by the images

$$v_i = h(dx_i) \in (R[t_1^{\pm 1}, \dots, t_r^{\pm 1}])^n \quad (i = 1, \dots, s).$$

The derivation $h \circ d$ of $F_s$ determined by $v_i$'s descends to a derivation of $\Gamma$ if and only if

$$h(dr_1) = \cdots = h(dr_t) = 0,$$

namely if and only if

$$\sum_{j=1}^{s} \Phi\left(\frac{\partial r_i}{\partial x_j}\right) v_j = 0 \quad (i = 1, \dots, t). \quad \square$$

For $1 \le j \le s$, let us denote by $M_j$ the $t \times (s-1)$ matrix obtained from $M$ by removing the $j$-th column. Now regard $M_j$ as a $tn \times (s-1)n$ matrix with coefficients in $R[t_1^{\pm 1}, \dots, t_r^{\pm 1}]$. For an $(s-1)n$-tuple of indices,

$$I = (i_1, \dots, i_{(s-1)n}) \quad (1 \le i_1 < \cdots < i_{(s-1)n} \le tn),$$

we denote by $M_j^I$ the $(s-1)n \times (s-1)n$ matrix consisting of the $i_k$-th rows of the matrix $M_j$ where $k = 1, \dots, (s-1)n$.

The following two lemmas form the foundation of our definition of twisted Alexander polynomial.

LEMMA 2. *$\det \Phi(1 - x_j) \neq 0$ for some $j$.*

Proof. We can take a generator $x_j$ such that $\alpha(x_j) = t_1^{e_1} \cdots t_r^{e_r} \neq 1$, since the homomorphism $\alpha$ is surjective. Then

$$\det \Phi(1 - x_j) = \det(1 - t_1^{e_1} \cdots t_r^{e_r} \rho \circ \phi(x_j))$$

is some non-zero Laurent polynomial. □

LEMMA 3. *$(\det M_j^I)(\det \Phi(1 - x_k)) = \pm (\det M_k^I)(\det \Phi(1 - x_j))$ for $1 \leq j < k \leq s$ and for any choice of the indices $I$. The sign in the formula is always a $+$ if the degree of the representation $\rho$ is even.*

*Proof.* By interchanging columns if necessary, we may assume that $j = 1$ and $k = 2$. Note that for $i = 1, \ldots, t$, the formula (2.4) implies

$$\sum_{j=1}^{s} \Phi\left(\frac{\partial r_i}{\partial x_j}\right) \Phi(1 - x_j) = 0,$$

hence

$$\Phi\left(\frac{\partial r_i}{\partial x_1}\right) \Phi(1 - x_1) = - \sum_{j=2}^{s} \Phi\left(\frac{\partial r_i}{\partial x_j}\right) \Phi(1 - x_j). \tag{3.3}$$

Let us denote by $\tilde{M}_2$ the matrix obtained from $M_2$ by altering the first $n$ columns by replacing the blocks $\Phi(\frac{\partial r_i}{\partial x_1})$ with $\Phi(\frac{\partial r_i}{\partial x_1})\, \Phi(1 - x_1)$.Thus we have

$$\det \tilde{M}_2^I = (\det M_2^I)(\det \Phi(1 - x_1)),$$

where $\tilde{M}_2^I$ is the matrix consisting of the rows of the matrix $\tilde{M}_2$ indicated by $I$. By (3.3) we may regard the first $n$ columns of $\tilde{M}_2$ as consisting of the blocks $-\sum_{j=2}^{s} \Phi(\frac{\partial r_i}{\partial x_j})\, \Phi(1 - x_j)$. To each of the first $n$ columns of $\tilde{M}_2$, we can add a linear combination of the other $(s - 2)n$ columns and reduce the matrix $\tilde{M}_2$ to $\tilde{M}_1$ whose first $n$ columns consist of the blocks $- \Phi(\frac{\partial r_i}{\partial x_2})\, \Phi(1 - x_2)$. The matrix $\tilde{M}_1$ can also be obtained by multiplying the first $n$ columns of the matrix $M_1$ by $-\Phi(1 - x_2)$ from the right. Therefore,

$$\begin{aligned} \det \tilde{M}_2^I &= \det \tilde{M}_1^I \\ &= \pm (\det M_1^I)(\det \Phi(1 - x_2)). \end{aligned}$$

This completes the proof of Lemma 3. □

COROLLARY 4. *If $\det \Phi(1 - x_j)$ and $\det \Phi(1 - x_k)$ are non-zero Laurent polynomials, then*

$$\frac{\det M_j^I}{\det \Phi(1 - x_j)} = \pm \frac{\det M_k^I}{\det \Phi(1 - x_k)}.$$

*The sign in the formula is always a $+$ if the degree of the representation $\rho$ is even.*

We denote by $Q_j(t_1, \ldots, t_r) \in R[t_1^{\pm 1}, \ldots, t_r^{\pm 1}]$ the greatest common divisor of $\det M_j^I$ for all the choices of the indices $I$. We remark that the Laurent polynomial ring $R[t_1^{\pm 1}, \ldots, t_r^{\pm 1}]$ over a unique factorization domain $R$ is again a unique factorization domain. The Laurent polynomial $Q_j(t_1, \ldots, t_r)$ is well-defined up to a factor of $\varepsilon t_1^{e_1} \cdots t_r^{e_r}$ where $\varepsilon \in R^\times$ is a unit of $R$ and $e_1, \ldots, e_r$ are integers. If $t < s - 1$ then we define $Q_j(t_1, \ldots, t_r)$ to be the zero polynomial.

COROLLARY 5. *If* $\det \Phi(1 - x_j)$ *and* $\det \Phi(1 - x_k)$ *are non-zero Laurent polynomials, then*

$$\frac{Q_j(t_1, \ldots, t_r)}{\det \Phi(1 - x_j)} = \varepsilon t_1^{e_1} \cdots t_r^{e_r} \frac{Q_k(t_1, \ldots, t_r)}{\det \Phi(1 - x_k)} \quad (\varepsilon \in R^\times, e_1, \ldots, e_r \in \mathbb{Z}).$$

*Definition.* By Lemma 2 we can always choose an index $j$ such that $\det \Phi(1 - x_j) \neq 0$. Then we define the twisted Alexander polynomial of the group $\Gamma$ associated to the representation $\rho$ to be the rational expression

$$\Delta_{\Gamma,\rho}(t_1, \ldots, t_r) = \frac{Q_j(t_1, \ldots, t_r)}{\det \Phi(1 - x_j)}.$$

This definition is obviously an abuse of the terminology "polynomial"; I will give some excuses later. Up to a factor of $\varepsilon t_1^{e_1} \cdots t_r^{e_r}$ ($\varepsilon \in R^\times, e_1, \ldots, e_r \in \mathbb{Z}$), the Alexander polynomial is in fact an invariant of the group $\Gamma$, the associated homomorphism $\alpha$, and the representation $\rho$. Namely, let $\Gamma_1$ and $\Gamma_2$ be finitely presentable groups with surjective homomorphisms $\alpha_1 : \Gamma_1 \to T_r$ and $\alpha_2 : \Gamma_2 \to T_r$ respectively. If there is an isomorphism

$$\psi : \Gamma_1 \to \Gamma_2$$

such that $\alpha_1 = \alpha_2 \circ \psi$, then for any representation

$$\rho : \Gamma_1 \to GL_n(R)$$

of $\Gamma_1$, we have

$$\Delta_{\Gamma_1,\rho}(t_1, \ldots, t_r) = \varepsilon t_1^{e_1} \cdots t_r^{e_r} \Delta_{\Gamma_2,\rho\circ\psi^{-1}}(t_1, \ldots, t_r) \quad (\varepsilon \in R^\times, e_1, \ldots, e_r \in \mathbb{Z}).$$

This is due to the following:

THEOREM 1. *The twisted Alexander polynomial* $\Delta_{\Gamma,\rho}(t_1, \ldots, t_r)$ *is independent of the choice of the presentation.*

*Proof.* Suppose that we start from the presentation (3.1), define the Alexander Matrix $M$ by using (3.2), then from it compute the twisted Alexander polynomial $\Delta_{\Gamma,\rho}(t_1, \ldots, t_r)$.

Now suppose instead that we use the presentation

$$P' = (x_1, \ldots, x_s \mid r_1, \ldots, r_t, r) \tag{3.4}$$

obtained from the presentation (3.1) by applying the Tietze transformation of type I. Namely

$$r = \prod_{k=1}^{p} w_k r_{i_k}^{\varepsilon_k} w_k^{-1},$$

where $1 \le i_k \le t$, $w_k \in F_s$, and $\varepsilon_k = \pm 1$ for $1 \le k \le p$. Applying (2.1) and (2.2) we can easily obtain

$$dr = \sum_{k=1}^{p} \left( \prod_{l=1}^{k-1} w_l r_{i_l}^{\varepsilon_l} w_l^{-1} \right) (u_k dr_{i_k} + (1 - w_k r_{i_k}^{\varepsilon_k} w_k^{-1}) dw_k),$$

where

$$u_k = \begin{cases} w_k & \text{if } \varepsilon_k = 1, \\ -w_k r_{i_k}^{-1} & \text{if } \varepsilon_k = -1. \end{cases}$$

Hence

$$\frac{\partial r}{\partial x_j} = \sum_{k=1}^{p} \left( \prod_{l=1}^{k-1} w_l r_{i_l}^{\varepsilon_l} w_l^{-1} \right) \left( u_k \frac{\partial r_{i_k}}{\partial x_j} + (1 - w_k r_{i_k}^{\varepsilon_k} w_k^{-1}) \frac{\partial w_k}{\partial x_j} \right).$$

Since $\Phi(r_i) = 1$ for all $i$, we obtain

$$\Phi\left(\frac{\partial r}{\partial x_j}\right) = \sum_{k=1}^{p} \varepsilon_k \Phi(w_k)\Phi\left(\frac{\partial r_{i_k}}{\partial x_j}\right). \tag{3.5}$$

Let us denote by $M'$ the Alexander matrix obtained from the presentation (3.4). Then the first $tn$ rows of $M'$ are exactly the matrix $M$, and (3.5) above shows that the last $n$ rows of the matrix $M'$ are linear combinations of the first $nt$ rows of $M$. We can then easily see that the twisted Alexander polynomial computed from the matrix $M'$ is the same as the one computed from $M$.

Next suppose that we perform the Tietze transformation of type II to the presentation (3.1) to obtain

$$P' = (x_1, \ldots, x_s, x \mid r_1, \ldots, r_t, xw^{-1}),$$

where $w \in F_s$. The Alexander matrix $M'$ obtained from this presentation $P'$ is of the form

$$M' = \begin{pmatrix} M & 0 \\ * & 1 \end{pmatrix}.$$

Suppose that $\det \Phi(1 - x_j) \neq 0$. Then the determinant of the matrix $M_j'^J$ consisting of the rows of $M_j'$ indicated by the $sn$-tuple

$$J = (i_1, \ldots, i_{sn}) \quad (1 \leq i_1 < \cdots < i_{sn} \leq (t+1)n)$$

can be non-zero only if $J$ is of the form

$$J = (i_1, \ldots, i_{(s-1)n}, tn + 1, \ldots, (t+1)n),$$

and then

$$\det M_j'^J = \det M_j^I$$

where $I = (i_1, \ldots, i_{(s-1)n})$. It is then obvious that the Alexander polynomial computed from the matrix $M'$ is the same as $\Delta_{\Gamma,\rho}(t_1, \ldots, t_r)$. This completes the proof of the theorem. □

Before closing this section, let us remark that the twisted Alexander polynomial does not depend on the choice of the basis for the representation space $V$: Two representations $\rho$ and $\rho'$ are said to be equivalent if there is an automorphism $\psi$ of the representation space $V$ such that $\rho'(\gamma) = \psi \circ \rho(\gamma) \circ \psi^{-1}$ for all $\gamma \in \Gamma$. Then the twisted Alexander polynomials for $\rho$ and $\rho'$ are the same;

$$\Delta_{\Gamma,\rho}(t_1, \ldots, t_r) = \Delta_{\Gamma,\rho'}(t_1, \ldots, t_r).$$

## 4. EXAMPLES

A few examples show what the twisted Alexander polynomial is like. Our first example is quite simple; namely the infinite cyclic group $\Gamma = \langle t \rangle$. The abelianization is the identity map; $\alpha = id : \Gamma \to \langle t \rangle$. Every complex linear representation

$$\rho : \Gamma \to GL_n(\mathbb{C})$$

is determined exactly by the image $A = \rho(t) \in GL_n(\mathbb{C})$ of the generator of $\Gamma$. It is easy to see that

$$\Delta_{\Gamma,\rho}(t) = \frac{1}{\det(1 - tA)}.$$

(This is the zeta function of the linear transformation $A \in GL_n(\mathbb{C})$.)

A not so simple example is the following. Consider a group $\Gamma$ given by

$$\Gamma = \langle x, y \mid xyx = yxy \rangle.$$

This group $\Gamma$ is isomorphic to the group of the trefoil knot $3_1$. It is also known as the braid group $B_3$ of 3 strings.

It is often more convenient to deal with relations rather than relators for computation purposes. A relation $u = v$ $(u, v \in F_s)$ corresponds to the relator $uv^{-1}$. From $d(uv^{-1}) = du - (uv^{-1})dv$, we easily get

$$\Phi\left(\frac{\partial}{\partial x_j}(uv^{-1})\right) = \Phi\left(\frac{\partial}{\partial x_j}(u - v)\right) \quad (j = 1, \ldots, s).$$

This shows that we may use $r = u - v$ instead of $r = uv^{-1}$ for the computation of the Alexander matrix.

Going back to the example, let us write

$$r = xyx - yxy.$$

The free derivatives of $r$ are

$$\frac{\partial r}{\partial x} = 1 - y + xy,$$

and

$$\frac{\partial r}{\partial y} = -1 + x - yx.$$

As the associated homomorphism we take the abelianization

$$\alpha : \Gamma \to \langle t \rangle.$$

It is given by $\alpha(x) = \alpha(y) = t$.

First, let us consider the trivial, 1-dimensional representation over $\mathbb{Z}$,

$$\rho_0 : \Gamma \to GL_n(\mathbb{Z}).$$

Namely, $\rho_0(x) = \rho_0(y) = 1$. The corresponding Alexander matrix is

$$\left(\Phi\left(\frac{\partial r}{\partial x}\right), \Phi\left(\frac{\partial r}{\partial y}\right)\right) = (1 - t + t^2, -1 + t - t^2).$$

We also have

$$\Phi(1 - x) = \Phi(1 - y) = 1 - t.$$

Therefore, the twisted Alexander polynomial of $\Gamma$ associated to $\rho_0$ is

$$\Delta_{\Gamma, \rho_0}(t) = \frac{1 - t + t^2}{1 - t}.$$

Next, we consider the 2-dimensional representation

$$\rho : \Gamma \to GL_2(\mathbb{Z}[s^{\pm 1}])$$

of $\Gamma$ over the Laurent polynomial ring $\mathbb{Z}[s^{\pm 1}]$ known as the reduced Burau representation of the braid group $B_3$. It is given by

$$\rho(x) = \begin{pmatrix} -s & 1 \\ 0 & 1 \end{pmatrix}, \quad \text{and} \quad \rho(y) = \begin{pmatrix} 1 & 0 \\ s & -s \end{pmatrix}.$$

We have

$$\det \Phi\left(\frac{\partial r}{\partial x}\right) = \det\begin{pmatrix} 1-t & -st^2 \\ -st+st^2 & 1+st-st^2 \end{pmatrix}$$
$$= (1-t)(1+st)(1-st^2),$$

and

$$\det \Phi(1-x) = \det\begin{pmatrix} 1+st & -t \\ 0 & 1-t \end{pmatrix}$$
$$= (1-t)(1+st).$$

Therefore, the twisted Alexander polynomial of $\Gamma$ associated to $\rho$ is

$$\Delta_{\Gamma,\rho}(t) = 1 - st^2.$$

## 5. TWISTED ALEXANDER POLYNOMIAL FOR LINKS

Let $L \subset S^3$ be an oriented link in the oriented 3-sphere. Recall that the Wirtinger presentation of the link group $\pi L = \pi_1(S^3 - L)$ is defined as follows: Given a regular projection of the link $L$, assign to each overpass a generator $x_i$, and to each crossing as in Fig. 1, a relator $x_i x_j x_k^{-1} x_j^{-1}$. (The orientation of the undercrossing arc is irrelevant.)

Thus we obtain a presentation of $\pi L$ with $s$ generators and $s$ relators,

$$(x_1, \ldots, x_s, | r_1, \ldots, r_s). \tag{5.1}$$

After some reordering of the indices, the relators satisfy

$$\prod_{i=1}^{s} r_i^{\pm 1} = 1. \tag{5.2}$$

This implies that any one of the relators $r_1, \ldots, r_s$ is a consequence of the other $s-1$ relators. We remove one of the relators, say $r_s$, and call the resulting presentation

$$(x_1, \ldots, x_s, | r_1, \ldots, r_{s-1})$$

the Wirtinger presentation of $\pi L$.

The abelianization of the link group $\pi L$,

$$\alpha : \pi L \to H_1(S^3 - L)$$

is given by assigning to each generator $x_i$ the meridian element $t_c \in H_1(S^3 - L)$ of the corresponding component $c$ of $L$.

Let $\rho$ be a linear representation of the group $\pi L$ over an integral domain $R$. In this case, since the matrix $M_j$ is a square matrix we can simply put

$$Q_j(t_1, \ldots, t_r) = \det M_j.$$

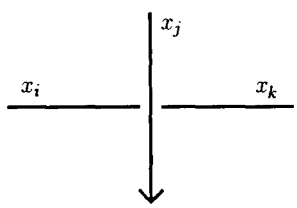



Fig. 1.

*Definition.* Choose an index $j$ $(1 \leq j \leq s)$. Then we call the rational expression

$$\Delta_{L,\rho}(t_1, \ldots, t_r) = \frac{\det M_j}{\det(1 - x_j)} \tag{5.3}$$

obtained from the Wirtinger presentation the twisted Alexander polynomial for the link $L$ associated to the representation $\rho$.

This is, of course, nothing but the twisted Alexander polynomial for the link group $\pi L$. The aim of this definition is the following:

THEOREM 2. *As an invariant of the oriented link type of $L$, the twisted Alexander polynomial $\Delta_{L,\rho}(t_1, \ldots, t_r)$ is well-defined up to a factor of $\varepsilon t_1^{ne_1} \cdots t_r^{ne_r} (\varepsilon \in R^\times, e_1, \ldots, e_r \in \mathbf{Z})$, where $n$ is the degree of the representation $\rho$.*

*Furthermore, if $\rho$ is a unimodular representation, i.e. a homomorphism to the special linear group $SL_n(R)$, then the twisted Alexander polynomial for the link $L$ is well-defined up to a factor of $\pm t_1^{ne_1} \cdots t_r^{ne_r}$ if $n$ is odd, and up to only $t_1^{ne_1} \cdots t_r^{ne_r}$ if $n$ is even.*

Before proving the theorem, let us introduce three new transformations for group presentations:

Ia. To replace one of the relators, $r_i$, by its inverse $r_i^{-1}$.

Ib. To replace one of the relators, $r_i$, by its conjugate $w r_i w^{-1} (w \in F_s)$.

Ic. To replace one of the relators, $r_i$, by $r_i r_k$ $(k \neq i)$.

If a presentation is transformable to another by a finite sequence of operations of types Ia, Ib, Ic, the Tietze transformation of type II, and their inverse operations, we say that the two presentations are strongly Tietze equivalent. This is in fact a stronger equivalence of group presentations; under these transformations the difference between the number of generators and the number of relators remains unchanged.

First, we prove:

LEMMA 6. *All the Wirtinger presentations of a given link $L$ are strongly Tietze equivalent to each other.*

*Proof.* We first remark that the Wirtinger presentations obtained from (5.1) by removing one relator are all strongly Tietze equivalent. This follows easily from (5.2).

The proof of the Lemma is based on the Reidemeister moves for oriented links, which can be stated as follows: A regular projection of a link $L$ can be transformed to any other regular projection of $L$ by applying a finite sequence of local operations of types shown in Fig. 2 called Reidemeister moves.

Let us consider the Reidemeister move of type (1). Let

$$(x_1, \ldots, x_{s-1}, x, y \mid r_1, \ldots, r_{s-1}, yx^{-1}) \tag{5.4}$$

be the Wirtinger presentation of a link $L$ associated to a projection containing a part which looks like the one in the left of Fig. 2(1). If we replace the part of the projection by the middle of Fig. 2(1), the Wirtinger presentation changes to

$$(x_1, \ldots, x_{s-1}, x \mid r'_1, \ldots, r'_{s-1}), \tag{5.5}$$

where the relators $r'_i$ $(i = 1, \ldots, s-2)$ are obtained from $r_i$ by replacing all the occurrences of the letter $y$ by the letter $x$.

The presentation (5.4) is transformable to (5.5) by the operations of types Ia, Ib, Ic, and II as follows: Suppose that the relator $r_i$ contains the letter $y$, and is written as

$$r_i = uy^{\pm 1}v \quad (u, v \in \langle x_1, \ldots, x_{s-2}, x, y \rangle).$$

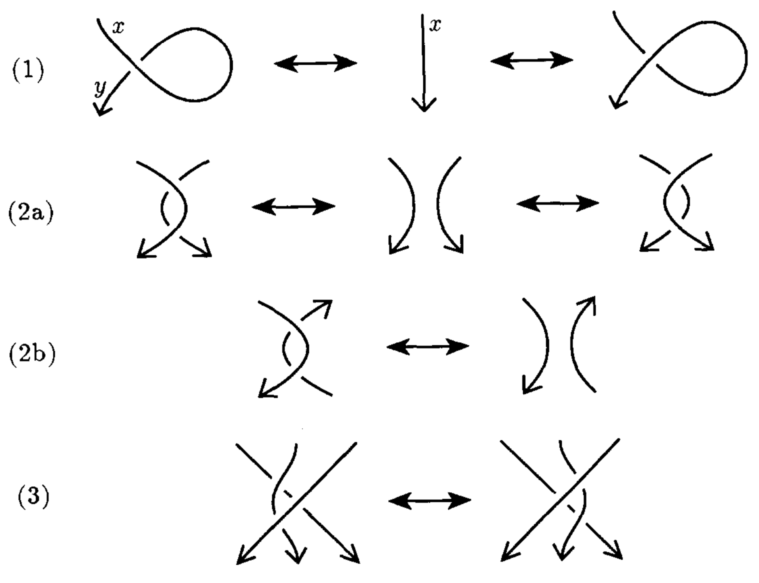



Fig. 2. Reidemeister moves for oriented links.

By applying the operation of type Ia if necessary, we may assume that the relator $r_i$ is of the form

$$r_i = uy^{-1}v.$$

We apply the operations of types Ib and Ic and replace $r_i$ by

$$v^{-1}((vr_iv^{-1})yx^{-1})v = ux^{-1}v.$$

This shows that we can change any occurrences of $y$ to $x$ in the relators $r_1, \ldots, r_{s-2}$ by the operations of types Ia, Ib and Ic. Thus we can transform (5.4) to

$$(x_1, \ldots, x_{s-1}, x, y \mid r'_1, \ldots, r'_{s-1}, yx^{-1}).$$

We can then apply the inverse operation of the Tietze transformation of type II to reduce it to (5.5).

The proofs for the other cases are as straightforward as the above, and are left as an exercise for the reader. □

*Proof of Theorem* 2. Let $M$ denote the Alexander matrix associated to the representation $\rho$ of a presentation

$$(x_1, \ldots, x_s \mid r_1, \ldots, r_{s-1}). \tag{5.6}$$

If we apply the operation of type Ia and replace a relator $r_i$ by its inverse $r_i^{-1}$, then the rows of the matrix $M$ corresponding to the blocks $\Phi(\frac{\partial r_i}{\partial x_j})$ $(j = 1, \ldots, s)$ are replaced by $\Phi(-\frac{\partial r_i}{\partial x_j})$. Therefore, det $M_j$ changes to $(-1)^n$ det $M_j$, where $n$ is the degree of the matrix $\Phi(\frac{\partial r_i}{\partial x_j})$, namely, the degree of the representation $\rho$.

Next, consider applying the operation of type Ib. Suppose that we replace $r_i$ by $wr_iw^{-1}$ $(w \in F_s)$. Since

$$\Phi\left(\frac{\partial}{\partial x_j}(wr_iw^{-1})\right) = \Phi(w)\Phi\left(\frac{\partial r_i}{\partial x_j}\right) \quad (j = 1, \ldots, s),$$

$\det M_j$ is replaced by $(\det\Phi(w))\det M_j$. Notice that $\varepsilon = \det(\rho\circ\phi(w))$ is a unit of $R$ since $\rho\circ\phi(w)\in GL_n(R)$, and $\alpha\circ\phi(w) = t_1^{e_1}\cdots t_r^{e_r}$ for some integers $e_1, \ldots, e_r$. Hence,

$$\begin{aligned}\det\Phi(w) &= (\alpha\circ\phi(w))^n\det(\rho\circ\phi(w))\\ &= \varepsilon t_1^{ne_1}\cdots t_r^{ne_r}.\end{aligned}$$

Suppose that we replace a relator $r_i$ by $r_i r_k$ $(k\neq i)$ and denote by $M'$ the corresponding Alexander matrix. Since

$$\Phi\left(\frac{\partial}{\partial x_j}(r_i r_k)\right) = \Phi\left(\frac{\partial r_i}{\partial x_j}\right) + \Phi\left(\frac{\partial r_k}{\partial x_j}\right) \quad (j = 1, \ldots, s),$$

we can obtain the matrix $M'$ simply by adding the rows corresponding to $r_k$ to the ones corresponding to $r_i$. We have therefore

$$\det M'_j = \det M_j.$$

Lastly, let $M'$ denote the Alexander matrix of the presentation obtained from (5.6) by applying the Tietze transformation of type II. As shown in the proof of Theorem 1, we have

$$\det M'_j = \det M_j.$$

Combining the above results with Corollary 4 and Lemma 6, we see that the twisted Alexander polynomial for $L$ defined by (5.3) is well-defined up to a factor of $\pm\,\varepsilon t_1^{ne_1}\cdots t_r^{ne_r}$, where $\varepsilon\in R^\times$ and $e_1, \ldots, e_r\in\mathbb{Z}$.

If $\rho$ is a unimodular representation, then since

$$\det\Phi(w) = (\alpha\circ\phi(w))^n,$$

the twisted Alexander polynomial for $L$ is well-defined up to a factor of

$$(-1)^{nk} t_1^{ne_1}\cdots t_r^{ne_r} \ (k, e_1, \ldots, e_r\in\mathbb{Z}).$$

This completes the proof of Theorem 2. $\square$

This proof also shows:

COROLLARY 7. *The twisted Alexander polynomial for a link L may be computed from any presentation which is strongly Tietze equivalent to the Wirtinger presentation.*

It still remains to justify the terminology "polynomial."

PROPOSITION 8. *Let $K\subset S^3$ be a knot, and*

$$\rho:\pi K\to GL_n(R)$$

*be a representation of the knot group $\pi K$ satisfyng the following condition*

(C) There is an element $\gamma$ of the commutator subgroup of $\pi K$ such that 1 is not an eigenvalue of $\rho(\gamma)$.

*Then, the twisted Alexander polynomial $\Delta_{K,\rho}(t)$ is a Laurent polynomial with coefficients in the field of quotients of R.*

*Proof.* Let

$$(x_1, \ldots, x_s \mid r_1, \ldots, r_{s-1})$$

be a Wirtinger presentation for the knot $K$, and $M$ the associated Alexander matrix. Choose an element $w\in F_s$ such that $\phi(w) = \gamma$. The presentation

$$(x_1, \ldots, x_s, x \mid r_1, \ldots, r_{s-1}, wx)$$

is easily seen to be strongly Tietze equivalent to the above Wirtinger presentation. We denote the Alexander matrix of this presentation by $M'$; it is of the form

$$M' = \begin{pmatrix} M & 0 \\ * & \Phi(w) \end{pmatrix}.$$

We have

$$\Phi(w) = \rho(\gamma) \in SL_n(R),$$

for $\gamma$ is an element of the commutator subgroup of $\pi K$. Since 1 is not an eigenvalue of this matrix,

$$\det \Phi(1 - w) = \det \tilde{\rho}(1 - \gamma)$$

is a non-zero element of $R$. The twisted Alexander polynomial is then written as

$$\Delta_{K,\rho}(t) = \frac{\det M'_s}{\det \tilde{\rho}(1 - \gamma)},$$

and is therefore a Laurent polynomial. □

PROPOSITION 9. *If $L$ is a link with two or more components, then for any representation*

$$\rho : \pi L \to GL_n(R),$$

*the twisted Alexander polynomial $\Delta_{L,\rho}(t_1, \ldots, t_r)$ is a Laurent polynomial with coefficients in the field of quotients of $R$.*

*Proof.* Let

$$(x_1, \ldots, x_s \mid r_1, \ldots, r_{s-1})$$

be a Wirtinger presentation for the link $L$, and $M$ the associated Alexander matrix. Suppose that $x_j$ and $x_k$ correspond to distinct components of $L$ whose meridian elements in $H_1(S^3 - L)$ are $t_a$ and $t_b$ respectively. Lemma 3 asserts that the polynomial $(\det M_j)(\det \Phi(1 - x_k))$ is divisible by the polynomial $\det \Phi(1 - x_j)$. Since $\det \Phi(1 - x_j)$ is a Laurent polynomial in $t_a$ while $\det \Phi(1 - x_k)$ is one in $t_b$, their greatest common divisor $\delta$ is an element of $R$. It follows that

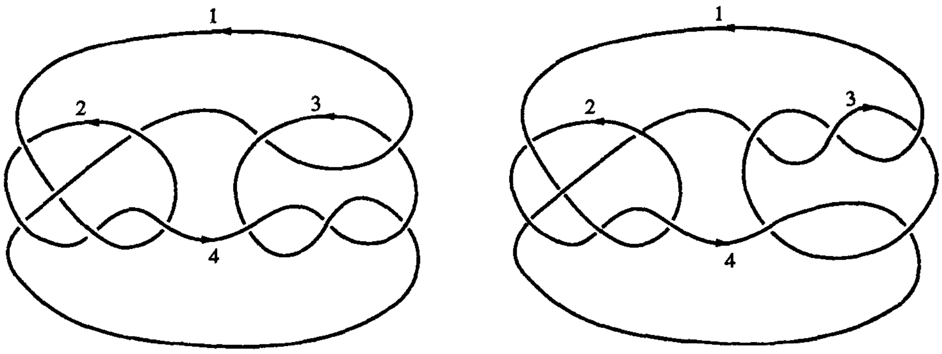
$$\frac{\delta \det M_j}{\det \Phi(1 - x_j)}$$

is a Laurent polynomial. □

## 6. KINOSHITA-TERASAKA AND CONWAY'S KNOTS

Kinoshita-Terasaka 11 crossing knot $KT$ shown in Fig. 3(a) is one of the classical examples of knots with trivial Alexander polynomial ([6]). In [2] J. Conway classified the 11 crossing knots and found another 11 crossing knot $C$ shown in Fig. 3(b) with trivial Alexander polynomial.

Besides their appearance, these two knots have a remarkable similarity. For instance, their Jones [5] and Homfly [4] polynomials coincide. In [8] R. Riley distinguished the two knots by computing some homology invariants of nonabelian coverings of the knot complement associated with homomorphisms of the knot group to $PSL_2(\mathbb{F}_7)$. However, he could not distinguish these knots by merely counting the homomorphisms of the knot groups to $PSL_2(\mathbb{F}_p)$ for primes $p \leq 31$. Recently, I wrote a computer program which




(a) Kinoshita-Terasaka knot $KT$ (b) Conway's knot $C$

Fig. 3.

outputs all the homomorphisms of a given knot group to $SL_2(\mathbb{F}_p)$,

$$\rho : \pi K \to SL_2(\mathbb{F}_p),$$

whose image of a meridian of $K$ is a matrix with trace 2; such homomorphisms are called parabolic representations. The results of application of the program to the knots $KT$ and $C$ endorse Riley's results. The number of equivalence classes of parabolic representations of the groups of these knots to $SL_2(\mathbb{F}_p)$ are exactly the same for all primes $p \leq 181$.

Here, we compute the twisted Alexander polynomial of the two knots associated to parabolic representations to $SL_2(\mathbb{F}_p)$. The Wirtinger presentation for $\pi KT$ is strongly Tietze equivalent to the one with generators $x_1, x_2, x_3, x_4$ and relations

$$\begin{cases} x_1x_2x_1^{-1} = x_4x_2x_4x_2^{-1}x_4^{-1}, \\ x_4x_2x_4^{-1} = x_2^{-1}x_3x_1x_3^{-1}x_2x_1x_2^{-1}x_3x_1^{-1}x_3^{-1}x_2, \\ x_1x_3x_1^{-1} = x_4x_3x_4x_3^{-1}x_4^{-1}. \end{cases} \tag{6.1}$$

That for $\pi C$ is strongly Tietze equivalent to the presentation with generators $x_1, x_2, x_3, x_4$ and relations

$$\begin{cases} x_1x_2x_1^{-1} = x_4x_2x_4x_2^{-1}x_4^{-1}, \\ x_4x_2x_4^{-1} = x_2^{-1}x_3^{-1}x_1^{-1}x_3x_1x_3x_2x_1x_2^{-1}x_3^{-1}x_1^{-1}x_3^{-1}x_1x_3x_2, \\ x_1x_3x_1^{-1} = x_4x_3^{-1}x_1x_3x_4^{-1}. \end{cases} \tag{6.2}$$

These knots $KT$ and $C$ are in fact 3-bridge knots, and their groups are generated by three elements; in both (6.1) and (6.2), it is easy to see that one can use the first two relations to write $x_4$ in terms of the other three generators. Therefore we can, if we like, reduce the presentations to ones with only three generators; but with much lengthier relations.

There are two non-trivial parabolic representations $\theta_1$ and $\theta_2$ of the group $\pi KT$ to $SL_2(\mathbb{F}_7)$, up to equivalence of representations. These representations, the images of the longitude $\ell$ commuting with $\phi(x_1)$, and the twisted Alexander polynomials are as follows:

$$\theta_1(\phi(x_1)) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \theta_1(\phi(x_2)) = \begin{pmatrix} 1 & 0 \\ 6 & 1 \end{pmatrix}, \quad \theta_1(\phi(x_3)) = \begin{pmatrix} 3 & 1 \\ 3 & 6 \end{pmatrix},$$

$$\theta_1(\phi(x_4)) = \begin{pmatrix} 3 & 4 \\ 6 & 6 \end{pmatrix}, \quad \theta_1(\ell) = \begin{pmatrix} 6 & 1 \\ 0 & 6 \end{pmatrix},$$

$$\Delta_{KT,\theta_1}(t) = 6 + 3t + 4t^2 + 6t^3 + 4t^4 + 3t^5 + 6t^6$$
$$= 6(1 + t^2)^2(1 + 4t + t^2),$$
$$\theta_2(\phi(x_1)) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \theta_2(\phi(x_2)) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \theta_2(\phi(x_3)) = \begin{pmatrix} 1 & 0 \\ 6 & 1 \end{pmatrix},$$
$$\theta_2(\phi(x_4)) = \begin{pmatrix} 3 & 4 \\ 6 & 6 \end{pmatrix}, \quad \theta_2(\ell) = \begin{pmatrix} 1 & 5 \\ 0 & 1 \end{pmatrix},$$
$$\Delta_{KT,\theta_2}(t) = 6 + 2t^2 + 5t^3 + 2t^4 + 6t^6$$
$$= 6(6 + t)^4(1 + 4t + t^2).$$

Notice that the twisted Alexander polynomials above are defined up to multiplication by a power of $t^2$.

The group $\pi C$ of the knot $C$ also has two nontrivial parabolic representations $\theta'_1$ and $\theta'_2$ up to equivalence. These representations and their twisted Alexander polynomials are given by

$$\theta'_1(\phi(x_1)) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \theta'_1(\phi(x_2)) = \begin{pmatrix} 1 & 0 \\ 6 & 1 \end{pmatrix}, \quad \theta'_1(\phi(x_3)) = \begin{pmatrix} 1 & 0 \\ 5 & 1 \end{pmatrix},$$
$$\theta'_1(\phi(x_4)) = \begin{pmatrix} 3 & 4 \\ 6 & 6 \end{pmatrix}, \quad \theta'_1(\ell) = \begin{pmatrix} 6 & 1 \\ 0 & 6 \end{pmatrix},$$
$$\Delta_{C,\theta'_1}(t) = 6 + 2t + 6t^3 + 4t^4 + 3t^5 + 4t^6 + 6t^7 + 2t^9 + 6t^{10}$$
$$= 6(1 + 4t + t^2)(2 + 5t + 5t^2 + 2t^3 + t^4)(4 + t + 6t^2 + 6t^3 + t^4),$$
$$\theta'_2(\phi(x_1)) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \theta'_2(\phi(x_2)) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \theta'_2(\phi(x_3)) = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix},$$
$$\theta'_2(\phi(x_4)) = \begin{pmatrix} 4 & 2 \\ 6 & 5 \end{pmatrix}, \quad \theta'_2(\ell) = \begin{pmatrix} 1 & 5 \\ 0 & 1 \end{pmatrix},$$
$$\Delta_{C,\theta'_2}(t) = 6 + 5t + 6t^2 + 3t^4 + 2t^5 + 3t^6 + 6t^8 + 5t^9 + 6t^{10}$$
$$= 6(3 + t)^2(5 + t)^2(6 + t)^4(1 + 4t + t^2).$$

Thus the groups $\pi KT$ and $\pi C$ are not isomorphic to each other.

Finding all the parabolic presentations of a group like the above to $SL_2(\mathbb{F}_p)$ for primes $p$ up to, say 31 takes several seconds on a Macintosh† IIfx. To compute the twisted Alexander polynomials, we used Mathematica‡; it takes about 10 seconds to compute one for the knot $KT$, and about 20 seconds for the knot $C$.

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‡ Mathematica is a trademark of Wolfram Research Inc.

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