INTRODUCTION TO NON-POSITIVE CURVATURE

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Abstract. Aspects of negative and non-positive curvature have been the focus of much research, especially in recent years. We will present a combinatorial condition which captures curvature upper bounds for general metric spaces and proceed to prove some basic results. Our eventual goal will be to address metric cube complexes and give a simple, combinatorial condition to determine when they are non-positively curved. These notes largely follow Martin Bridson and Andre Haefliger's *Metric Spaces of Non-positive Curvature*, chapters 1 and 2.

1. Model Spaces

For each $\kappa \in \mathbb{R}$ and each $n \in \mathbb{N}$ we want space whose sectional curvature is constantly κ. We break this in to three possibilities, depending on κ. (These are like the 8 model geometries in three dimensions)

- If $\kappa = 0$ then the obvious choice of homogeneous, isotropic metric space is $\mathbb{E}^n$, the Euclidean n-space with the standard Euclidian metric.
- If $\kappa = 1$, then M_1^n is the standard unit n-sphere with the round metric. More precisely,
$$\mathbb{S}^n = \{(x_1, \dots, x_{n+1} : x_1^2 + \cdots + x_n^2 = 1\}$$
and for $A, B \in S^n$ we say $\cos d(A, B) = \langle A \mid B \rangle$.
- If $\kappa = -1$ we use the standard hyperbolic n-space. Again, more precisely, we take $\mathbb{H}^n = \{(x_1, \dots, x_n\} : x_n > 0\}$ with the Riemannian metric
$$ds^2 = \frac{dx_1^2 + \cdots + dx_n^2}{x_n^2}.$$

For $\mathbb{H}^n$, we can also use the hyperboloid model
$$\mathbb{I}^n = \{(x_1, \dots, x_{n+1}) : x_1^2 + x_2^2 + \cdots + x_n^2 - x_{n+1}^2 = 1\}$$
with the metric for $A, B \in \mathbb{I}^n$ given by $\cosh d(A, B) = \langle A \mid B \rangle_{(n,1)}$, where the subscript means the standard quadratic form of signature $(n, 1)$.

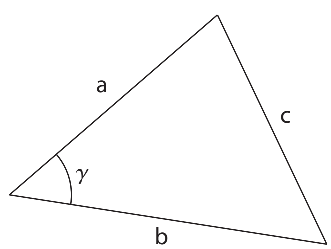
Now, if $\kappa < 0$, we take $\mathbb{H}^n$ and scale the metric by $\frac{1}{\sqrt{-\kappa}}$. So $d_\kappa(A, B) = \frac{d_{-1}(A,B)}{\sqrt{-\kappa}}$. This is the same as replacing the 1 in the hyperboloid model with $\frac{1}{\kappa}$.

If $\kappa > 0$ we take $\mathbb{S}^n$ and scale the metric by $\frac{1}{\sqrt{\kappa}}$. So $d_\kappa(A,B) = \frac{d_1(A,B)}{\sqrt{\kappa}}$. This is the same as the sphere of radius $\frac{1}{\sqrt{\kappa}}$.

Remark 1.1. Call the model spaces about M^n_κ. The case $\kappa > 0$ behaves very differently form the case $\kappa \leq 0$.

- If $\kappa \leq 0$, then M^n_κ is contractible. This fails to be true of $\kappa > 0$
- If $\kappa > 0$, then M^n_κ is compact, and so has finite diameter $D_\kappa = \frac{\pi}{\sqrt{\kappa}}$. If $\kappa \leq 0$ M^n_κ is non-compact and has infinite diameter.

Remark 1.2. For each κ we also have a law of cosines, as follows:



- $\kappa > 0$:
$$\cos\left(\sqrt{\kappa}c\right) = \cos\left(\sqrt{\kappa}a\right)\cos\left(\sqrt{\kappa}b\right) - \sin\left(\sqrt{\kappa}a\right)\sin\left(\sqrt{\kappa}b\right)\cos\gamma$$
- $\kappa = 0$:
$$c^2 = a^2 + b^2 - 2ab\cos\gamma$$
- $\kappa < 0$:
$$\cosh\left(\sqrt{-\kappa}c\right) = \cosh\left(\sqrt{-\kappa}a\right)\cosh\left(\sqrt{-\kappa}b\right) - \sinh\left(\sqrt{-\kappa}a\right)\sinh\left(\sqrt{-\kappa}b\right)\cos\gamma.$$

2. Basic Concepts in General Metric Spaces

The basic idea in general metric spaces is to compare distances (and triangles) with those in the model spaces mentioned above. In everything below, let X be a metric space. Consider two geodesics $c\colon [0,a] \to X$ and $c'\colon [0,a'] \to X$ with $c(0) = c'(0)$. Let $t \in [0,a]$, $t' \in [0,a']$, we can form a triangle in $\mathbb{E}^2$ $\Delta(0,x,y)$ so that $d(0,x) = t$, $d(0,y) = t'$ and $d(x,y) = d(c(t), c'(t'))$. Denote the angle at 0 by $\overline{\angle}_{c(0)}(c(t), c'(t'))$.

Definition 2.1. The *upper angle* (or Alexandroff angle) between the geodesics c, c' is

$$\angle(c,c') = \limsup_{t,t'\to 0} \overline{\angle}_{c(0)}(c(t), c'(t')) = \lim \epsilon \to 0 \sup_{0<t,t'<\epsilon} \overline{\angle}_{c(0)}(c(t), c'(t')).$$

If $\lim_{t,t'\to 0} \overline{\angle}_{c(0)}(c(t), c'(t'))$ exists, then we say that angle exists in the strict sense.

Remark 2.2. In each of the M^n_κ's, this corresponds to the usual sense of angle, using the tangent space to the point $c(0)$. Recall that X need not be a smooth manifold, or any space resembling one.

Proposition 2.3. *Let $p \in X$ and take three geodesics c, c' and c'' emanating from p (defined on some common time interval). Then*

$$\angle(c', c'') \le \angle(c, c') + \angle(c, c'')$$

Proof. Suppose not. Then there exists $\delta > 0$ so that

$$\angle(c', c'') > \angle(c, c') + \angle(c, c'') + 3\delta.$$

By definition of angle there exists $\epsilon > 0$ so that

(1) $\overline{\angle}_p(c(t), c'(t')) < \angle(c, c') + \delta$, for all $t, t' < \epsilon$
(2) $\overline{\angle}_p(c(t), c''(t'')) < \angle(c, c'') + \delta$, for all $t, t'' < \epsilon$
(3) $\overline{\angle}_p(c'(t'), c''(t'')) > \angle(c, c'') - \delta$, for some $t', t'' < \epsilon$.

Let t', t'' be as in the third statement. We form a triangle in $\mathbb{E}^2$ with vertices $0, x', x''$ so that $d(0, x') = t'$, $d(0, x'') = t''$ and and angle at zero of measure α, where

$$\overline{\angle}_p(c'(t'), c''(t'')) > \alpha > \angle(c', c'') - \delta.$$

We can do this by our choice of $t', t'' < \epsilon$. By the law of cosines, this means that $d(x', x'') < d(c'(t'), c''(t''))$ and we also know that $\alpha > \angle(c, c') + \angle(c, c'') + 2\delta$.
Choose $x \in [x', x'']$ so that the angle, α' between $[0, x']$ and $[0, x]$ is larger than $\angle(c, c') + \delta$ and α'' between $[0, x'']$ and $[0, x]$ larger than $\angle(c, c'') + \delta$. Let $t = d(0, x)$. Then $t \le \max(t', t'') < \epsilon$, so the first and second statements above apply. In particular

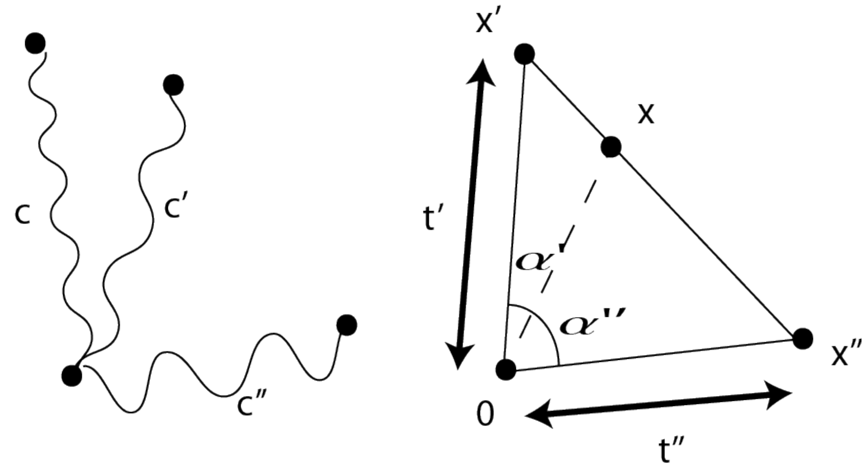
$$\overline{\angle}_p(c(t), c'(t')) < \alpha', \text{ and} \overline{\angle}_p(c(t), c''(t'')) < \alpha''$$

Again, by the law of cosines, we see that $d(c(t), c'(t')) < d(x, x')$ and $d(c(t), c''(t'')) < d(x, x'')$. Putting this together:

$$d(c'(t'), c''(t'')) > d(x', x'') = d(x, x') + d(x, x'') > d(c(t), c'(t') + d(c(t), c''(t'')),$$

contradicting the triangle inequality in X. □

Angles are clearly positive and symmetric, thus the above shows that the angle between two geodesic germs defines a pseudometric on the space of all geodesic germs at a point.



(A) Picture in X (B) Picture in $\mathbb{E}^2$

It is NOT generally a metric. For example, consider $\mathbb{R}^2$ with the l^∞ metric. $(x(t), y(t))$ is a geodesic if and only if for all t $|\dot{x}(t)| > |\dot{y}(t)|$ or vice-versa. It is parameterized proportional to arc length if and only if the coordinate with larger derivative has derivative 1. Consider the two geodesics emanating from zero $(t, [t(1-t)]^2)$ and $(t, [t(1-t)]^3)$ on the interval $[0, \frac{1}{3}]$. Even though these geodesics are clearly different, and their germs at zero are different, the angle, γ, between them is zero. s, t positive the distance between the two points is either $|s-t|$ or $|t(1-t)^2 - s(1-s)^3|$. In the first case, using the law of cosines we see that

$$\cos\gamma = \lim_{s,t\to 0} \frac{1}{2st}[t^2 + s^2 - (t-s)^2] = 1.$$

In the latter case we need to use the sandwich rule. For $0 < s, t < 1$

$$\frac{1}{2st}\left[2st(1-t)^2(1-s)^3\right] \leq \frac{1}{2st}\left[t^2 + s^2 - \left(t^2(1-t)^2 - s^3(1-s)^3\right)^2\right] \leq \frac{1}{2st}[t^2+s^2-(t-s)^2].$$

The right inequality happens because when $d_\infty((t, t^2(1-t)^2), (s, s^3(1-s)^3) = |t^2(1-t)^2 - s^3(1-s)^3|$ if and only if $\left(t^2(1-t)^2 - s^3(1-s)^3\right)^2 \geq (t-s)^2$. This implies that $\cos\gamma = 1$, and so $\gamma = 0$. There is nothing special about 2 and 3 here, we can choose any $n > m \geq 2$.

The proof of the angle triangle inequality brought up an important concept, comparing distances in X with distances in a model space.

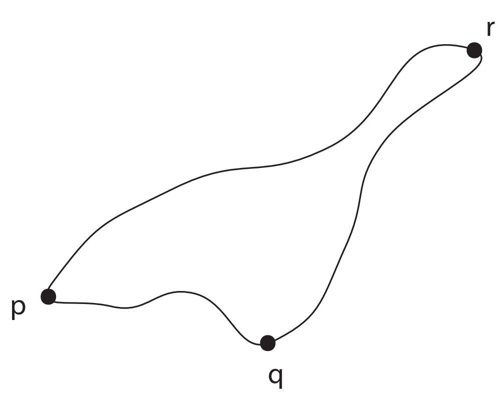
Definition 2.4. Let X be a metric space with $p, q, r \in X$. A comparison triangle in M_κ^2 is a triple of point $\overline{p}, \overline{q}, \overline{r}$ such that

$$d(\overline{p}, \overline{q}) = d(p, q)$$
$$d(\overline{q}, \overline{r}) = d(q, r)$$

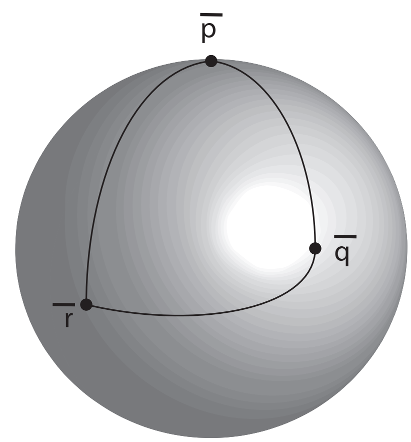
$$d(\overline{r}, \overline{p}) = d(r, p).$$

Lemma 2.5. *Let κ be a real number and let p, q, r be three points in X. Let $d(p,q) + d(q,r) + d(r,p) < 2D_\kappa$ (note that if $\kappa < 0$, $D_\kappa = \infty$). Then there exist $\overline{p}, \overline{q}, \overline{r} \in M_\kappa^2$ which form a comparison triangle for p, q, r.*

Proof. For the sake of notation, let $a = d(p,q)$, $b = d(q,r)$ and $c = d(r,p)$ and WLOG assume that $a \leq b \leq c$. By the triangle inequality, $c \leq a + b$, so $c \leq D_\kappa$.
Take two points $\overline{p}$ and $\overline{q}$ distance a from one another in M_κ^2. Our goal is to find an angle γ so that if we take a geodesic of length b starting at $\overline{q}$ that makes and angle of γ with $[\overline{p}, \overline{q}]$, the distance between its end point and $\overline{p}$ will be c. Starting at $\gamma = 0$ we find that the triangle is degenerate and the distance between the endpoints is $b - a \leq c$. If we let $\gamma = \pi$ then the distance is $a + b \geq c$. Since distance from $\overline{p}$ is a continuous function on M_κ^2 and geodesics vary continuously with their endpoints, there must exist an angle $0 < \gamma < \pi$ so that the geodesic originating at $\overline{q}$ that make angle γ with $[\overline{p}, \overline{q}]$, terminating at $\overline{r}$ has $d(\overline{q}, \overline{r}) = c$. □



(A) Triangle in X



(B) Reference Triangle in M_κ^2

Along the same lines, we have an important lemma regarding gluing triangles in M_κ^2. In light of the above lemma about the existence of comparison triangles, the following is important in establishing some results, like the Cartan-Hadamard Theorem.

Lemma 2.6. *(Alexandrov's Lemma) Consider four distinct points $A, B, B', C \in M_\kappa^2$ with $d(C,B) + d(C,B') + d(A,B) + d(A,B') < 2D_\kappa$. Suppose that B and B' lie on opposite sides of the line through A and C.*
Consider the geodesic triangles $\Delta(A,B,C)$ and $\Delta'(A,B',C)$ with angles α, β, γ (resp α', β', γ') at the vertices A, B, C (resp. A, B', C).
Let $\overline{\Delta}$ be a triangle in M_κ^2 with vertices $\overline{A}$, $\overline{B}$, and $\overline{B'}$ where $d(\overline{A}, \overline{B}) = d(A,B), d(\overline{A}, \overline{B'}) = d(A,B')$ and $d(\overline{B}, \overline{B'} = d(B,C) + d(C,B')$. Let $\overline{C}$ be the point on $[\overline{B}, \overline{B'}]$ at distance $d(B,C)$ from $\overline{B}$. Let $\overline{\alpha}, \overline{\beta}, \overline{\beta'}$ be the angles of $\overline{\Delta}$. If $\gamma + \gamma' > \pi$ then:

1. $d(B,C) + d(B',C) \leq d(B,A) + d(B',A)$
2. $\overline{\alpha} > \alpha + \alpha', \overline{\beta} \geq \beta, \overline{\beta'} \geq \beta'$, *and* $d(A,C) \leq d(\overline{A}, \overline{C})$. *Futher any one equality implies the others and occurs if and only if* $\gamma + \gamma' = \pi$

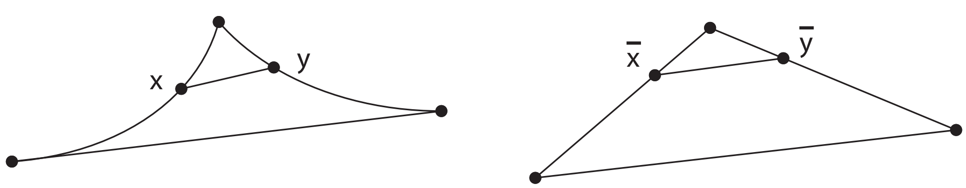
For a proof, see Bridson and Haefliger, Chapter I.2, Lemma 2.14.

3. Definitions of $CAT(\kappa)$

Definition 3.1. Let X be a metric space and let Δ be a geodesic triangle in X with perimeter less than $2D_\kappa$ with comparison triangle $\overline{\Delta}$ (unique up to isometry) in M_κ^2. Δ satisfies the $CAT(\kappa)$ inequality if for all $x, y \in \Delta$ with comparison points $\overline{x}, \overline{y} \in \overline{\Delta}$:

$$d(x,y) \leq d(\overline{x}, \overline{y}).$$

If X is D_κ geodesic and all geodesic triangles with perimeter less than $2D_\kappa$ satisfy the $CAT(\kappa)$ inequality, then we say that X is $CAT(\kappa)$



In essence, triangles in a $CAT(\kappa)$ space are thinner than triangles in the associate model space.

If a metric space X is locally $CAT(\kappa)$ we say that it has curvature $\leq \kappa$. If X has curvature ≤ 0 we say that it is *non-positively curved.*

This definition was first introduced that A.D. Alexandrov in '51 and the terminology CAT was coined by Gromov in '87 honoring Cartan, Alexandrov and Toponogov.

Classically, a C^3 Riemannian manifold has curvature $\leq \kappa$ if and only if all of its sectional curvatures are $\leq \kappa$.

Before we go any further, we should mention one basic, very important result about $CAT(\kappa)$ spaces.

Proposition 3.2. *Let X be $CAT(\kappa)$, then X is uniquely D_κ geodesic. Further, geodesics depend continuously on their endpoints.*

Proof. Take two geodesics $[x, y]$ and $[x, y]'$ and let $t \in [x, y]$. Consider the geodesic triangle with sides $[x, t], [t, y], [x, y]'$. Take $t' \in [x, y]'$ with $d(x, t') = d(x, t)$. Then in the comparison triangle, which is necessarily degenerate, $\overline{t} = \overline{t'}$. Thus $d(t, t') = 0$ by the $CAT(\kappa)$ inequality. Thus $[x, y] = [x, y]'$. □

Focusing on just the non-positive case, we some results about $CAT(0)$ spaces, and will mention when results apply for $\kappa > 0$.

Definition 3.3. We say that function $f : I \to \mathbb{R}$ is convex on an interval I if for any $t, t' \in I$ and $s \in [0, 1]$,

$$f((1 - s)t + st') \leq (1 - s)f(t) + sf(t')$$

.

A function f on a metric space is convex if for any geodesic path $c : I \to X$ parameterized proportional to arc length the map $t \mapsto f(c(t))$ is convex.

Proposition 3.4. *Let X be a $CAT(0)$ space, then the distance function $d : X \times X \to \mathbb{R}$ is convex. This means that given any pair of linearly reparameterized geodesics $c, c' : [0, 1] \to X$ the following holds for all $t \in [0, 1]$:*

$$d(c(t), c'(t)) \leq (1 - t)d(c(0), c'(0)) + td(c(1), c'(1)).$$

Proof. Consider the special case where $c(0) = c'(0)$. Consider the geodesic triangle $\Delta(c(0), c(1), c'(1))$ in X. Form the comparison triangle $\overline{\Delta}$ in $\mathbb{E}^2$. Take $t \in [0, 1]$. Euclidean geometry tells us that $d(\overline{c(t)}, \overline{c'(t)}) = td(\overline{c(1)}, \overline{c'(1)})$ and the $CAT(0)$ inequality for X tells us that:

$$d(c(t), c'(t)) \leq d(\overline{c(t)}, \overline{c'(t)}) = td(\overline{c(1)}, \overline{c'(1)}) = d(c(1), c'(1)).$$

Now, if $c(0) \neq c'(0)$ consider the geodesic c'' from $c(0)$ to $c'(1)$. Let $t \in [0, 1]$. Using the analysis above on the pairs c, c'' and c', c'' (but backwards) we see that $d(c(t), c''(t)) \leq td(c(1), c''(1))$ and $d(c'(t), c''(t)) \leq (1 - t)d(c''(0), c'(0))$. So

$$d(c(t), c'(t)) \leq d(c(t), c''(t)) + d(c''(t) + c'(t)) \leq t(c(1), c'(1)) + (1 - t)d(c(0), c'(0)).$$

□

Corollary 3.5. *$CAT(0)$ spaces are contractible.*

Proof. Let X be a $CAT(0)$ space and let $x \in X$. Consider the function

$$f: X \times [0,1] \to X$$

given by $(y,t) \mapsto ty$, where ty is the point on the geodesic $[x,y]$ at distance $td(x,y)$ from y. We only need to show that this map is continuous.
First, for each $t \in [0,1]$ the function $c_t(y) = ty$ should be continuous. To see this, note that by convexity of the metric $d(ty, ty') \leq td(y,y')$. Now to see that $c_t(y)$ defines a continuous family, take $t_n \to t_\infty$ in $[0,1]$. Then, in the topology of uniform convergence on compact sets, $c_{t_n} \to c_{t_\infty}$, since $d(ty, t'y) \leq |t - t'|d(x,y)$ for all $t, t' \in [0,1]$ and $y \in X$. □

Note that if $\kappa > 0$, then balls of radius D_κ are convex, in the sense that any two points in the ball can be joined by a geodesic entirely contained in that ball. Thus $CAT(\kappa)$ spaces are always locally contractible.

The $CAT(\kappa)$ condition has many other formulations, or varying use. The following proposition gives a sampling of these.

Proposition 3.6. *Fix $\kappa < 0 \in \mathbb{R}$ and let X be a geodesic metric space. The following are equivalent:*

1. *X is $CAT(\kappa)$*
2. *For every geodesic triangle $\Delta([p,q],[q,r],[r,p])$ in X and every point $x \in [q,r]$ the following inequality is satisfied for $\overline{x}$ the comparison point for x in the comparison triangle $\overline{\Delta}(p,q,r)$:*
$$d(p,x) \leq d(\overline{p}, \overline{x}).$$
3. *For every geodesic triangle $\Delta([p,q],[q,r],[r,p])$ in X and every pair of points $x \in [p,q]$ and $y \in [p,r]$ with $x, y \neq p$, the angles at the vertices corresponding to p in the reference triangles $\overline{\Delta}(p,q,r)$, $\overline{\Delta}(p,x,y)$ satisfy:*
$$\angle_p^{(\kappa)}(x,y) \leq \angle_p^{(\kappa)}(q,r).$$
4. *The Alexandrov angle between the sides in any geodesic triangle in X with distinct vertices is not larger than the angle between the corresponding sides of a comparison triangle in M_κ^2.*
5. *For every geodesic triangle $\Delta([p,q],[q,r],[r,p])$ in X with $p \neq q$ and $p \neq r$, if γ denotes the Alexandrov angle between $[p,q]$ and $[p,r]$ at p and if $\Delta(\hat{p},\hat{q},\hat{r})$ is a geodesic triangle in M_κ^2 with $d(\hat{p},\hat{q}) = d(p,q)$ and $d(\hat{p},\hat{r}) = d(p,r)$ and $\angle_{\hat{p}}(\hat{q},\hat{r}) = \gamma$, then $d(q,r) \geq d(\hat{q},\hat{r})$.*

One of the important points in this proposition is that the $CAT(\kappa)$ condition can be restated just in terms of Alexandrov angles. In fact, statement 3 implies that angles in a $CAT(\kappa)$ space (and so a space with curvature $\leq \kappa$) always exist in the strict sense. If $\kappa > 0$ these also hold so long as the geodesic triangles considered have perimeter less than $2D_\kappa$.

Proof. Clearly (1) implies (2), (3) implies (4) and (4) is equivalent to (5), by the law of cosines. We show that (1) is equivalent to (3) and that (2) implies (3) and (4) implies (2).

Consider a geodesic triangle in X, as in (3). Take x, y as in (3) as well (which we can do WLOG by change the labels on the vertices). Let $\overline{\Delta}(p,q,r)$ be a reference triangle. Then $d(x,y) \leq d(\overline{x},\overline{y})$ if and only if $\angle_p^{(\kappa)}(x,y) \leq \angle_p^{(\kappa)}(q,r)$ by the law of cosines in M_κ^2. Thus (1) is equivalent to (3).

To show that (2) implies three, we consider three triangles in M_κ^2: $\overline{\Delta}(p,q,r)$, with comparison points $\overline{z}$, $\overline{\Delta}(p,x,y)$, with comparison points $\overline{z}'$, and $\overline{\Delta}(p,x,r)$ with comparison points $\overline{z}''$. Let α, α', α'' be the angles at p in each of these triangles. By condition (2):

$$d(\overline{x}'',\overline{y}'') \geq d(x,y) = d(\overline{x}',\overline{y}').$$

By the law of cosines, $\alpha'' > \alpha'$.
Now, also by condition (2),

$$d(\overline{x}'',\overline{r''}) = d(x,r) \leq d(\overline{x},\overline{r}),$$

and again by the law of cosines, $\alpha > \alpha'' > \alpha$, as desired.

Now to show (4) implies (2), let $\Delta([p,q],[q,r],[r,p])$ be a geodesic triangle in X and let $x \in [q,r]$, $x \neq q,r$. Let γ and γ' be the Alexandrov angles that a geodesic [p,x] makes with the subsegments of $[q,r]$: $[q,x]$ and $[x,r]$ respectively. Then $\gamma + \gamma' \geq \pi$. Consider the comparison triangle in M_κ^2: $\overline{\Delta}(p,q,r)$. Let $\overline{\beta}$ be the angle at $\overline{q}$. Also consider the two comparison triangles $\overline{\Delta}(p,q,x)$ and $\overline{\Delta}(p,x,r)$. We can arrange these two so that they share the segment $[\tilde{p},\tilde{x}]$ and $\tilde{q}$ and $\tilde{r}$ lie on opposite sides. Then by condition (4),

$$\tilde{\gamma} + \tilde{\gamma'} \geq \gamma + \gamma' \geq \pi.$$

Using Alexandrov's lemma, this means that $\tilde{\beta} \geq \overline{\beta}$ and by the law of cosines,

$$d(\overline{p},\overline{x}) \geq d(\tilde{x},\tilde{p}) = d(x,p).$$

□

If $\kappa = 0$ there is one more condition which is important (it comes in to play when considering products of metric spaces. It is the CN inequality of Bruhat and Tits. For all $p,q,r \in X$ and $m \in X$ with $d(q,m) = d(r,m) = \frac{1}{2}d(q,r)$ we have that:

$$d(p,q)^2 + d(p,r)^2 \geq 2d(m,p)^2 + \frac{1}{2}d(q,r)^2.$$

Equality holds in the $\mathbb{E}^2$ case, giving the implication in one direction.

There is one last obvious proposition that we can leave off with regarding the relationships between the $CAT(\kappa)$ conditions. In particular, if $\kappa < \kappa'$ triangles in M_κ^2

are thinner than triangles in $M^2_{\kappa'}$. That, and a look at the laws of cosines leads to the following.

Proposition 3.7. *A metric space X is $CAT(\kappa)$ if and only if it is $CAT(\kappa')$ for all $\kappa' > \kappa$.*

4. Examples of $CAT(\kappa)$ spaces

The first examples one can think of of $CAT(\kappa)$ spaces are convex subsets of M^2_κ, since they inherit their geodesics from M^2_κ and their induced length metric is identical to the restriction metric.

The second examples are C^3 Riemannian manifolds with curvature $\leq \kappa$.

Let X be complement of the quadrant $\{(x, y) \colon x > 0, y > 0\}$ with the induced length metric. It is not hard to see that X is a $CAT(0)$ space. However, the complement of an octant in $\mathbb{E}^3$ with the induced length metric is not $CAT(\kappa)$ for any κ, since it is not uniquely geodesic, nor r-locally uniquely geodesic for any $r > 0$. To see this, determine that there are two geodesics between the points $(\alpha, \alpha, -2\alpha)$ and $(-\alpha, -\alpha, \alpha)$ for all $\alpha > 0$.

Consider a metric space X so that between any two points x, y there is a unique path $c \colon [0, 1] \to X$ which is embedded. Such a metric space is called an $\mathbb{R}$-tree. A metric tree is an obvious example of this, as is $\mathbb{R}^2$ with the "Paris Railway Metric". $\mathbb{R}$-trees are $CAT\kappa$ for every $\kappa \in \mathbb{R}$ since all triangles are tripods. We sometimes call them $CAT(-\infty)$. Every space which is $CAT(\kappa)$ for all κ is, in fact, an $\mathbb{R}$-tree.

If you take the universal cover of a metric space with curvature $\leq \kappa \leq 0$, it is $CAT(\kappa)$. This is known as the Cartan-Hadamard theorem. This implies that smooth manifolds that admit Riemannian metrics of non-positive curvature have trivial higher homotopy groups.

If X_1 and X_2 are both $CAT(0)$ spaces, then $X_1 \times X_2$ is $CAT(0)$. This follows from the CN-inequality or direct use of the $CAT(0)$ inequality, but in $\mathbb{E}^4 = M^4_0$. Note that this makes the phenomenon unique for $\kappa = 0$!

5. Projection to Convex Subsets

6. M_κ Polyhedral Complexes

Our goal is now to build $CAT(0)$ spaces from simple pieces, in this case polyhedra.

Definition 6.1. An M_κ-polyhedral complex is a space built of pieces that are isometric to convex hulls of finitely many points in M^n_κ (polyhedra) glued isometrically along faces.

This definition is similar to the definition of simplicial complex in topology, and so should not seem unfamiliar. Each face of a polyhedral cell is again a polyhedral cell. Our main focus will be on (M_0-) cubical complexes, i.e. their only shapes are isometric to standard n-cubes. Further, we have the following theorem of Bridson on such metric spaces.

Theorem 6.2. *Let K be an M_κ polyhedral complex. Let Shapes(K) denote the set of all isometry classes of polyhedral cells in K. If Shapes(K) is finite, then K is a complete geodesic metric space.*

Corollary 6.3. *Let X be a cubical complex. If the dimension of X is finite then X is a complete geodesic metric space.*

The study of these spaces often relies on understanding the local structure at a vertex of the complex.

Definition 6.4. Let X be an M_κ polyhedral complex and let $v \in X$ be a vertex. The link of v, denoted $Lk(v, X)$ is the space of all germs of unit speed geodesics emanating from v.

This definition is not very clear, as this is also the definition of the space of directions at a point in any metric space (there we have to mod out by the relation $\angle(c, c') = 0$.

More precisely, if X is locally finite, we think of $Lk(v, X)$ as a polyhedral complex. For each n-dimensional polyhedral cell incident at v we obtain an $(n-1)$-polyhedral cell as follows:

- For each edge incident at v we get a vertex in $Lk(v, X)$;
- Two vertices are joined by an edge if and only if they are edges in a common 2-cell
- For each m-cell, put an $(m-1)$-cell whose boundary are the $(m-2)$-cells corresponding to the faces.

The Alexandrov angle at v defines a metric on $Lk(v, X)$.

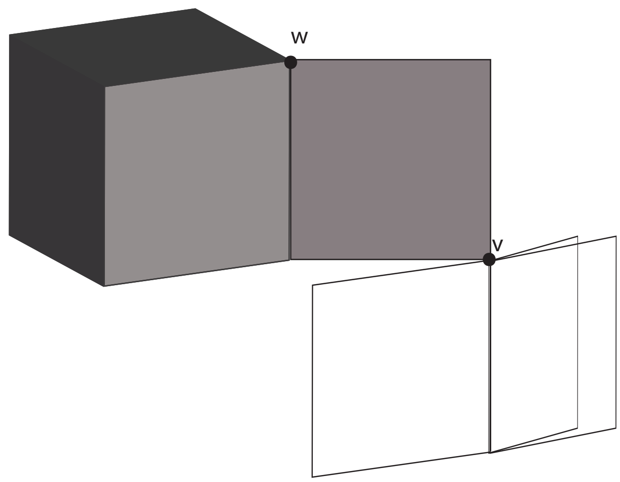
The open star of v, denoted $st(v)$ is the union of all open polyhedral cells containing v. The closed star of v, denoted $St(v)$ is the union of all closed polyhedral cells containing v. Define $\epsilon(v)$ be the shortest distance between v and a face in $St(v)$ that does note contain v.

Theorem 6.5. *If $\epsilon(v) > 0$, then $B(v, \epsilon(v)/2)$ is isometric to the κ-cone on $Lk(v, X)$.*

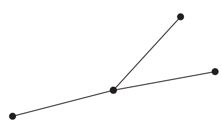
Refer to Bridson & Haefliger for a definition of κ-cone.

Determining when a polyhedral complex has curvature $\leq \kappa$ is generally hard. In particular, vertices pose a major problem. In fact, vertices pose the only problem, but this is not immediate.

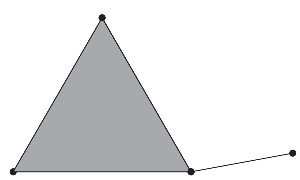
Using work of Berestovskii, and the above theorem, Gromov proved a very elegant condition on a cube complex to determine when it is non-positively curved. To



(A) Cube Complex



(B) Link of v



(C) Link of w

understand the condition, we note first that in a cube complex the link of every vertex is a (M_1-) simplicial complex.

A simplicial complex is a flag complex if it has "no missing simplices". This means that for subset $\{v_0, \ldots, v_n\}$ of the vertices which are pairwise joined by edges, there is an n-simplex with vertices $\{v_0, \ldots, v_n\}$. In particular, every 1-dimensional complex with girth ≥ 4 is a flag complex.

Theorem 6.6. *(Gromov's Link Condition) Let X be a cube complex. X is non-positively curved if any only if for each vertex v, $Lk(v, X)$ is a flag complex.*

We give an idea of why this condition should hold. Consider a vertex v whose link is a triangle, but with no 2-simplex. Then in X, the star of v is top corner of an empty 3-cube. A full empty 3-cube is homeomorphic to a sphere. If X were non-positively curved, this cannot happen. Having the top corner introduces part of a sphere though, and so some amount of positive curvature.