Scissors Congruence &
Hilbert’s 3rd Problem

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CSI Math Club Talk

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A **polygonal decomposition** of a polygon $P$ in the Euclidean plane is a finite collections of polygons $P_1, P_2, \ldots, P_n$ whose union is $P$ and which pairwise intersect only in their boundaries.
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**Example:** Tangrams
Scissors Congruence

Polygons $P$ and $Q$ are **scissors congruent** if there exist polygonal decompositions $P_1, \ldots, P_n$ and $Q_1, \ldots, Q_n$ of $P$ and $Q$ respectively such that $P_i$ is congruent to $Q_i$ for $1 \leq i \leq n$. In short, two polygons are scissors congruent is one can be cut up and reassembled into the other. Let us denote scissors congruence by $\sim_{sc}$. We will write $P \sim_{sc} P_1 + P_2 + \ldots + P_n$. 

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The idea of scissors congruence goes back to Euclid. By “equal area” Euclid meant scissors congruent (not in that terminology). In fact Euclid’s proof of the Pythagorean Theorem partitions the three squares into triangles with equal areas. Euclid’s “geometric algebra” will also remind you of scissors congruence (groups).
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Scissors Congruent proof of the Pythagorean Theorem.
Properties of Scissors Congruence

- If $P \sim_{sc} Q$ then $\text{Area}(P) = \text{Area}(Q)$. 

Transitivity follows by juxtaposing the two decompositions of $Q$ and using the resulting common sub-decomposition of $Q$ to reassemble into $P$ and $R$, thus showing that $P \sim_{sc} R$. 

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![Diagram](image)
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![Diagram of scissors congruence](image)
Properties of Scissors Congruence

Theorem (Bolyai-Gerwien 1833)

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We will see two proofs of this theorem.
Step 1: Every polygon has a polygonal decomposition into triangles, in fact into acute angled triangles.
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For a polygon, choose a line of slope $m$ which is distinct from the slopes of all its sides. Lines of slope $m$ through the vertices of the polygon decompose it into triangles and trapezoids, which again can be decomposed into acute angled triangles.
Step 2: Any two parallelograms with same base and height are scissors congruent. The same is true for triangles.

Proof: Let $ABCD$ be a rectangle with base $AB$ and height $AD$. Let $ABXY$ be a parallelogram with height $AD$. Assume $|DY| \leq |DC|$. Then

$$ABCD \sim_{sc} AYD + ABCY \sim_{sc} ABCY + BXC \sim_{sc} ABXY.$$
If $|DY| > |DC|$, then cutting along the diagonal $BY$ and regluing the triangle $BXY$, we obtain the scissors congruent parallelogram $ABYY_1$ such that $|DY_1| = |DY| - |DC|$. Continuing this process $k$ times, for $k = \lceil |DY|/|DC| \rceil$, we obtain the parallelogram $ABY_{k-1}Y_k$ such that $|DY_k| < |DC|$, which is scissors congruent to $ABCD$ as above.
Since any triangle is scissors congruent to a parallelogram with the same base and half height, this implies that any two triangles with same base and height are scissors congruent. ■
First Proof

Step 3: Any two triangles with same area are scissors congruent.

Proof: By Step 2, we can assume both the triangles are right angles triangles.

\[\text{Area}(\triangle ABC) = \text{Area}(\triangle AXY)\]
\[\Rightarrow \frac{|AB|}{|AC|} = \frac{|AY|}{|AX|}\]
\[\Rightarrow \frac{|AY|}{|AC|} = \frac{|AB|}{|AX|}\]
\[\Rightarrow \triangle ABY \sim \triangle AXC\]

This implies \( BY \) is parallel to \( XC \). Hence triangles \( \triangle BYC \) and \( \triangle BYX \) have same base and same height which implies by Step 2 that they are scissors congruent.

\[\triangle ABC \sim \text{sc} \triangle ABY + \triangle BYC \sim \text{sc} \triangle ABY + \triangle BYX \sim \text{sc} \triangle AXY.\]

\[\blacksquare\]
Step 3: Any two triangles with same area are scissors congruent.

Proof: By Step 2, we can assume both the triangles are right angles triangles.

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\text{Area}(ABC) = \text{Area}(AXY)
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\frac{|AB||AC|}{2} = \frac{|AY||AX|}{2}
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|AY| = |AB|
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This implies $BY$ is parallel to $XC$. Hence triangles $BYC$ and $BYX$ have same base and same height which implies by Step 2 that they are scissors congruent.

\[
ABC \sim_{sc} ABY + BYC \sim_{sc} ABY + BYX \sim_{sc} AXY.
\]
First Proof

To complete the proof, any triangle $T$ is scissors congruent to a right triangle with height 2 and base equal to the area of $T$, which is scissors congruent to a rectangle with unit height and base equal to area of $T$. Let's denote such a rectangle by $R_x$ where $x$ is its area (= length of the base).
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Thus for any polygon $P$,

\[
P \sim_{sc} T_1 + \ldots + T_n \text{ by Step 1} \\
\sim_{sc} R_{\text{Area}(T_1)} + \ldots + R_{\text{Area}(T_n)} \text{ by Step 3} \\
\sim_{sc} R_{\text{Area}(T_1)} + \ldots + \text{Area}(T_n) \text{ by laying rectangles side by side} \\
\sim_{sc} R_{\text{Area}(P)} \text{ by Step 1}
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Thus polygons with equal area are scissors congruent to the same rectangles and hence to each other.
Second Proof

**Step 1 & 2** same as before.

**Step 3:** A rectangle is scissors congruent to a square of the same area.

*Proof:*

![Diagram of a rectangle and a square with labels a, b, x, y, and x](image)

where

\[ x = a - \frac{\sqrt{a(b - a)}}{a}(b - \sqrt{a(b - a)}), \quad y = \sqrt{ab} \]

We need to verify the equation

\[
\frac{x\sqrt{a(b - a)}}{\sqrt{ab}} + \sqrt{(b - \sqrt{a(b - a)})^2 + (a - x)^2} = \sqrt{ab}
\]
From Steps 1, 2 & 3 we know that any triangle $T$ is scissors congruent to a square, denoted by say $S_{\text{Area}}(T)$. So for any polygon $P$,

\[ P \sim_{sc} T_1 + \ldots + T_n \text{ by Step 1} \]
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A polyhedron is a solid in $\mathbb{E}^3$ whose faces are polygons.

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If \( P \sim_{sc} Q \) then \( \text{Volume}(P) = \text{Volume}(Q) \).
Scissors Congruence in 3 dimensions

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Hilbert’s Third Problem

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Are polyhedra in $\mathbb{R}^3$ of same volume scissors congruent?

Hilbert made clear that he expected a negative answer.
The negative answer to Hilbert's Third problem was provided in 1902 by Max Dehn. Dehn showed that the regular tetrahedron and the cube of the same volume were not scissors congruent.
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Dehn defined a new invariant of scissors congruence, now known as the Dehn invariant.

**Dehn invariant**

For an edge \( e \) of a polyhedron \( P \), let \( \ell(e) \) and \( \theta(e) \) denote its length and dihedral angles respectively. The Dehn invariant \( \delta(P) \) of \( P \) is

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\delta(P) = \sum_{\text{all edges } e \text{ of } P} \ell(e) \otimes \theta(e) \in \mathbb{R} \otimes (\mathbb{R}/\pi \mathbb{Q})
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The $\otimes$ symbol takes care that $\delta(P)$ does not change when you cut along an edge or cut along an angle i.e. $\delta(P)$ is an invariant of scissors congruence.
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- In $\delta(P)$, dihedral angles which are rationals multiples of $\pi$ are $0$!
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\[\delta(\text{unit cube}) \neq 6 \times a \otimes \arccos\left(\frac{1}{3}\right) = \delta(\text{tetrahedra})\]

Thus the unit cube and the unit tetrahedra are not scissors congruent!
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Further Comments

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- Does volume and Dehn invariant determine scissors congruence in $\mathbb{E}^3$? Yes they do! Sydler answered this question in 1965. This question is known as the “Dehn invariant sufficiency” problem.
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“Dehn invariant sufficiency” is still open for 3-dimensional spherical geometry $S^3$ and hyperbolic geometries $H^3$ and in higher dimensions.
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- Dupont, Sah, Parry, Suslin etc gave relations between scissors congruences and $K$-theory of fields.
- Neumann used a “complexified” Dehn invariant in $\mathbb{H}^3$ to define invariants of hyperbolic 3-manifolds.
References

4. Scissors Congruence by Efton Park, Seminar Notes, Texas Christian University.
6. Tangram pictures taken from the iPhone application LetsTans http://www.letstans.com/.