Scissors Congruence & Hilbert's 3rd Problem

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CSI Math Club Talk

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Scissors Congruence

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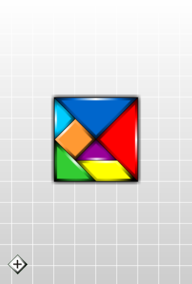
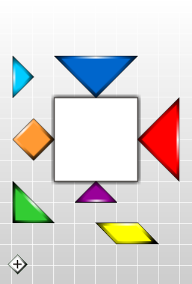
Example: Tangrams





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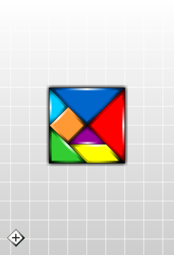
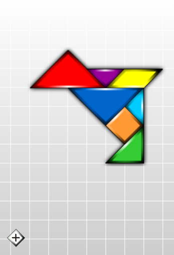
Example: All the polygons below are scissors congruent.

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Two pictures of Euclid

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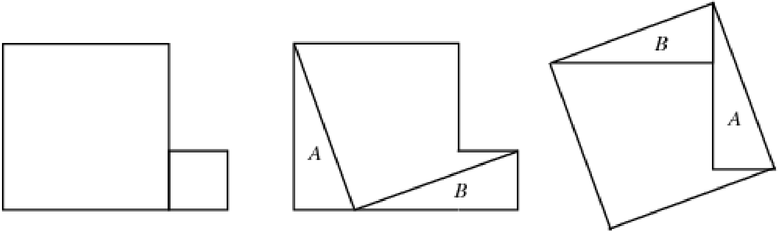
Two pictures of Euclid

The idea of scissors congruence goes back to Euclid. By "equal area" Euclid meant scissors congruent (not in that terminology). In fact Euclid's proof of the Pythagorean Theorem partitions the three squares into triangles with equal areas. Euclid's "geometric algebra" will also remind you of scissors congruence (groups).

Pythagorean Theorem

Scissors Congruent proof of the Pythagorean Theorem.



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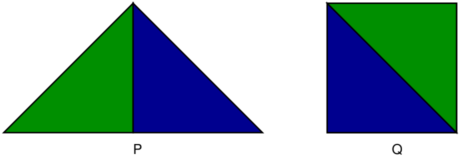

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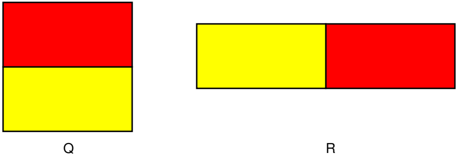

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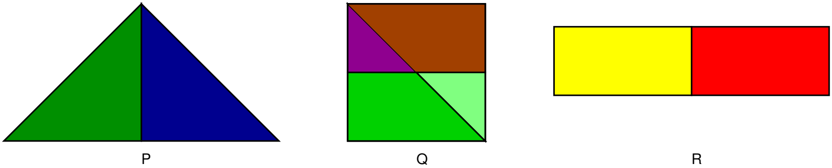

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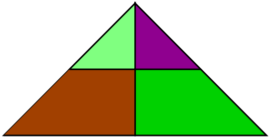
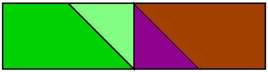


P



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We will see two proofs of this theorem.

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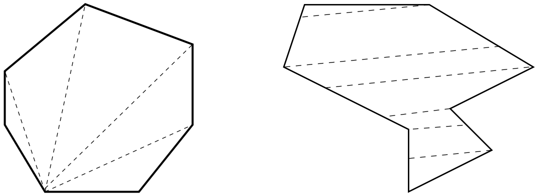

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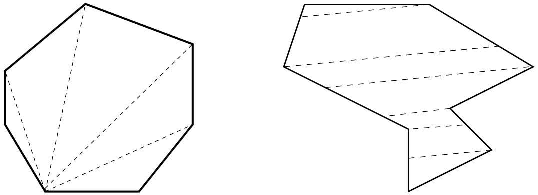


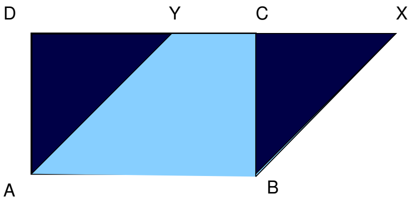
For a polygon, choose a line of slope m which is distint from the slopes of all its sides. Lines of slope m through the vertices of the polygon decompose it into triangles and trapezoids, which again can be decomposed into acute angled triangles. ■

First Proof

Step 2: Any two parallelograms with same base and height are scissors congruent. The same is true for triangles.

Proof: Let $ABCD$ be a rectangle with base AB and height AD. Let $ABXY$ be a parallelogram with height AD. Assume $|DY| \leq |DC|$. Then

$$ABCD \sim_{sc} AYD + ABCY \sim_{sc} ABCY + BXC \sim_{sc} ABXY.$$



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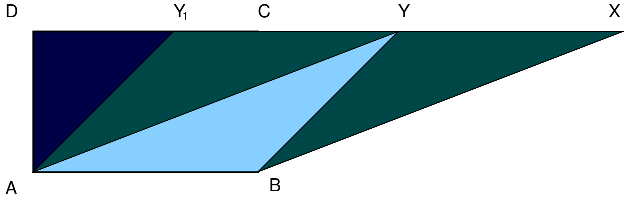
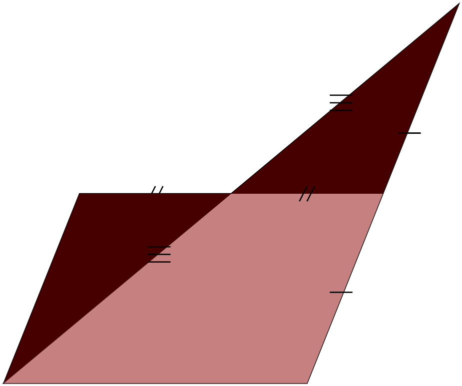
If $|DY| > |DC|$, then cutting along the diagonal BY and regluing the triangle BXY, we obtain the scissors congruent parallelogram $ABYY_1$ such that $|DY_1| = |DY| - |DC|$. Continuing this process k times, for $k = [|DY|/|DC|]$, we obtain the parallelogram $ABY_{k-1}Y_k$ such that $|DY_k| < |DC|$, which is scissors congruent to $ABCD$ as above.



First Proof

Since any triangle is scissors congruent to a parallelogram with the same base and half height, this implies that any two triangles with same base and height are scissors congruent. ■.



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\begin{aligned}
& \mathrm{Area}(ABC) = \mathrm{Area}(AXY) \\
\implies & \frac{|AB||AC|}{2} = \frac{|AY||AX|}{2} \\
\implies & \frac{|AY|}{|AC|} = \frac{|AB|}{|AX|} \\
\implies & ABY \sim AXC \text{ SAS test}
\end{aligned}
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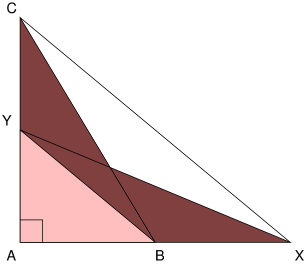

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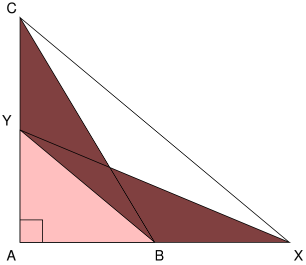

This implies BY is parallel to XC. Hence triangles BYC and BYX have same base and same height which implies by Step 2 that they are scissors congruent.

$$ABC \sim_{sc} ABY + BYC \sim_{sc} ABY + BYX \sim_{sc} AXY.$$

First Proof

To complete the proof, any triangle T is scissors congruent to a right triangle with height 2 and base equal to the area of T, which is scissors congruent to a rectangle with unit height and base equal to area of T. Lets denote such a rectangle by R_x where x is its area ($=$ lenght of the base).

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Thus for any polygon P ,

$$\begin{aligned}
P &\sim_{sc} T_1 + \ldots + T_n \text{ by Step 1} \\
&\sim_{sc} R_{\mathrm{Area}(T_1)} + \ldots + R_{\mathrm{Area}(T_n)} \text{ by Step 3} \\
&\sim_{sc} R_{\mathrm{Area}(T_1)+\ldots+\mathrm{Area}(T_n)} \text{ by laying rectangles side by side} \\
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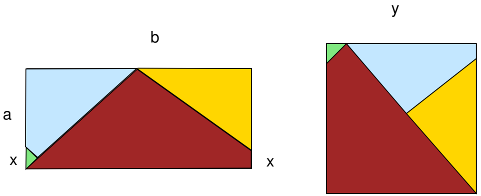
Thus polygons with equal area are scissors congruent to the same rectangles and hence to each other. ■

Step 1 & 2 same as before.

Step 3: A rectangle is scissors congruent to a square of the same area.

Proof:



$$\text{where } x = a - \frac{\sqrt{a(b-a)}}{a}(b - \sqrt{a(b-a)}), \;\; y = \sqrt{ab}$$

We need to verify the equation

$$\frac{x\sqrt{a(b-a)}}{\sqrt{ab}} + \sqrt{(b - \sqrt{a(b-a)})^2 + (a-x)^2} = \sqrt{ab}$$

Second Proof

From Steps 1,2 & 3 we know that any triangle T is scissors congruent to a square, denoted by say $S_{\mathrm{Area}(T)}$. So for any polygon P,

$$\begin{aligned} P &\sim_{sc} T_1 + \ldots + T_n \text{ by Step 1} \\ &\sim_{sc} S_{\mathrm{Area}(T_1)} + \ldots + S_{\mathrm{Area}(T_n)} \text{ by Step 3} \\ &\sim_{sc} S_{\mathrm{Area}(T_1)+\ldots+\mathrm{Area}(T_n)} \text{ by Pythagorean Theorem} \\ &\sim_{sc} S_{\mathrm{Area}(P)} \text{ by Step 1} \end{aligned}$$

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Scissors Congruence in 3 dimensions

A polyhedron is a solid in $\mathbb{E}^3$ whose faces are polygons.

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Anybody interested in making an animation of this ? Please let me know

Hilbert's Third Problem

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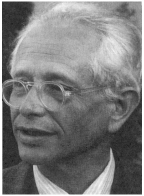
Hilbert made clear that he expected a negative answer.

Solution to Hilbert's Third Problem



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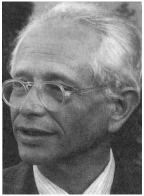

The negative answer to Hilbert's Third problem was provided in 1902 by Max Dehn.

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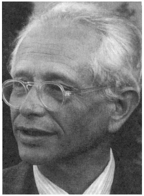
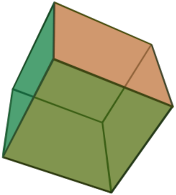
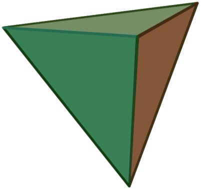


The negative answer to Hilbert's Third problem was provided in 1902 by Max Dehn.

Dehn showed that the regular tetrahedron and the cube of the same volume were not scissors congruent.



$\not\sim_{sc}$



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Dehn defined a new invariant of scissors congruence, now known as the Dehn invariant.

Dehn invariant

For an edge e of a polyhedron P, let $\ell(e)$ and $\theta(e)$ denote its length and dihedral angles respectively. The Dehn invariant $\delta(P)$ of P is

$$\delta(P) = \sum_{\text{all edges } e \text{ of } P} \ell(e) \otimes \theta(e) \in \mathbb{R} \otimes (\mathbb{R}/\pi\mathbb{Q})$$

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The $\otimes$ symbol takes care that $\delta(P)$ does not change when you cut along an edge or cut along an angle i.e. $\delta(P)$ in an invariant of scissors congruence.

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Dehn's solution

- In $\delta(P)$, dihedral angles which are rationals multiples of π are 0 !
- $\delta(\text{unit cube}) = 12 \times 1 \otimes (\pi/2) = 0$ since $\pi/2 = 0 \in \mathbb{R}/\pi\mathbb{Q}$
- For a regular tetrahedra with unit volume, the lengths of all its sides is some positive number a and all its dihedral angles are α where $\cos(\alpha) = 1/3$.
- $\delta(\text{tetrahedra}) = 6 \times a \otimes \arccos(\frac{1}{3})$
- $\dfrac{\arccos(\frac{1}{3})}{\pi}$ is irrational ! (needs proof)
- $\delta(\text{unit cube}) = 0 \neq 6 \times a \otimes \arccos(\frac{1}{3}) = \delta(\text{tetrahedra})$
- Thus the unit cube and the unit tetrahedra are not scissors congruent !

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- Neumann used a "complexified" Dehn invariant in $\mathbb{H}^3$ to define invariants of hyperbolic 3-manifolds.

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