*SEIFERT:*
*TOPOLOGY OF 3-DIMENSIONAL*
*FIBERED SPACES*

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# *TOPOLOGY OF 3-DIMENSIONAL FIBERED SPACES**

The subject of this paper is related to the homeomorphism problem for 3-dimensional closed manifolds. The fundamental theorem for 2-manifolds tells us how many topologically distinct 2-manifolds there are. The methods for its proof cannot yet be applied to 3 or more dimensions. There are two ways to approach the 3-dimensional problem. The first one is to examine fundamental regions (Diskontinuitätsbereiche) of groups acting on a 3-dimensional metric space (Bewegungsgruppen). In the 2-dimensional case, every closed surface is a fundamental region of a fixed-point-free action; however, there are 3-manifolds for which this is not true. The fundamental regions of 3-dimensional spherical actions are endowed with a certain fibration: the fibers are trace curves (Bahnkurven) of a continuous action on the hypersphere; examples will be given in §3 and can also be found in DB II.[1] This leads us to the second way: instead of investigating a complete system of *topological* invariants of 3-dimensional manifolds, we search for a system of invariants for *fiber preserving* maps of *fibered* 3-manifolds. This task is completely solved in this paper. These invariants refer of course to the fibering of the manifold, not to the manifold itself, so that so far the question remains whether two spaces with different fibrations can be homeomorphic. Furthermore there are 3-manifolds that do not admit a fibration (§15). Even so, in many cases the fiber invariants can be used to decide whether 3-manifolds are homeomorphic. Examples for this are given in §12–§14 and in DB II.

A knowledge of the topology of surfaces, the fundamental group, and the

*Reprinted from H. Seifert, *Acta Mathematica* **60** (1933), 147–288 (translated by Wolfgang Heil).

[1]Cf. W. Threlfall and H. Seifert, Topologische Untersuchungen der Diskontinuitätsbereiche endlicher Bewegungsgruppen des dreidimensionalen sphärischen Raumes. *Math. Ann.* **107**. This will be referred to as DB II; the first part in *Math. Ann.* **104** will be cited as DB I.

homology group is assumed. The spaces of line elements[2] (Linienelemente) provide introductory examples. Other examples are given in this paper.

## 1. Fibered Spaces

We define a manifold[3] to be a set of points such that for each point there is a system of subsets, called *neighborhoods*, which satisfy the axioms (1)–(4) below.

(1) Hausdorff axioms:

(a) Each point $P$ has at least one neighborhood $U(P)$; each neighborhood of $P$ contains $P$.
(b) If $U(P)$ and $V(P)$ are neighborhoods of $P$, then there exists a neighborhood $W(P) \subset U(P) \cap V(P)$.
(c) If $Q$ lies in $U(P)$, then there exists a neighborhood $U(Q)$ of $Q$ which is contained in $U(P)$.
(d) For two distinct points there exist disjoint neighborhoods.

A system of neighborhoods satisfying these axioms is called a *topological space*. Two equivalent systems of the same point set determine the same topological space. Here systems are equivalent if each neighborhood $U(P)$ of one system contains a neighborhood $U'(P)$ of the other system, and vice versa. A subset of a topological space is *open* if it contains for each of its points a neighborhood of this point. The system of all open subsets of a topological space is a system of neighborhoods, which is equivalent to all other systems of neighborhoods of this space. From now on we always choose this system of neighborhoods.

(2) Each point of $M$ has a neighborhood homeomorphic to an open 3-ball in 3-dimensional Euclidean space.

(3) If an arbitrary neighborhood is assigned to each point, then countably many of these cover the manifold. If already finitely many suffice to cover the manifold, it is called *closed*, otherwise *open*.[4]

(4) The manifold is *connected*, i.e., any two points can be connected by an arc, or equivalently, the manifold is not the union of two disjoint open sets.

[2] W. Threlfall, Räume aus Linienelementen. *Jahresber. Deutsch. Math.-Verein.* **42** (1932), 88–110.

[3] Cf. H. Kneser, Topologie der Mannigfaltigkeiten. *Jahresber. Deutsch. Math.-Verein.* **34** (1926), 1.

[4] Instead of (3) we could require the second Hausdorff countability axiom in addition to (1) and (2): There exists an equivalent system of neighborhoods that consists of countably many distinct point sets. The following axiom would do just as well: The manifold can be covered with countably many subsets, each of which is homeomorphic to an open 3-dimensional Euclidean ball.

In combinatorial topology manifolds are required to admit a *triangulation*. This requirement is redundant for our purpose, since fibered spaces can be triangulated, as will be shown in §4. One could say a manifold is fibered if it is decomposed into curves, called *fibers*, such that each point lies on exactly one fiber and a neighborhood of each point can be mapped homeomorphically onto a neighborhood of a point in a Euclidean space in such a way that fibers are mapped to line segments of a bundle of parallel lines. This requirement is a local one. But even if we postulated this for all points of the manifold, we would still find this definition of a fibered manifold to be too general.

In the present paper we consider only those fibered manifolds which satisfy in addition to the four manifold axioms the three following axioms which relate to properties of the fibering in the large. (We call these manifolds *fibered spaces*.)

(5) The manifold can be decomposed into fibers, where each fiber is a simple closed curve.

(6) Each point lies on exactly one fiber.

(7) For each fiber $H$ there exists a *fiber neighborhood*, that is, a subset consisting of fibers and containing $H$, which can be mapped under a *fiber preserving* map onto a *fibered solid torus*, where $H$ is mapped onto the "middle fiber."

A *fibered solid torus* is obtained from a fibered cylinder $D^2 \times I$ where the fibers are the lines $x \times I$, $x \in D^2$, by rotating $D^2 \times 1$ (but keeping $D^2 \times 0$ fixed) through an angle of

$$2\pi(\nu/\mu)$$

and then identifying $D^2 \times 0$ and $D^2 \times 1$ (i.e., $x \times 0$ is identified with $\rho(x) \times 1$, where $\rho$ is the rotation). Here $\nu, \mu$ are coprime integers. Without loss of generality we can assume that

$$\mu > 0 \qquad \text{and} \qquad 0 \leqslant \nu \leqslant \tfrac{1}{2}\mu.$$

For if $\nu$ is replaced by $\nu + k\mu$ or by $-\nu$, then the new solid torus can be mapped onto the old one by a fiber preserving map.

A map is *fiber preserving* if it (1) is a homeomorphism and (2) maps fibers to fibers. Two solid tori which are homeomorphic under a fiber preserving map will not be distinguished.

When identifying the cylinder $D^2 \times I$ with the solid torus the lines (fibers) of $D^2 \times I$ are decomposed into classes such that each class contains exactly $\mu$ lines, which match together to give one fiber of the solid torus, except that the class containing the axis of $D^2 \times I$ consists of the axis alone, which also makes up a fiber. If $\mu = 1$, we call the solid torus an *ordinary solid torus*.

The fiber neighborhoods are (in contrast to point neighborhoods) closed sets: each fiber neighborhood contains its boundary torus.

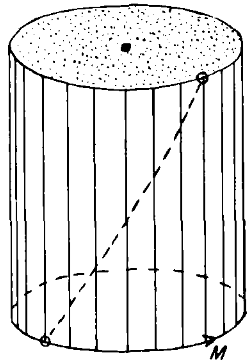


FIG. 1

A *meridian* $M$ of a solid torus $V$ is a simple closed oriented curve on the boundary torus $T$ which is not contractible on $T$ but contractible in $V$. A homeomorphism of $V$ onto itself maps a meridian to a meridian. If we forget about orientation, we can map a meridian onto any other meridian under a continuous deformation of $T$. In Fig. 1, e.g., the oriented boundary curve of the bottom surface $D^2 \times 0$ is a meridian. A *longitude* $B$ of the solid torus is a simple closed curve on $T$ which intersects $M$ in exactly one point.

$B$ is determined (modulo deformations of $T$) up to its orientation and multiples of $M$. Any pair of meridian and longitude can be mapped onto another such pair by a topological map of the solid torus onto itself; however, even though any meridian can be mapped onto any other by a deformation of $T$, this is not necessarily true for longitudes. The topological map of the solid torus, which sends a longitude to another which is not homologous (on $T$), cannot be obtained by a deformation of the identity.

We now orient a fiber $H$ of a solid torus. Thus, if we have chosen a fiber $H$, a meridian $M$, and a longitude $B$ on the boundary $T$ of a given fibered solid torus $V$, we can just as well choose instead of $H, M, B$ any other system $H', M', B'$ which is related to the first system as follows:

$$H \sim \varepsilon_1 H', \tag{1}$$

$$M \sim \varepsilon_2 M', \tag{2}$$

$$B \sim \varepsilon_3 B' + xM'. \tag{3}$$

Here $\varepsilon_i = \pm 1$; $x$ is an integer. Instead of the equal sign we have chosen the homology sign, which denotes *homology on* $T$. For homology is all that matters to us and we allow, for example, that $H'$ be a fiber disjoint to $H$ and $M'$ be a meridian obtained from $M$ by a deformation of $T$.

Throughout, we write relations of the homology group additively and relations of the fundamental group multiplicatively.[5]

The numbers $\mu$ and $\nu$ not only determine the fibered solid torus $V$, but

[5] Cf. B. L. Van der Waerden, "Moderne Algebra I." p. 19. Berlin, 1930.

conversely $V$ determines $\mu$ and $\nu$ uniquely, i.e., two fibered solid tori can be mapped onto each other by a fiber preserving map iff they have the same defining numbers $\mu, \nu$. For, choosing the longitude $B$ suitably (shortest path on $\partial D^2 \times I$ from a point $x \in \partial D^2 \times 0$ to its equivalent point on $\partial D^2 \times 1$, the dotted line in Fig. 1) and orienting $M$ and $H$ suitably, we have on $T$ the homology

$$H \sim \nu M + \mu B, \tag{H}$$

which means precisely that $\mu$ and $\nu$ are the defining numbers of the fibered solid torus. If we were to choose instead of $H, M, B$ an arbitrary system $H', M', B'$ of the fibered solid torus, then we would get

$$H' \sim nM' + mB' \tag{H'}$$

since $M'$ and $B'$ are a fundamental system[6] of curves on $T$ which is a basis for the homology. Here $m$ and $n$ are coprime integers since the fiber is a simple closed curve, and $m \neq 0$ since it is not homologous to the meridian. On the other hand, we can express the homology (H) in terms of $H', M', B'$ via the formulas (1), (2), (3):

$$\varepsilon_1 H' \sim (\varepsilon_2 \nu + x\mu)M' + \varepsilon_3 \mu B'.$$

Therefore

$$\varepsilon_1\left[(\varepsilon_2 \nu + x\mu)M' + \varepsilon_3 \mu B'\right] \sim nM' + mB'.$$

Comparing the coefficients, we see that $\mu$ and $\nu$ are determined by $m$ and $n$. To see this, note that $|\mu| = |m|$, also $\mu > 0$, so $\mu = |m|$; also $\nu$ is equal to $|n|$, reduced modulo $m$ to a number in the interval $[-\frac{1}{2}m, \frac{1}{2}m]$. Thus the numbers $\mu$ and $\nu$ are characteristic for the given fibered solid torus.

Meridian and longitude are already defined on a nonfibered solid torus. We need to define still another curve, the *crossing curve* $Q$ (Querkreis), presuming the fibering. It is a simple closed curve on $T$ that intersects each fiber of $T$ in exactly one point. It is therefore (except for its orientation and multiples of the fiber) determined by the fibering of $T$, i.e., if $Q$ and $Q'$ are two crossing curves, we have the formula

$$Q \sim \varepsilon_4 Q' + yH' \tag{4}$$

in addition to the transformation formulas (1)–(3). The fiber $H$ and crossing curve $Q$ are a fundamental system of curves on $T$ similar to meridian and longitude, i.e., any other closed curve on $T$ is homologous to a linear combination of $H$ and $Q$.

The boundary of an arbitrary fibered solid torus is a fibered torus. Therefore the boundaries of any two fibered solid tori can be mapped onto each other under a fiber preserving homeomorphism. The fibered solid torus

[6] Meridian and longitude are also called a canonical system of curves or a pair of conjugate Rückkehrschnitte.

is determined by the fibering of its boundary torus only if on this torus a closed curve $M$ is distinguished as meridian. Of course, $M$ must satisfy the conditions to be a simple closed curve not homologous to zero (on $T$) and not homologous to a fiber. If on a fibered torus the fiber $H$ is oriented and a crossing curve $Q$ is chosen, $M$ can be expressed [with coprime integers $\alpha$ ($\neq 0$) and $\beta$] as follows:

$$M \sim \alpha Q + \beta H.$$

We claim that the fibered solid torus is uniquely determined by the fibering of its boundary and by $M$, hence by $\alpha$, $\beta$. We show this by computing the characteristic numbers $\mu, \nu$. If

$$B \sim \rho Q + \sigma H$$

is a longitude on the fibered torus, we can assume (choosing orientation of $B$ suitably) that

$$\begin{vmatrix} \alpha & \beta \\ \rho & \sigma \end{vmatrix} = 1 \tag{5}$$

since both of $Q, H$ and $M, B$ are a fundamental system of curves on the torus. Then

$$H \sim \alpha B - \rho M$$

$\rho$ is determined by $\alpha$ and $\beta$ up to multiples of $\alpha$ by (5). As before from (H′), the last equation gives us now the characteristic numbers $\mu$ and $\nu$ uniquely: $\mu = |\alpha|$, $\nu =$ the absolute value of the number $\rho$, reduced mod $\alpha$ to $[-\frac{1}{2}\alpha, \frac{1}{2}\alpha]$. In particular if the meridian is a crossing curve we have an ordinary fibered solid torus.

The simplest example of a fibered space is $S^1 \times S^2$. It is obtained from $S^2 \times I$ by identifying the points $x \times 0$ and $x \times 1$. Figure 2 shows a cross section through the center point of $S^2 \times I \subset R^3$. The fibers correspond to the radii of the hollow ball. We have a fibered space, since each fiber has a fiber neighborhood which can be mapped onto a fibered solid torus with the numbers $\mu = 1$, $\nu = 0$.

## 2. Orbit Surface

The most important concept in the study of fibered spaces is that of the *orbit surface* (Zerlegungsfläche). Every fibered space $F$ has an orbit surface $f$. Now $f$ is not a subset of the space $F$ and can in general not be embedded in $F$,[7] but is defined as follows: there is a one-to-one correspondence between the fibers of $F$ and the points of $f$.[8] Since each point of $F$ lies on exactly one

[7]Our definition of Zerlegungsfläche is not related to G. D. Birkhoff's surface of section, Dynamical systems with two degrees of freedom [*Trans. Amer. Math. Soc.* **18** (1917), 268; cf. also L. Bieberbach, "Differentialgleichungen," p. 136. Berlin, 1923].

[8]The orbit surface thus indicates how the manifold is "decomposed" into fibers [cf. H. Tietze and L. Vietoris, *Encykl. Math. Wiss.* (III) **AB 13** (1930), 178].

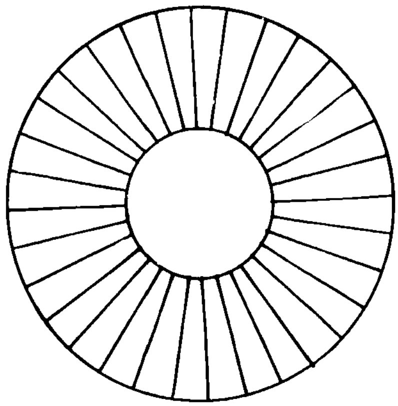

FIG. 2

fiber, it follows that each point of $F$ has exactly one image on $f$. The neighborhoods of $f$ are defined as images of the neighborhoods in $F$ (i.e., of the open subsets of $F$). The following can be proved:

(1) $f$ is a Hausdorff space.

(2) Each point of $f$ has a neighborhood homeomorphic to an open 2-cell. (For the proof use the fact that each fiber neighborhood can be mapped topologically onto a solid torus.)

(3) Any covering of $f$ by neighborhoods has a countable subcovering. $f$ is an open or closed manifold if $F$ is open or closed, respectively.

(4) $f$ is connected.

(1)–(4) imply that $f$ is triangulable, by a theorem of T. Radó.[9] Therefore we can apply all the theorems of the theory of 2-manifolds. If $F$ is closed, then $f$ is an orientable surface of genus $p$ (number of handles) or a nonorientable surface of genus $k$ (number of cross-caps). In the example $S^1 \times S^2$, the orbit surface is a 2-sphere which can be embedded into $S^1 \times S^2$ so that each fiber meets it in exactly one point.

Any closed or open, orientable or nonorientable surface $f$ is the orbit surface of some fibered space, for example of the product $f \times S^1$ (the fibers are $x \times S^1$, $x \in f$). Here the orbit surface can again be embedded into the fibered space, as above. In §3 we shall give an example where this is no longer possible.

We use throughout the following notation. Passing from the fibered space $F$ to the orbit surface $f$ we pass from capital letters to small letters. Thus to the fiber $H$ of the space $F$ corresponds the point $h$ of the orbit surface $f$.

If $\Omega_H$ is a fiber neighborhood of the fiber $H$, we call its image $\omega_h$ an *orbit neighborhood* (Zerlegungsumgebung) of the image point $h$ of $H$. The *orbit neighborhood* is obtained from the meridian disk of the fiber neighborhood,

[9] T. Radó, Über den Begriff der Riemannschen Fläche, *Acta Univ. Szeged.* **2** (1925), 101.

i.e., from the bottom disk of the cylinder of Fig. 1, by identifying points which belong to the same fiber. Therefore, the orbit neighborhood is a circle sector of an angle $2\pi/\mu$ whose boundary radii have been identified, or in other words: it is the orbit surface of a cyclic rotation group of order $\mu$ of the disk about its center point. Hence the orbit neighborhood can be mapped homeomorphically onto a disk with boundary; hence it is a 2-cell. The orbit neighborhoods are just like the fiber neighborhoods *closed* point sets. They satisfy the neighborhood axioms only after removing their boundary curves.

The orbit neighborhoods satisfy the following:

LEMMA 1. *If $\omega_h$ is an orbit neighborhood of the point h and if e is a 2-cell contained in $\omega_h$ such that h is not on the boundary of e, then e is also an orbit neighborhood* (a) *of h, if h is an interior point of e,* (b) *of each interior point of e, if h does not belong to e. The fiber neighborhoods E (resp. $\Omega_H$) which map onto e (resp. $\omega_h$) are in case* (a) *homeomorphic under a fiber preserving map; in case* (b) *E is an ordinary fibered solid torus.*

*Proof.* (a) The fibers that map to the points of $e$ constitute a fibered subset $E$ of $\Omega_H$ which contains the fiber $H$ in its interior. If we think of $\Omega_H$ as a fibered cylinder with boundary disks identified under a rotation, we obtain the orbit neighborhood $\omega_h$ (Fig. 3) from the meridian disk $\tilde{\omega}_{\tilde{h}}$ of $\Omega_H$ (Fig. 4) if we identify those points of $\tilde{\omega}_{\tilde{h}}$ which are equivalent under the cyclic rotation group of order $\mu$ acting on $\tilde{\omega}_{\tilde{h}}$.

The points of $\tilde{\omega}_{\tilde{h}}$ which map to points of $e$ constitute a 2-cell $\tilde{e}$ (shaded in Fig. 4) which contains the center point $\tilde{h}$ of $\tilde{\omega}_{\tilde{h}}$ in its interior and which is mapped to itself under the cyclic rotation group. The subspace $E$ of $\Omega_H$ consists of the lines parallel to the axis of the cylinder $\Omega_H$ which pass through the points of $\tilde{e}$. We shall show that we can map $\tilde{e}$ onto $\tilde{\omega}_{\tilde{h}}$ under an

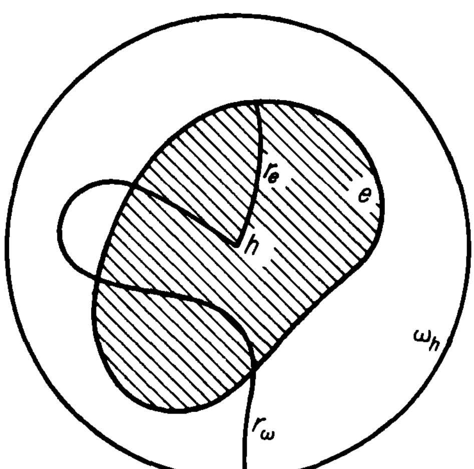


FIG. 3

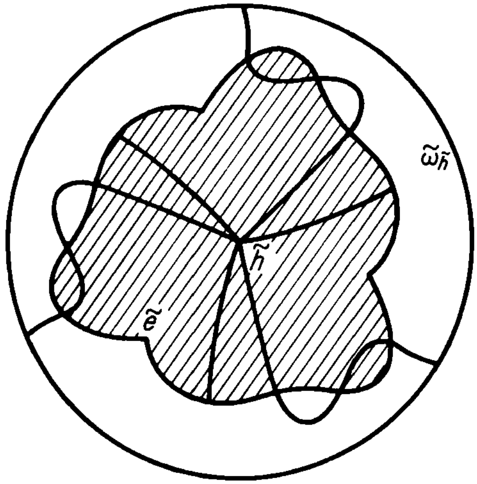


FIG. 4

orientation preserving homeomorphism $\tilde{a}$ keeping $\tilde{h}$ fixed and such that any $\mu$ points which are equivalent under the cyclic rotation group are again mapped onto $\mu$ such points. Taking the corresponding map on the lines of $E$ and $\Omega_H$, we obtain a topological map of $E$ onto $\Omega_H$ which maps fibers to fibers and keeps the middle fiber $H$ fixed, as claimed in the lemma.

The map $\tilde{a}$ is obtained as follows: Let $a$ be an orientation preserving map that maps $e$ onto $\omega_h$ and keeps $h$ fixed, let $r_e$ be a simple arc from $h$ to the boundary of $e$, and let $r_\omega$ be the image of $r_e$ which is a simple arc from $h$ to the boundary of $\omega_h$. Now $\tilde{e}$ (resp. $\tilde{\omega}_{\tilde{h}}$) is decomposed by the $\mu$ (pre-)images of $r_e$ (resp. $r_\omega$) into $\mu$ consecutive sectors

$$\tilde{e}^1, \tilde{e}^2, \ldots, \tilde{e}^\mu \qquad (\text{resp.}\ \ \tilde{\omega}^1, \tilde{\omega}^2, \ldots, \tilde{\omega}^\mu)$$

which are cyclically interchanged by the rotation group. The map $a$ determines a map of the sector $\tilde{e}^i$ onto the sector $\tilde{\omega}^i$ and hence a map $\tilde{a}$ of $\tilde{e}$ onto $\tilde{\omega}_{\tilde{h}}$, as required.

(b) In this case, to the 2-cell in $\omega_h$ there correspond in $\tilde{\omega}_{\tilde{h}}$ now $\mu$ disjoint 2-cells $\tilde{e}^1, \tilde{e}^2, \ldots, \tilde{e}^\mu$ which are interchanged under the cyclic rotation group. The fiber set $E$ corresponding to $e$ is in the cylinder $\Omega_H$ made up of $\mu$ congruently fibered cylinders which lie over $\tilde{e}^1$ to $\tilde{e}^\mu$. Now $E$ is obtained from these pieces by pasting them together (one after the other) and finally identifying top and bottom disks under the identity map. Therefore $E$ is an ordinary fibered solid torus, in which we can take each inner fiber as the middle fiber.

From Lemma 1 we obtain

LEMMA 2. *If $\Omega_H^1$ and $\Omega_H^2$ are two fiber neighborhoods of the fiber $H$, they are homeomorphic under a fiber preserving map which keeps $H$ fixed.*

*Proof.* On the orbit surface there exists a 2-cell $e$ containing $h$ and lying in the interior of the intersection of the orbit neighborhoods $\omega_h^1$ and $\omega_h^2$. By Lemma 1, $e$ is the image of a fiber neighborhood $E$ of the fiber $H$, and $E$ can be mapped under a fiber preserving map (keeping $H$ fixed) to each of $\Omega_H^1$ and $\Omega_H^2$, respectively.

This lemma implies that for a given fiber $H$ the numbers $\mu, \nu$ are the same for all fiber neighborhoods of $H$; hence they are an invariant of $H$. If $\mu > 1$, we call $H$ an *exceptional fiber of order* $\mu$ of the space; if $\mu = 1$, an *ordinary* fiber. If a fiber in the neighborhood of an exceptional fiber $H$ of order $\mu$ approaches $H$, its limit runs $\mu$ times around $H$. In a fibered solid torus all the fibers are ordinary fibers, except possibly for the middle fiber. In a fiber neighborhood of an exceptional fiber $H$ of order $\mu$ we have that $\mu \cdot H$ is homologous to an ordinary fiber. The points of the orbit surface that are images of exceptional fibers are *exceptional points*; as points of the orbit surface, they cannot be distinguished from ordinary points.

THEOREM 1. *A closed fibered space contains at most finitely many exceptional fibers.*

For otherwise there would exist a point of the space such that any neighborhood of it meets infinitely many exceptional fibers. The fiber through this point would not have a fiber neighborhood.

## 3. Fiberings of $S^3$

Before studying fiberings in general, we construct examples of fiberings of $S^3$ with exceptional fibers. We think of $S^3$ as lying in $R^4$, where it is a hypersurface with the equation

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1,$$

where $x_1, x_2, x_3, x_4$ are Cartesian coordinates. The *fibers* are the trace curves of certain groups of rigid motions in a single variable (eingliedrigen) of the hypersphere into itself. As hypersphere curves of $R^4$ they are given by the equations

$$\begin{aligned}
x_1' &= \phantom{-}x_1 \cos mt + x_2 \sin mt, \\
x_2' &= -x_1 \sin mt + x_2 \cos mt, \\
x_3' &= \qquad\qquad\qquad\qquad\qquad\quad x_3 \cos nt + x_4 \sin nt, \\
x_4' &= \qquad\qquad\qquad\qquad\qquad\, -x_3 \sin nt + x_4 \cos nt.
\end{aligned}$$

Here $m$ and $n$ are coprime positive integers; $t$ is a continuous parameter. The trace curves are closed curves which are traversed once if $t$ runs from 0 to $2\pi$.

We visualize the sphere by projecting it stereographically from the north pole $(0,0,0,1)$ into the equator plane $x_4 = 0$. The equator plane is a

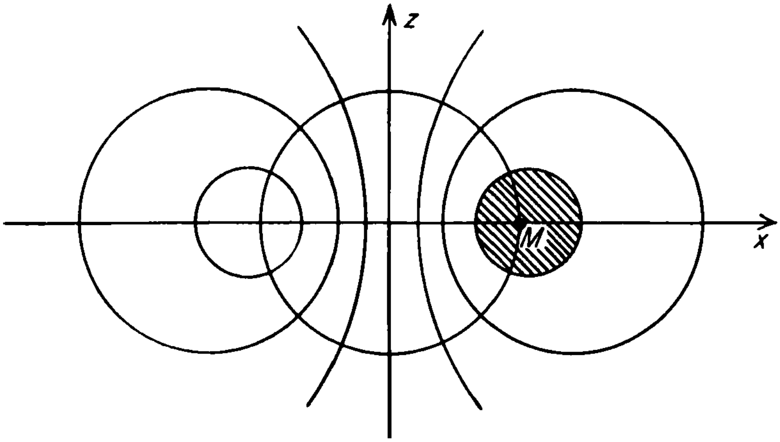


FIG. 5

3-dimensional Euclidean space with the Cartesian coordinates $x, y, z$ which we close to the conformal space by adjoining one single point of infinity, the image of the north pole.[10] Each point $(x_1, x_2, x_3, x_4)$ distinct from the north pole has a unique image with coordinates $x, y, z$; the $x$-, $y$-, and $z$-axes are identified with the $x_1$-, $x_2$-, and $x_3$-axes of $R^4$. The Euclidean space has now in addition to the Euclidean metric (from $R^4$) a spherical metric which comes from the stereographic projection of the hypersphere. The projection transforms the rigid motions of the hypersphere into conformal (or spherical-rigid) motions which permute diametrical balls of the unit sphere $x^2 + y^2 + z^2 = 1$. In particular, the above described continuous group is mapped into a group which sends the $z$-axis and the unit circle $x^2 + y^2 = 1$, $z = 0$, to itself.

Then the $\infty^1$ tori, which have the $z$-axis as axis of rotation and which intersect each of the spheres through the unit circle orthogonally, are all mapped into themselves. Figure 5 shows a section of the torus with the $x, z$-plane. Each of the tori bounds a solid torus which contains the unit circle in its interior and is fibered by the trace curves of the group of motions. For a half-plane bounded by the $z$-axis is under a motion of the group rotated about the $z$-axis. The circular section of the half-plane with a solid torus (shaded in the figure) is spherical-rigidly rotated about its spherical center $M$ about the angle $2\pi n/m$ during the time that the half-plane is rotated once about the $z$-axis. The characteristic numbers $\mu$ and $\nu$ of the fibered solid torus are therefore $\mu = m$ and $\nu =$ absolute value of $n$, reduced mod $m$ to $[-\frac{1}{2}m, \frac{1}{2}m]$.

The part of the hypersphere lying outside the torus considered is also a solid torus fibered by trace curves which has the $z$-axis as middle fiber. For under the rigid motion $x_1' = x_3$, $x_2' = x_4$, $x_3' = x_1$, $x_4' = x_2$, that is, under the

[10] Cf. DB II §7, §1, and §2.

corresponding spherical motion of the conformal space, the unit circle and $z$-axis are interchanged. The characteristic numbers of this solid torus are $\mu = n$ and $\nu = m$ (reduced mod $n$ to the interval $[-\frac{1}{2}n, \frac{1}{2}n]$). The unit circle is therefore an exceptional fiber of multiplicity $m$, and the $z$-axis an exceptional fiber of multiplicity $n$. Each other trace curve is an ordinary fiber since it is contained in a fibered solid torus neighborhood of the unit circle. Each ordinary fiber wraps $m$ times about the $z$-axis and $n$ times about the unit circle; hence it is knotted, namely, a torus knot[11] if $m$ and $n$ are different from 1.

The orbit surface of a hypersphere fibration is always the 2-sphere. For each closed curve in $S^3$ can be deformed into a point; therefore the same holds for the orbit surface. Since $S^3$ is closed, so is the orbit surface (§2); hence it can only be the 2-sphere. Here is a direct verification in the case that $m = n = 1$, in which case there are no exceptional fibers. In this case the trace curves are circles, which include the $z$-axis and the unit circle. Each circle intersects the interior of the unit circle exactly once, except for the unit circle. If a point in the interior of the unit disk approaches the boundary, then the trace curve through this point approaches the unit circle. Thus one has to close the interior of the unit circle with one single point, the image point of the unit circle, to obtain the orbit surface. This completion gives us the 2-sphere.

The orbit surface cannot be embedded into the hypersphere so that each fiber intersects it in its image point, because a 2-sphere in $S^3$ intersects any closed curve in an even number of points.[12]

In §11 we shall show that the fiberings described above are the only possible fiberings of the hypersphere; i.e., any fibering of $S^3$ can be mapped to one of these under a fiber preserving map.

## 4. Triangulations of Fibered Spaces

The fibered spaces are defined as topological spaces via point sets, but it is well known that there are also other, purely combinatorial, definitions of manifolds which use different things for their construction, namely, cells of dimensions 0 to 3. A combinatorial manifold determines a topological manifold if we fill in the cells (which can be chosen to be simplexes) with points. In 2 dimensions, any topological space satisfying the corresponding axioms (1)–(4) of §1 can be triangulated (see Footnote 9), and therefore one can base theorems about 2-manifolds on the topological or the combinatorial definition, whichever is more convenient. In three and more dimensions,

[11] K. Reidemeister, Knoten und Gruppen. *Abh. Math. Sem Univ. Hamburg* **5** (1927), 19.

[12] Since each point of the hyperspere is mapped to a point of the orbit surface, we have a map of $S^3$ onto $S^2$. It is the same map which H. Hopf investigates in "Über die Abbildungen der 3-dimensionalen Sphäre auf die Kugelfläche" [*Math. Ann.* **104** (1931), 637–665].

however, it is not yet proved that a manifold satisfying axioms (1)–(4) of §1 can be triangulated.* Therefore it is important to know that fibered spaces can be triangulated, so that we can use both the methods of point set and combinatorial topology. We now present a lemma which is useful but not necessary for the proof of the triangulation of fibered spaces.

LEMMA 3. *If $\omega$ is a (closed) 2-cell on the orbit surface $f$ which contains no exceptional points, then $\omega$ is an orbit neighborhood of each of its interior points. If $\omega$ contains exactly one exceptional point in its interior, then $\omega$ is an orbit neighborhood of this exceptional point.*

*Proof.* Let $h$ be the exceptional point, or if $\omega$ has no exceptional points, let $h$ be an arbitrary interior point of $\omega$. Take a triangulation of $\omega$ which is so small that each 2-simplex is covered by an orbit neighborhood. Furthermore we require that $h$ lie in the interior of a 2-simplex. Such a triangulation exists, for mapping $\omega$ onto a disk of $R^2$, we find a positive radius $\varepsilon$ such that a disk of radius $\varepsilon$ about an arbitrary point $p$ of $\omega$ is covered by an orbit neighborhood (which is not necessarily the orbit neighborhood of $p$). If the $\varepsilon$-disk is not contained in the disk, we consider only the part belonging to $\omega$. If there did not exist such an $\varepsilon$, there would exist a sequence of disks whose radii and center points converge to 0 and a point $p_0$, respectively, and each of which could not be covered with an orbit neighborhood. Then we could take a disk of radius $\rho > 0$ about $p_0$ which is covered by the orbit neighborhood of $p_0$. This disk contains almost all disks of the sequence, almost all of which can therefore be covered by one orbit neighborhood. This contradiction assures the existence of an $\varepsilon$ as above. We now triangulate $\omega$ so small that each 2-simplex can be covered by a disk of radius $\varepsilon$. Then we apply Lemma 1 to the $\varepsilon$-disks and find that all 2-simplexes are orbit neighborhoods. The corresponding fiber neighborhoods are ordinary fibered solid tori, except possibly for the orbit neighborhood $\Delta_H$ of the fiber $H$ which is mapped into the 2-simplex $\delta_H$ containing $h$. Now, as is well known, there is a sequence of 2-cells $\omega_1 = \delta_h, \omega_2, \ldots, \omega_\sigma = \omega$, which all are 2-simplexes of the triangulation of $\omega$ and such that each is obtained from its predecessor by adjoining an adjacent 2-simplex along one or two edges, a fact which, by the way, may not be true in 3 dimensions. The corresponding fiber sets $\Omega_1 = \Delta_H, \Omega_2, \ldots, \Omega_\sigma = \Omega$ are fiber neighborhoods of $H$. For as $\omega_i$ is obtained from $\omega_{i-1}$ by pasting on a 2-simplex $\delta$ along a single 1-cell $s$ (which may consist of one or two edges of $\delta$), we obtain $\Omega_i$ from $\Omega_{i-1}$ by pasting an ordinary fibered solid torus $\Delta$ fiber preservingly to $\Omega_{i-1}$ along a fibered annulus $S$. It is easy to see that this gives us again a fibered solid torus.

THEOREM 2. *Every fibered space can be triangulated.*

*Proof.* We take a triangulation of the orbit surface such that the exceptional points are contained in the interior of the 2-simplexes and such

* *Translator's note*: This paper was printed December 14, 1932.

that no 2-simplex contains more than one exceptional point. By Lemma 3 each 2-simplex is an orbit neighborhood. The fibered space is therefore decomposed into a finite or countable number of fibered solid tori. Two adjacent such solid tori have a fibered annulus in common, which is mapped onto a 1-simplex of the orbit surface and which can be mapped onto a rectangle of $R^2$ after removing a spanning arc. We can therefore speak about straight lines in such an annulus. These are lines which map to straight lines of the rectangle. Now we triangulate each of the fibered solid tori so that the triangulation of the three annuli which make up the boundary of the solid torus is "linear." On each of these annuli there are now two triangulations which come from the triangulations of the two adjacent solid tori and which can be replaced by a common subdivision since they are linear. This gives us a decomposition of the fibered space into cells. From this we can deduce a simplicial triangulation by barycentric subdivision.

## 5. Drilling and Filling (Surgery)

An essential aid for the classification of fibered spaces will be the method of *drilling out* exceptional fibers and replacing the *drill hole* by ordinary fibered solid tori. To drill out a fiber $H$ from a fibered space $F$ means to remove from $F$ the interior points of a fiber neighborhood $\Omega_H$ of $H$. This results in a fibered space $\bar{F}$ *with boundary*. The boundary is a fibered torus. The orbit surface $\bar{f}$ of $\bar{F}$ is obtained from the orbit surface $f$ of $F$ by removing the interior points of the orbit neighborhood $\omega_h$ into which the fiber neighborhood $\Omega_H$ is mapped.

We first show that the space $\bar{F}$ is independent of the choice of the fiber neighborhood of the fiber $H$ and second that $\bar{F}$ is independent of the choice of $H$ if $H$ is an ordinary fiber. Then we get back fibered spaces $F$ by *closing* an arbitrary fibered space $\bar{F}$ with boundary with suitable fibered *torus seals* (Verschluss ring).

LEMMA 4. *If $\Omega$ and $\Omega'$ are two fiber neighborhoods of a fiber $H$ in a fibered space $F$, there exists a fiber preserving deformation of $F$ which sends $\Omega$ to $\Omega'$ and leaves $H$ fixed.*

*Proof.* Between $\Omega$ and $\Omega'$ we put a fiber neighborhood $\Omega_1$ of $H$ which lies in the interior of $\Omega$ and $\Omega'$ and show that there exists a fiber preserving deformation of $F$ that keeps $H$ fixed and sends $\Omega$ to $\Omega_1$. Then there is also a deformation which sends $\Omega'$ to $\Omega$ since $\Omega'$ is not distinguished from $\Omega$. The required deformation is the first deformation followed by the inverse of the second. The existence of such a fiber neighborhood $\Omega_1$ follows from Lemma 1 since for any two orbit neighborhoods $\omega$ and $\omega'$ of $h$ there exists an orbit neighborhood $\omega_1$ of $h$ which lies in the interior of $\omega$ and $\omega'$.

We now take another orbit neighborhood $\omega_a$ of $h$ which contains $\omega$ in its interior. This is possible; one can choose for $\omega_a$ a 2-cell which contains $\omega$ in

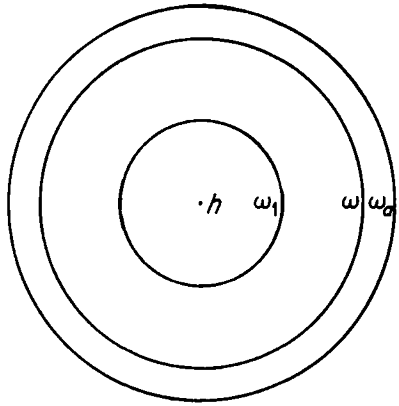


FIG. 6

its interior and contains no exceptional points except $h$. This 2-cell exists since the orbit neighborhoods are closed and exceptional points have no accumulation points, and it is an orbit neighborhood by Lemma 3.

To get a model, we map $\omega_a$ onto a disk of $R^2$, with the image of $h$ as center point, and such that $\omega$ and $\omega_1$ are mapped to concentric circles (Fig. 6). Now we perform on the disk a deformation which sends $\omega_1$ to $\omega$ (for example, by radially blowing up $\omega_1$). This deformation of the orbit neighborhood of $h$ corresponds to a fiber preserving deformation of the fiber neighborhood $\Omega_a$ of $H$ which keeps $H$ and the boundary of $\Omega_a$ pointwise fixed. We obtain this deformation of $\Omega_a$ by cutting $\Omega$ into a Euclidean cylinder and transferring the deformation of $\omega_a$ to all meridian disks which are $\mu$-fold branched covering surfaces of $\omega_a$.

Lemma 4 implies that the fibered space $\bar{F}$, which is obtained from $F$ by drilling out a fiber $H$, is independent of the choice of the (infinitely many) fiber neighborhoods of $H$.

LEMMA 5. *The fibered space with boundary $\bar{F}$, which is obtained from $F$ by drilling out an ordinary fiber $H$, is independent of the choice of the ordinary fiber $H$.*

*Proof.* If $H$ and $H'$ are two ordinary fibers of $F$, $h$ and $h'$ their image points on the orbit surface $f$, there exists a 2-cell $\omega$ which contains $h$ and $h'$ in its interior and contains no points which are images of exceptional fibers. Then there exists a deformation of $\omega$ which sends $h$ to $h'$ and keeps the boundary of $\omega$ fixed. By Lemma 3, $\omega$ is an orbit neighborhood of each of its interior points and hence the image of an ordinary fibered solid torus $\Omega$. The deformation of $\omega$ corresponds to a fiber preserving deformation of $\Omega$ which sends $H$ to $H'$ and leaves the boundary torus of $\Omega$ pointwise fixed.

The same arguments apply to the drilled-out space $\bar{F}$ and show that the space obtained from $F$ by drilling out an arbitrary number of ordinary fibers is independent from the choice of the ordinary fibers which are drilled out. The only requirement is that the drilled-out fiber neighborhoods be mutually disjoint.

From the fibered space with boundary $\bar{F}$ that is obtained from $F$ by drilling out a fiber we can construct new (closed) fibered spaces by closing the boundary torus $\bar{\Pi}$ of $\bar{F}$ with a fibered solid torus, the *torus seal* $V$. This is achieved by a fiber preserving pasting of the boundary torus $\Pi$ of $V$ to the torus $\bar{\Pi}$. Given the torus seal $V$, this closing can be made in infinitely many essentially different ways. But the closing is completely determined if one knows the image $\bar{M}$ of a meridian curve $M$ of $V$ on the torus $\bar{\Pi}$. Obviously, $\bar{M}$ can neither be null homologous nor homologous to a fiber on $\bar{\Pi}$ since otherwise this would be true for $M$ on $\Pi$; furthermore, $\bar{M}$ is without singular points. These are all requirements for $\bar{M}$. For we have

LEMMA 6. *If on the boundary torus $\bar{\Pi}$ of a fibered space with boundary $\bar{F}$ we have a simple closed curve $\bar{M}$ on $\bar{\Pi}$ which is neither homologous to 0 nor to a fiber, then there exists exactly one fibered solid torus $V$ whose boundary torus $\Pi$ can be mapped under a fiber preserving map onto $\bar{\Pi}$ such that $\bar{M}$ is homotopic to 0 in $V$. The thus resulting (closed) fibered space $F_1$ is uniquely determined by $\bar{F}$ and the homology class of $\bar{M}$ on $\bar{\Pi}$.*

*Proof.* (a) First we show that there exists one and only one fibered solid torus $V$ that satisfies the requirements of the theorem. If $\bar{Q}$ is a crossing curve, $\bar{H}$ an oriented fiber on $\bar{\Pi}$, we have

$$\bar{M} \sim \alpha\bar{Q} + \beta\bar{H} \qquad (\alpha = 0, (\alpha, \beta) = 1).$$

In §1 it was shown that there exists exactly one fibered solid torus $V$ with meridian $M$, fiber $H$, and suitable chosen crossing curve $Q$ such that on the boundary $\Pi$ of $V$ we have

$$M \sim \alpha Q + \beta H.$$

We can map $\Pi$ onto $\bar{\Pi}$ under a fiber preserving map such that $Q$ goes to $\bar{Q}$ and $H$ to $\bar{H}$. For we can cut $\Pi$, $\bar{\Pi}$ along $Q$ and $H$, $\bar{Q}$ and $\bar{H}$, respectively, into two rectangles which are hatched by the fibers and we can map these rectangles onto each other under a fiber preserving map. Then $\bar{M}$ is mapped to $M$, and thus $\bar{M}$ becomes a meridian of $V$.

(b) We now show that the fibered space $F_1$ is uniquely determined by $\bar{F}$ and the homology class of $\bar{M}$ (on $\bar{\Pi}$). All possible fiber preserving maps of $\bar{\Pi}$ onto $\Pi$ under which $\bar{M}$ becomes homotopic to 0 in $V$ are obtained from a single such map followed by a fiber preserving map $A_\Pi$ from $\Pi$ onto $\Pi$ which maps the meridian $M$, or more precisely its homology class, to itself or its negative. We shall have proved the independence of the resulting fibering $F_1$ from the choice of the above maps once we have shown that we can extend $A_\Pi$ to a fiber preserving map $A_V$ of $V$ onto $V$ whose restriction to $\Pi$ is $A_\Pi$.

We first check how the homology classes of $\Pi$ are transformed under $A_\Pi$. Let $H$, $Q$, and $M$ be fiber, crossing curve, and meridian curve on $\Pi$, respectively, with an arbitrary but fixed orientation, and let

$$M \sim \alpha Q + \beta H.$$

Because of the transformations (4) in §1 we can choose $Q$ a priori such that $\alpha > 0$ and $0 \leqslant \beta < \alpha$; of course, since $M$ is a simple closed curve, $\alpha$ and $\beta$ are coprime. Let $H'$, $Q'$, $M'$ be the images of these curves under $A_\Pi$. Since $A_\Pi$ is fiber preserving, we have from §1

$$H' \sim \varepsilon_1 H, \qquad Q' \sim \varepsilon_2 Q + \lambda H \qquad (\varepsilon_1, \varepsilon_2 \pm 1). \tag{1}$$

The meridian curve $M$ is mapped under $A_\Pi$ into

$$M' \sim \alpha Q' + \beta H' \sim \varepsilon_2 \alpha Q + (\varepsilon_1 \beta + \alpha\lambda) H.$$

Now we must have that $M' \sim \varepsilon_3 M$, hence

$$\varepsilon_2 \alpha Q + (\varepsilon_1 \beta + \alpha\lambda) H \sim \varepsilon_3(\alpha Q + \beta H).$$

Comparing coefficients, we get $\varepsilon_2 = \varepsilon_3$ and

$$\alpha\lambda + \varepsilon_1 \beta = \varepsilon_2 \beta. \tag{2}$$

If $\alpha > 2$, this implies $\lambda = 0$ and for (1) there are only the two possibilities

$$\alpha > 2 \qquad \begin{cases} (1) & H' \sim H, \quad Q' \sim Q \\ (2) & H' \sim -H, \quad Q' \sim -Q. \end{cases}$$

For $\alpha = 2$ we must have $\lambda = +1, -1$, or 0, since $0 < \beta < \alpha$. Thus there are 4 possibilities

$$\alpha = 2 \qquad \begin{cases} (1) & H' \sim H, \quad Q' \sim Q \\ (2) & H' \sim -H, \quad Q' \sim -Q \\ (3) & H' \sim -H, \quad Q' \sim Q + H \\ (4) & H' \sim H, \quad Q' \sim -Q - H. \end{cases}$$

For $\alpha = 1$ we again get $\lambda = 0$ and we obtain the four possibilities

$$\alpha = 1 \qquad \{ H' \sim \pm H, \quad Q' \sim \pm Q$$

with all four combinations of the signs.

The map $A_V$ which we have to construct will be the composition of two fiber preserving maps $A_V = J_V \cdot B_V$.[13] $B_V$ is an arbitrary fiber preserving map which transforms the homology classes on $\Pi$ in the same way as $A_\Pi$ does. $J_V$ maps each class to itself. We cut $V$ into a right circular cylinder. In case that $H' \sim -H$, $Q' \sim -Q$ we let $B_V$ be a rotation of $\Pi$ about a line orthogonal to the cylinder axis. Then $B_V$ is fiber preserving and sends each homology class on $\Pi$ to its negative. In the case $\alpha = 1$ we obtain the desired map $B_V$ by the

[13] $J_V \cdot B_V$ is the map obtained by first applying $B_V$, then $J_V$.

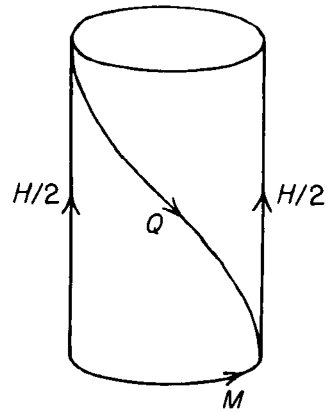


FIG. 7

rotation as in the previous case or by a reflection on a plane which is orthogonal to or passes through the cylinder axis. In the case $\alpha = 2$, the fiber is made up of two lines lying diametrical to the middle fiber. Since $M \sim 2Q + H$, the crossing curve appears as in Fig. 7. A transformation (3) is obtained by reflecting the cylinder at the plane orthogonal to the cylinder axis and going through its center point; a transformation (4) is obtained by reflecting at a plane which goes through the axis.

It remains to be shown that for an arbitrary fiber preserving map $J_\Pi$ of $\Pi$ onto itself which maps each homology class of $\Pi$ to itself, there exists a fiber preserving map $J_V$ of $V$ onto itself whose restriction to $\Pi$ is $J_\Pi$. We show first that $J_\Pi$ can be deformed to the identity by a fiber preserving deformation. We can show this, e.g., by first taking a rigid translation of the fiber into itself such that the image $Q'$ of $Q$ is mapped onto $Q$. Such a deformation is possible since by hypothesis $Q'$ is homologous to $Q$ on the boundary torus. This is followed by a fiber preserving deformation which interchanges the fibers and such that the composition keeps $Q$ pointwise fixed. The map $J_\Pi$ so deformed appears in the fibered rectangle, which is obtained from $\Pi$ by cutting along a fiber $H$ and $Q$, as a fiber preserving map $C$ which leaves the two parallel edges $Q$ pointwise fixed and which translates the inner points only along their fibers. To transform this map of the rectangle into the identity by a fiber preserving map, we proceed as in the proof of the Tietze deformation theorem by Alexander. We complete the rectangle to a strip by the region which is shaded in Fig. 8 and define a map $C'$ of this strip which coincides with $C$ in the rectangle and is the identity in the shaded region. Let $T(t)$ be a stretching of the band upward which leaves the lower boundary $Q$ of the band fixed: the ordinate $\xi$ of a point of the band should go over to $t\xi$. Then $T(t)^{-1}C'T(t) = C'(t)$ is a topological map of the strip, which maps the rectangle fiber preservingly into itself for $t \geqslant 1$. For $t = 1$ this map coincides with $C$ in the rectangle. As $t \to \infty$, $C'(t)$ continuously approaches the identity.

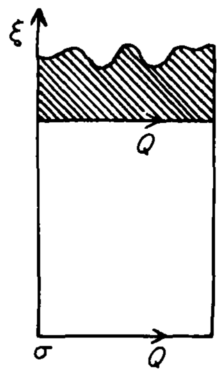


FIG. 8

Thus $C$ and therefore $J_\Pi$ is deformed into the identity by a fiber preserving deformation.

We now describe this deformation by a parameter $\tau$ which decreases from 1 to $\frac{1}{2}$. Let the map corresponding to $\tau$ be $J_\Pi(\tau)$. To extend $J_\Pi$ to the desired map $J_V$, we cut $V$ to a cylinder (of radius 1) and introduce cylindrical coordinates $z, \varphi, \rho$. Then $\rho = \text{const}$ gives a concentric torus of radius $\rho$. We map each of the tori onto itself under a fiber preserving map. The boundary torus is mapped under $J_\Pi = J_\Pi(1)$. If the map $J_\Pi(\tau)$ in the coordinates $z, \varphi$ is given by

$$\left.\begin{aligned} z' &= z'(z,\varphi,\tau) \\ \varphi' &= \varphi'(z,\varphi,\tau) \end{aligned}\right\}, \qquad (J_\Pi(\tau))$$

the map $J_V$ for $1 \geqslant \rho \geqslant \frac{1}{2}$ is defined by

$$\left.\begin{aligned} z' &= z'(z,\varphi,\rho) \\ \varphi' &= \varphi'(z,\varphi,\rho) \\ \rho' &= \rho \end{aligned}\right\}, \qquad (J_V)$$

whereas for $\frac{1}{2} \geqslant \rho \geqslant 0$ it is the identity. This construction of the map $A_V$ completes the proof of Lemma 6.

Instead of constructing $A_V$ as above, we could have described this map directly in terms of cylindrical coordinates. For if

$$\left.\begin{aligned} \bar{z} &= \bar{z}(z,\varphi) \quad [= z'(z,\varphi,1)] \\ \bar{\varphi} &= \bar{\varphi}(z,\varphi) \quad [= \varphi'(z,\varphi,1)] \end{aligned}\right\}, \qquad (J_\Pi)$$

describes the map $J_\Pi$ of the torus $\Pi$ in terms of cylindrical coordinates, then the desired map $A_V$ is given in the range $1 \geqslant \rho \geqslant \frac{1}{2}$ by

$$\left.\begin{aligned} z' &= 2(\rho - \tfrac{1}{2})\bar{z} - 2(\rho - 1)z \\ \varphi' &= 2(\rho - \tfrac{1}{2})\bar{\varphi} - 2(\rho - 1)\varphi \\ \rho' &= \rho \end{aligned}\right\}, \qquad (A_V)$$

and for $\frac{1}{2} \geqslant \rho \geqslant 0$ it is the identity. However, since it is not quite easy to demonstrate that this map $A_V$ is a homeomorphism, we have chosen the method above.

## 6. Classes of Fibered Spaces

If $w$ is a path on the orbit surface $f$ from a point $h_1$ to a point $h_2$, we can in the fibered space deform the fiber $H_1$ into the fiber $H_2$ over fibers so that the image on $f$ runs along $w$. The path $w$ does not determine the mapping of $H_1$ to $H_2$ pointwise, but during the deformation the fiber can be translated in itself. But the map of $H_1$ to $H_2$ is determined up to orientation preserving autohomeomorphisms of $H_2$. Therefore, if $H_1$ is oriented, then the orientation is translated uniquely to $H_2$ along the path $w$. We shall take up this point more closely at the end of this section.

If $w'$ is another path of $h_1$ to $h_2$, the translation of a fixed orientation of $h_1$ along $w'$ can lead to a different result as translation along $w$. However, the end orientations agree if $w$ can be deformed to $w'$ on the orbit surface. In particular, if $w$ is a closed curve on $f$, it is possible that running along $w$ the orientation of the fiber is preserved or changed. Depending on whether we have the first or second case, we associate the value $+1$ or $-1$ to the curve $w$. Since this value is invariant under deformations of the curve, to each element of the fundamental group there corresponds a unique value. To the product $a \cdot b$ of two elements of the fundamental group corresponds the product of the two corresponding values; the inverse of $a$ has the same value as $a$. This implies that the value of a curve is determined already by its homology class. For each null homologous curve has value $+1$ since it represents an element of the commutator subgroup of the fundamental group, and is therefore a product of commutators, and each commutator $aba^{-1}b^{-1}$ has value $+1$. Therefore the values of all curves are known if the values of a fundamental system of curves of the fundamental group, or even the homology group, are known.

We say that two fibered spaces $F$ and $F'$ belong to the same *class* if their orbit surfaces $f$ and $f'$ can be mapped onto each other under a homeomorphism such that each curve is mapped to one with the same value. The class of a fibered space is therefore determined by its "*valuated orbit surface*." Two fibered spaces belong certainly to different classes if their orbit surfaces are not homeomorphic. However, spaces belonging to different classes may have the same orbit surface. We shall give a complete enumeration of the classes in §7 and §8. For example, for the projective plane there are two classes, depending on whether the orientation of the fiber is preserved or reversed along the projective line. For a simply connected surface there is only one class since each closed curve on it is null homologous, hence has value $+1$.

If we drill out a fiber of the space and replace the drill hole by a new torus seal as in §5 , the class of the fibered space is not changed. For the class is already determined if we know the value of one curve in each homology class. The representatives of the homology classes can then be chosen so that they are not affected by the drilling and filling, i.e., this process of changing the space does not affect the valuation of the curves, as it does not affect the orbit surface.

If we drill out all the exceptional fibers from a fibered space $F$ and fill in the drill holes with ordinary torus seals, we obtain from $F$ by this process (but not in a unique way) another space $F_0$ which has no exceptional fibers and belongs to the same class as $F$. Conversely, we can get back $F$ from $F_0$. Therefore we first would like to characterize all spaces without exceptional fibers belonging to the same class. To this end, we cut the orbit surface $f$ of a space $F_0$ into the fundamental polygon $\mathfrak{v}$, where we have to require that $f$ be closed, hence $F$ be a closed space. We adopt this restriction from now on. We change the fundamental polygon to a polygon $\bar{\mathfrak{v}}$ by cutting off the vertices, which means that we change the surface $f$ to a punctured surface $\bar{f}$ by cutting out a 2-cell which contains the vertex $h$ of $\mathfrak{v}$. Figure 9 shows the punctured fundamental polygon of the orientable surface of genus $p = 2$. We can think of $\bar{f}$ as the orbit surface of a space $\bar{F}_0$ which is obtained from $F_0$ by drilling out a fiber $H$. Then $\bar{F}_0$ is uniquely determined by $F_0$ since $\bar{F}_0$ does not depend on the choice of the drilled out ordinary fiber, by Lemma 5 (§5). Now we triangulate $\bar{f}$ using the edges of the polygon $\bar{\mathfrak{v}}$ (dotted lines of Fig. 9). The fibers of $\bar{F}_0$ which map to points of a 2-simplex of the triangulation constitute an ordinary fibered solid torus, by Lemma 3 (§4). As in the proof of Lemma 3 we can build up the polygon $\bar{\mathfrak{v}}$ step by step from 2-simplexes so that after each step we obtain a 2-cell. This construction corresponds to a construction of $\bar{F}_0$ from ordinary fibered solid tori, which gives us an ordinary fibered solid torus $\bar{V}$. The edges of $\bar{\mathfrak{v}}$ correspond in $\bar{V}$ to fibered annuli. If two edges $a'$ and $a''$ in $\bar{\mathfrak{v}}$ are identified with an arc $a$ of $\bar{f}$, we have to identify the corresponding

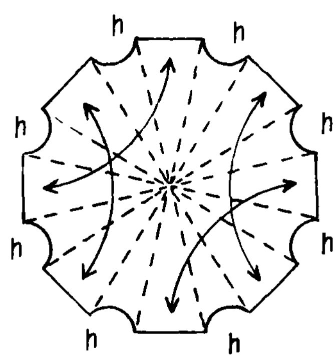


FIG. 9

annuli $A'$ and $A''$ in $\bar{V}$ with a fibered annulus $A$ of $\bar{F}_0$ under a fiber preserving map. If we identify in this way all the corresponding annuli of $\bar{V}$, we get $\bar{F}_0$. If we know how two edges $a'$ and $a''$ of $\bar{v}$ are identified (under an orientation preserving or orientation reversing map) and whether the fiber orientation is preserved or reversed along a closed curve of $\bar{f}$ which intersects the edges of $\bar{v}$ only in one point of the edge $a$, then the identification of the annuli $A'$ and $A''$ is uniquely determined up to an orientation preserving and fiber preserving map of one of the annuli onto itself, say $A'$. This map of $A'$ can be induced by a fiber preserving map of the solid torus $\bar{V}$ which keeps all the other annuli (which correspond in pairs) fixed. The map of $A'$ to $A'$ has therefore no effect on the closing of $\bar{V}$ to $\bar{F}_0$. All fibered spaces with boundary obtained in this way can be mapped onto $\bar{F}_0$ under a fiber preserving map.

This shows that all closed fibered spaces $F_0$ without exceptional fibers which belong to the same class give the same fibered space (with boundary) $\bar{F}_0$ after drilling out an arbitrary fiber. If we drill out $r + 1$ fibers instead of just one, we again obtain the same fibered space (bounded by $r + 1$ tori), namely, the sapce obtained from $\bar{F}_0$ by drilling out $r$ fibers. As the proof of Lemma 5 (§5) shows, it does not matter which fibers of $\bar{F}_0$ are drilled out. We sum up:

THEOREM 3. *Each class of closed fibered spaces determines (and is determined by) a unique fibered space with boundary, the* ***classifying space*** $\bar{F}_0$. *The classifying space is the only fibered space with boundary and without exceptional fibers which has as orbit surface the punctured valuated orbit surface which characterizes the class. From* $\bar{F}_0$ *we obtain all spaces of the class by drilling out a finite number r of fibers and closing the* $r + 1$ *boundary tori with arbitrary torus seals*. The enumeration of all classes will be given in Theorem 7, §8.

So far, we started with a given fibered space $F$ and defined its class, i.e., its valuated orbit surface. Now we start with an arbitrary valuated closed surface and show that it is the valuated orbit surface of a class. We cut the given surface $f$ into the fundamental polygon $v$ as above and puncture it by cutting off the vertices of $v$ to get $\bar{v}$. The ordinary fibered solid torus $\bar{V}$ which has $\bar{v}$ as meridian disk can be made into a fibered space (with boundary) $\bar{F}_0$ by identifying under a fiber preserving map any two annuli $A'$ and $A''$ on the boundary on $\bar{V}$ which map to corresponding edges $a'$ and $a''$ of $\bar{v}$ such that a fiber of $A'$ is identified with a fiber of $A''$ if the point of $a'$ is identified with the corresponding point of $a''$. Then there exist essentially two distinct maps of $A'$ to $A''$. For if we orient the the fibers of $\bar{V}$ simultaneously so that any two oriented fibers on $\bar{V}$ are homologous, we can map $A'$ to $A''$ under a map which preserves and under a map which reverses the fiber orientation. In the first case the orientation of the fiber is preserved along a curve which goes from a point of $A'$ through the interior of $\bar{V}$ to the equivalent point of $A''$; in the second case it is reversed. If we identify in this way any two annuli of $\bar{V}$ which correspond to two equivalent edges of $\bar{v}$ under one of the two maps, we get a space with a boundary $\Pi_0$ which consists of fibers. These boundary fibers correspond to the boundary curve $\bar{v}$. Therefore $\Pi_0$ is a torus or a Klein bottle. To show that $\Pi_0$ is a torus, we observe that if we run along the boundary curve of $\bar{f}$, we cross each edge of the polygon $v$ exactly twice. In both cases the fiber orientation is either preserved or reversed so that if we run once along the boundary curve the fiber orientation is preserved; but this is the case only for the torus. The space obtained from $\bar{V}$ under the identifications is therefore a fibered space (with boundary) without exceptional fibers. Its orbit surface is the punctured surface $f$, whose valuation was obtained from an arbitrary valuation of the edges of a fundamental polygon (namely the fundamental polygon dual to $v$). This proves

THEOREM 4. *For an arbitrary valuated closed surface there is a corresponding class of fibered spaces. A valuation of the surface is obtained by an arbitrary valuation of a canonical system of fundamental curves, i.e., the edges of a Poincaré fundamental polygon of the surface.*

We proved the last remark by constructing for any arbitrarily given valuation of the fundamental curves a space $\bar{F}_0$ whose orbit surface is the given punctured surface; the valuation of the orbit surface determined by $\bar{F}_0$ agrees for the fundamental curves with the arbitrarily given valuation. One could easily have shown directly that an arbitrary valuation of the fundamental curves, i.e., of the generators of the fundamental group, leads to a well-defined valuation of the whole group since each generator appears exactly twice in the single relation of the fundamental group, and therefore an arbitrary valuation of the generators gives a well-defined valuation of the single defining relation and hence of each relation between elements of the fundamemtal group.

Theorems 3 and 4 give us the tools to determine complete invariants of fibered spaces under fiber preserving maps. We now describe in detail the translation of the fiber orientation along a path which was used in the definition of class. If $w$ is a path on the orbit surface from a point $h_1$ to a point $h_2$ and if $s$ is a continuous parameter from 0 to 1 on $w$, we have for each value $s$ of the parameter a point $h(s)$ of $f$ and hence a fiber $H(s)$. Orient each fiber $H(s)$ arbitrarily. If the same fiber $H$ belongs to different values $s$, which happens if $w$ has multiple points, we give $H$ the same number of mutually independent orientations. A fiber neighborhood of $H(s)$ or, more precisely, the corresponding orbit neighborhood on $f$ cuts out from $w$ a neighborhood of the point $h(s)$. If for each value of $s$ all the fibers corresponding to the path near $h(s)$ are homologous in the fiber neighborhood of $H(s)$, where a $\mu$-fold exceptional fiber counts $\mu$ times, we say that *the fibers are oriented simultaneously along* $w$. It is clear that there exists such a simultaneous orientation of the fibers along $w$ if $w$ is covered by one orbit neighborhood $\omega$; because then we need only orient all the fibers which map to points of $w$ so that they are homologous in $\Omega$.[14] In the general case we decompose $w$ into finitely many pieces so that each piece lies in the interior of an orbit neighborhood. The fibers of the individual pieces can be oriented simultaneously so that each fiber at the intersection of two pieces gets the same orientation from the two pieces. Then all the fibers are oriented simultaneously along $w$. The fibers can be oriented simultaneously along $w$ apparently only in two opposite ways; the orientation along $w$ is determined by the orientation of a single fiber, e.g., the initial fiber. Under a simultaneous orientation of the fibers of $w$, the orientation of the first fiber is translated along $w$ to the last fiber.

If $w$ and $w'$ are homotopic curves of the orbit surface which both go from $h_1$ to $h_2$, and if the fiber $H_1$ is oriented, then the translation of the orientation along $w$ and $w'$ to $H_2$ gives the same result; i.e., the fiber orientation is preserved under translation along the closed path $ww'^{-1}$. For $ww'^{-1}$ bounds a singular 2-cell on $f$, i.e., the continuous image of a 2-cell $e$. We triangulate $e$ so small that the image of each 2-simplex is contained in an orbit neighborhood on $f$. Since the path $ww'^{-1}$ can be built up from boundary paths of 2-simplexes by canceling out edges which are traveled in opposite directions, and since the fiber orientation is preserved along a closed path which lies in an orbit neighborhood, the fiber orientation is preserved along $ww'^{-1}$.

We now want to solve the problem whether and in how many different ways the orbit surface $\bar{f}_0$ can be embedded in the classifying space $\bar{F}_0$ so that each fiber intersects it exactly in its image point. To this end, we cut $\bar{f}_0$ into a fundamental polygon $\bar{u}$ which, in contrast to the fundamental polygon $\bar{v}$ above, contains the hole of $\bar{f}_0$ in its interior, i.e., $\bar{u}$ is a punctured 2-cell. This corresponds to a cutting of $\bar{F}_0$ into a fibered hollow torus $\bar{U}$. The "inner" boundary surface $\Pi_0$ of $\bar{U}$ is mapped onto the boundary of the hole of $\bar{u}$, whereas the "outer" boundary $\Sigma$ is decomposed into an even number $2j$ of pairwise equivalent fibered annuli which map onto edges of the polygon $\bar{u}$. Suppose we have succeeded in embedding $\bar{f}_0$ into $\bar{F}_0$; then $\bar{f}_0$ appears in $\bar{U}$ necessarily as an annulus which meets $\Sigma$ in a crossing curve $Q$ and $\Pi_0$ in a crossing curve $Q_0$. If $Q_1', \ldots, Q_j'', Q_1'', \ldots, Q_j''$ are the $2j$ oriented edges which make up $Q$ and which correspond to the $2j$ lateral surfaces of $\Sigma$, then if two such lateral surfaces (annuli) $A_i'$ and $A_i''$ are identified, the two edges $Q_i'$ and $Q_i''$ which they contain have to be identified under an orientation preserving or reversing map. (Conversely,) a crossing curve $Q$ with this property can always be found on $\Sigma$ by choosing the crossing lines $Q_1', Q_2', \ldots, Q_j'$ arbitrarily, but such that their end points go to the same

[14] In this case we say that the fibers of $\Omega$ are oriented simultaneously. More generally we talk about a simultaneous orientation of all fibers of a fibered space if in each fiber neighborhood any two fibers are homologous, where a $\mu$-fold execptional fiber counts $\mu$ times. Not every fibered space admits a simultaneous orientation of fibers but only the spaces of the classes Oo and Nn I of p. 391.

point under the identification of the lateral surfaces. Then $\bar{f}_0$ can be embedded into $\bar{F}_0$; for example, we can cut $\bar{U}$ into a hollow cylinder (annulus $\times$ I) and draw from the points of $Q$ radii which lie orthogonal to the cylinder axis. These radii in $\bar{U}$ make up the required orbit surface.

Suppose now we have $\bar{f}_0$ embedded into $\bar{F}_0$ in a different way, with crossing curves $Q^*$ and $Q_0^*$ instead of $Q$ and $Q_0$. The lines $Q_i'$ and $Q_i'^*$ of $Q$ (resp. $Q^*$), which lie in the same lateral side $A_i'$ of $\Sigma$, have (after choosing an orientation of $\Sigma$) a certain intersection number[15] $\gamma_i'$; here we assume that $Q_i'$ and $Q_i'^*$ have no common endpoints, which can be achieved by a small deformation of the embedded orbit surfaces. Since under the identification of the corresponding lateral sides $A_i'$ and $A_i''$ the lines $Q_i'$ and $Q_i'^*$ are identified with the lines $Q_i''$ and $Q_i''^*$ (resp. with $-Q_i''$ and $Q_i''^*$), the intersection number is $\gamma_i'' = -\gamma_i'$ or $= +\gamma_i'$, depending on whether $A_i'$ and $A_i''$ form an association of type one or two.[16] $\gamma = \sum_{i=1}^{j} \gamma_i' + \gamma_i''$, i.e., the intersection number of $Q$ and $Q^*$ is 0 if all the lateral sides of $\sum$ are identified in the first way, i.e., if $\bar{F}_0$ is orientable. Otherwise we can choose $Q^*$ such that $\gamma$ is a given even number. Therefore, if $\bar{F}_0$ is orientable, $Q$ can be deformed into $Q^*$ and hence $Q_0$ into $Q_0^*$, i.e., on the boundary surface $\Pi_0$ of $\bar{F}_0$ there exists a crossing curve $Q_0$ which is determined up to orientation and deformations, such that $Q_0$ is the intersection of $\Pi_0$ and the orbit surface $\bar{f}_0$ is embedded in $\bar{F}_0$. If $\bar{F}_0$ is nonorientable, there are besides $Q_0$ infinitely many crossing curves $Q_0^*$ which can be the intersection of $\bar{f}_0$ and $\Pi_0$. They all differ from $Q_0$ by an even multiple of the fiber. If we cut the fibered torus $\times$ I, $\bar{U}$ along the embedded orbit surface $\bar{f}_0$, we obtain a drilled-out fibered prism in which bottom and top surface are equivalent and the lateral surfaces are pairwise equivalent. We shall use this representation of the classifying space in §10 to determine the fundamental group.

## 7. The Orientable Fibered Spaces

Our task to determine all fibered spaces and to characterize them by invariants splits into two parts: first, to determine all the classes; second to list all spaces of a given class. We first solve this problem for orientable spaces.

First suppose the orbit surface is orientable of genus $p$. Since the space is orientable, the fiber orientation is preserved along any curve of the surface. For if $w$ is a closed curve of value $-1$ on the orbit surface (which misses exceptional points), there is a fiber preserving deformation of the space which traces the fiber $H$ along the curve $w$. This is so because $w$ can be covered with finitely many orbit neighborhoods without exceptional points (Heine–Borel).

[15] O. Veblen, "Analysis Situs," 2nd ed., Amer. Math. Soc. Colloq. Publ. No. 5, Part 2. Amer. Math. Soc., New York, 1931.

[16] H. Tietze, Topologische Invarianten, *Monatsh. Math. Phys.* **19** (1907). [See Seifert and Threlfall, this Lehrbuch p. 220.]

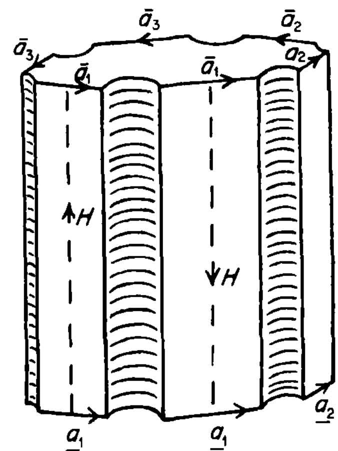


FIG. 10

Inside each orbit neighborhood one can apply the fiber preserving deformation of the proof of Lemma 5 and thus deform the fiber step by step along $w$ into its initial position. In particular we can choose the deformation such that an orbit neighborhood $\omega$ of the point $h$ comes back to itself, since along $w$ the orientation of the surface is not changed because it is orientable. The corresponding fibered solid torus $\Omega$ is then mapped onto itself under an orientation reversing map. But, by a well-known theorem, the orientation of an orientable space is not reversed under a deformation. Therefore, all curves have value $+1$, and there is for each orientable orbit surface a single class of orientable fibered spaces. Now the fibered topological product of a punctured surface of genus $p$ and $S^1$ is an orientable fibered space whose orbit surface is the punctured surface of genus $p$ and all of whose curves are of value $+1$. Since this space has no exceptional fibers, it is the classifying space $\bar{F}_0$.

*Even if the orbit surface is nonorientable, there is only one corresponding class of orientable spaces.* As in the above case we first observe that the fiber orientation is preserved along an orientation preserving curve of the orbit surface. But if $w$ is an orientation reversing curve of the orbit surface, then the space is orientable only if the fiber orientation is reversed along $w$. Therefore the valuation is determined by the surface. The classifying space is in this case not the topological product of the punctured surface of genus $k$ and $S^1$, but has to be constructed by the method of §6. Figure 10 shows it for $k = 3$. In the prism we have to identify bottom and top disks under a translation. The two lateral surfaces in which we have drawn the fiber $H$ are to be identified so that the edge $\underline{a}_1$ of one surface is identified with the edge $\overline{a}_1$ of the other surface. Similarly we have to identify the other four unshaded lateral sides of the prism. The six shaded sides become the boundary torus of the classifying space and the bottom surface becomes the orbit surface.

This finishes off the determination of the class and we now proceed to determine the invariants of an orientable fiber space $F$. We orient the space and the invariants depend on the orientation. We shall obtain the invariants by drilling out the exceptional fibers of $F$ and replacing them by ordinary solid tori whose meridians are uniquely determined by the fibered space $F$ up to orientation. In this way we get from the oriented space $F$ a unique oriented space $F_0$ without exceptional fibers. Let $C_1$ be an exceptional fiber of $F$ and $\Omega_1$ a fiber neighborhood of $C_1$. The solid torus $\Omega_1$ gets a certain orientation from $F$, which induces on the boundary torus $\Pi_1$ of $\Omega_1$ a certain orientation o. On $\Pi_1$ we choose an oriented crossing curve $Q$ and an oriented fiber $H$. These two curves determine an orientation o′ on $\Pi_1$. For, cutting $\Pi_1$ along $Q$ and $H$ into a rectangle, a certain orientation of it is determined by the sequence $QHQ^{-1}H^{-1}$. By reversing the orientation of one of the curves $Q$ and $H$, we reverse the orientation o′. But o′ is not changed by reversing the orientation of both curves simultaneously. We now orient $Q$ and $H$ so that o′ agrees with o. This can be expressed by saying that using the orientation o the curves $Q$ and $H$ shall have intersection number $+1$. Another pair of curves $Q_1$ and $H_1$ which determines the same orientation o′ = o on $\Pi_1$ is related to $Q$ and $H$ (on $\Pi_1$) as follows:

$$H \sim \varepsilon H_1, \qquad Q \sim \varepsilon Q_1 + yH_1 \qquad (\varepsilon = \pm 1, \quad y \text{ arbitrary integer}). \tag{1}$$

For if $Q_1, H_1$ determine the same orientation as $Q, H$, the determinant of the transformation must have value $+1$. This implies that in the transformation formulas (1) and (4) of §1, $\varepsilon_1 = \varepsilon_4$ $(= \varepsilon)$. The meridian curve $M_1$ of the solid torus $\Omega_1$ can now be expressed in terms of $Q$ and $H$ as

$$M_1 \sim \alpha Q + \beta H \sim \varepsilon\alpha Q_1 + (\alpha y + \varepsilon\beta)H_1 = \alpha_1 Q_1 + \beta_1 H_1. \tag{2}$$

We can choose $Q_1$ and $H_1$ such that

$$\alpha_1 > 1 \qquad \text{and} \qquad 0 < \beta_1 < \alpha_1, \tag{3}$$

which determines $\varepsilon$ and $y$. If instead of $M_1$ we choose the meridian curve with opposite orientation, we only have to reverse both the orientations of $Q_1$ and $H_1$ to obtain the same homology $M_1 \sim \alpha_1 Q_1 + \beta_1 H_1$. Hence the numbers $\alpha_1$, $\beta_1$ are determined uniquely by the nonoriented meridian of $\Omega_1$ and the crossing curve $Q_1$ is determined up to its orientation. We now drill out $\Omega_1$ and replace the drill hole with a new torus seal $V_1$ which has $Q_1$ as meridian curve. Then $V_1$ is an ordinary fibered solid torus since the meridian is a crossing curve. Thus we have derived an orientable fibered space $F_1$ from $F$ which is uniquely determined by $F$, the orientation of $F$, and the drilled-out exceptional fiber. For $F_1$ is independent of which fiber neighborhood $\Omega_1$ of $C_1$ is drilled out because by Lemma 4 (§5) we can deform an arbitrary fiber neighborhood of the fiber $C_1$ onto another under a fiber preserving deformation of $F$.

We now apply this construction to $F_1$, i.e., we drill out an exceptional fiber $C_2$ and obtain the pair $\alpha_2$, $\beta_2$ as additional invariants of the oriented space $F$.

Continuing in this way, we finally obtain an oriented space $\overline{F}_0$ without exceptional fibers which is determined by $F$ and its orientation. $F_0$ is independent of the order in which we have drilled out the exceptional fibers of $F$ because we can drill them all out at the same time by choosing the fiber neighborhoods sufficiently small.

From $F_0$ we drill out an arbitrary fiber neighborhood $V_0$ and obtain the class space $\overline{F}_0$ of $F$. It inherits the orientation from $F$. Since $\overline{F}_0$ is orientable there is a distinguished crossing curve $Q_0$ on the boundary torus $\Pi_0$ of $\overline{F}_0$ which is determined up to orientation and deformations as the boundary of the orbit surface $\bar{f}_0$ embedded in $\overline{F}_0$ (see §6). We orient $Q_0$ and a fiber $H_0$ of $\Pi_0$ so that they give on $\Pi_0$ the same orientation as that induced by $V_0$. The meridian curve $M_0$ of $V_0$, which is a crossing curve, is in the system $Q_0, H_0$ of the form

$$M_0 \sim Q_0 + bH_0. \tag{4}$$

The integer $b$ is determined by the oriented space $F_0$, hence by $F$ and its orientation.

This gives us a complete system of invariants of $F$, by the following:

THEOREM 5. *An orientable fibered space $F$ together with its orientation is determined by a one-to-one correspondence by a system of invariants*

$$(\mathrm{O}, \mathrm{o}; p \mid b; \alpha_1, \beta_1; \alpha_2, \beta_2; \ldots; \alpha_r, \beta_r)$$

*or*

$$(\mathrm{O}, \mathrm{n}; k \mid b; \alpha_1, \beta_1; \alpha_2, \beta_2; \ldots; \alpha_r, \beta_r).$$

*Here* O *means that $F$ is orientable*; o (*resp.* n) *means that the orbit surface is orientable* (*resp. nonorientable*). *$p$ and $k$ are the genus* [*number of handles* (*resp. cross-caps*)] *of the orientable* (*resp. nonorientable*) *orbit surface. The three symbols to the left of the bar determine therefore the class of $F$. The number $b$ determines uniquely the construction of the space without exceptional fibers $F_0$ from the class space $\overline{F}_0$. The numbers $\alpha_i$, $\beta_i$ determine uniquely* (*one-to-one*) *the exceptional fibers in $F$.*

The theorem tells us when two orientable fibered spaces with given orientations are homeomorphic under an orientation and fiber preserving map. Theorem 6 shows how the invariants change if the orientation is reversed.

We have seen how to find the system of invariants for a given oriented space $F$. To show that this system is complete, we construct conversely to a given system of invariants a unique oriented space $F$. The numbers $p$ (resp. $k$) determine the class (see p. 384) and hence by Theorem 3 (§6) the class space $\overline{F}_0$. We can orient $\overline{F}_0$ arbitrarily since there exists a fiber preserving and orientation reversing map of $\overline{F}_0$ onto itself (reflection of the solid torus $\overline{V}$ of §6 on a meridian disk). This determines the crossing curve $Q_0$ of the boundary torus $\Pi_0$ of $\overline{F}_0$ and a fiber $H_0$ up to simultaneous reversion of their orientation. $b$ determines $M_0 \sim Q_0 + bH_0$ up to orientation and therefore the closing of $\bar{F}_0$ to $F_0$ uniquely. From $F_0$ we have to drill out $r$ arbitrary fibers; the resulting space which is bounded by $r$ tori is independent of the choice of the drilled-out fibers by Lemma 5. On each of the boundary tori there is a distinguished (up to orientation) crossing curve $Q_i$, namely, the meridian of the drilled-out solid torus, and the orientation of $F_0$ therefore determines a pair of curves $Q_i, H_i$ up to simultaneous reversion of orientation. This determines uniquely the meridian $M_i \sim \alpha_i Q_i + \beta_i H_i$ of the new torus seal (up to orientation) and therefore uniquely the closing of $F_0$ to $F$.

We now describe for an orientable fibered space $F$ a useful "diagram" $\bar{V}_0$ which together with $\bar{F}_0$ determines the space. Choose in $F$ disjoint fiber neighborhoods $\Omega_i$ of the exceptional fibers. Then the ordinary torus seals $V_i$ which replace the drill holes in $F_0$ are disjoint. We can choose the fiber neighborhood $V_0$, which we removed from $F_0$ to obtain the class space $\bar{F}_0$, in such a way that it contains all torus seals $V_i$ in its interior by Lemma 3. The fibered space with boundary $\bar{V}_0$ that is obtained from $V_0$ after removing the $V_i$, and which is the topological product of $S^1$ and a disk punctured $r$ times, is the *diagram* of the fibered space $F$ if the distinguished crossing curve $Q_0$ of $\bar{F}_0$ is drawn on the boundary torus $\Pi_0$ of $\bar{V}_0$, and the meridian curves $M_i$ of the drill holes $\Omega_i$ are drawn on the remaining $r$ boundary tori $\Pi_i$. Obviously $Q_0$ determines how one has to glue on the class space $\bar{F}_0$ [which is determined by $p$ (resp. $k$)] to the boundary torus $\Pi_0$. By Lemma 6, $M_i$ determines the filling in of the drill hole $\Omega_i$. Furthermore, if we orient $\bar{V}_0$, we get an orientation of $F$.

To obtain the invariants $b; \alpha_1, \beta_1; \ldots; \alpha_r, \beta_r$ of $F$ from the diagram $\bar{V}_0$, we orient the fibers of $\bar{V}_0$ simultaneously, i.e., so that they are homologous in $\bar{V}_0$. Then the orientation of the fibers $H_0, H_1, \ldots, H_r$ on the boundary tori $\Pi_0, \Pi_1, \ldots, \Pi_r$ is determined. Hence the crossing curves $Q_1, \ldots, Q_r$ on the boundary tori are determined together with their orientation by requiring that the orientation on $\Pi_i$ which is induced by $Q_i$ and $H_i$ shall be opposite to the orientation induced by $\bar{V}_0$, and by requiring that the numbers $\alpha_i, \beta_i$ in

$$M_i \sim \alpha_i Q_i + \beta_i H_i \qquad (\text{on } \Pi_i) \tag{5}$$

satisfy $\alpha_i > 1$, $0 < \beta_i < \alpha_i$. The $Q_i$ are meridians of the torus seals $V_i$. Closing $\bar{V}_0$ with the $V_i$, we obtain an ordinary solid torus $V_0$ with the meridian

$$M_0 \sim Q_0 + bH_0 \qquad (\text{on } \Pi_0) \tag{6}$$

and it is easily proved that

$$M_0 \sim Q_1 + Q_2 + \cdots + Q_r \qquad (\text{in } \bar{V}_0)$$

and hence

$$-Q_0 + Q_1 + Q_2 + \cdots + Q_r \sim bH_0 \qquad (\text{in } \bar{V}_0). \tag{7}$$

Figure 11 shows $\bar{V}_0$ with $r = 3$, $b = 4$.

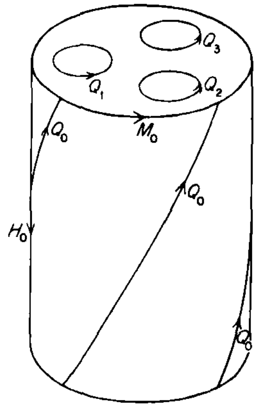


FIG. 11

We now want to find out how the invariants are changed if the orientation of $F$ is reversed. In the diagram $\overline{V}_0$ only the orientation is reversed, but not the curves $M_i$ and $Q_0$. It is useful to reverse the orientations of the fibers of $\overline{V}_0$ simultaneously; let $H_0', H_1', \ldots, H_r'$ be the fibers $H_0, H_1, \ldots, H_r$, but with opposite orientation:

$$H_i' \sim -H_i \qquad (\text{on } \Pi_i,\ i = 0, 1, \ldots, r). \tag{8}$$

We have to replace the $Q_1, Q_2, \ldots, Q_r$ by the crossing curves $Q_1'$, $Q_2', \ldots, Q_r'$. Then

$$Q_i' \sim Q_i + y_i H_i \qquad (\text{on } \Pi_i,\ i = 1, 2, \ldots, r). \tag{9}$$

The sign of $Q_i$ is $+1$ since the determinant of the transformation of the pair (8) and (9) has value $-1$, so that the orientation on $\overline{V}_0$ is reversed and hence the orientation of $\Pi_i$. For the same reason

$$Q_0' \sim Q_0 \qquad (\text{on } \Pi_0). \tag{10}$$

Then we have for the meridian $M_i$

$$M_i \sim \alpha_i Q_i + \beta_i H_i \sim \alpha_i Q_i' + (\alpha_i y_i - \beta_i) H_i' = \alpha_i' Q_i' + \beta_i' H_i'.$$

The requirement $\alpha_i' > 1$ and $0 < \beta_i' < \alpha_i'$ gives us $\alpha_i' = \alpha_i$ and $\beta_i' = \alpha_i - \beta_i$, i.e., $y_i = 1$. $b'$ is [as $b$ from (7)] now determined by

$$-Q_0' + Q_1' + \cdots + Q_r' \sim b' H_0'. \tag{11}$$

Using (7)–(10), we get $b' = -r - b$.

THEOREM 6. *The oriented fibered space $F$ with invariants*

$$(\mathrm{O}, \mathrm{o}; p \mid b; \alpha_1, \beta_1; \ldots; \alpha_r, \beta_r)$$

[*resp.*

$$(\mathrm{O}, \mathrm{n}; k \mid b; \alpha_1, \beta_1; \ldots; \alpha_r, \beta_r)]$$

*has after reversing its orientation the invariants*

$$(\mathrm{O}, \mathrm{o}; p \mid -r-b; \alpha_1, \alpha_1-\beta_1; \ldots; \alpha_r, \alpha_r-\beta_r)$$

[*resp.*

$$(\mathrm{O}, \mathrm{n}; k \mid -r-b; \alpha_1, \alpha_1-\beta_1; \ldots; \alpha_r, \alpha_r-\beta_r)].$$

If we had normed the numbers $\beta_i$ to the interval

$$-\tfrac{1}{2}\alpha_i < \beta_i \leqslant \tfrac{1}{2}\alpha_i$$

instead of norming to $0 < \beta_i < \alpha_i$ by (3), the invariants $b, \beta_1, \ldots, \beta_r$ would only change their signs if the orientation of $F$ were reversed, in the case that no exceptional fibers of order 2 were present, i.e., all $\alpha_i > 2$. But if $\alpha_1 = 2, \ldots, \alpha_s = 2$, only the last $r-s$ invariants $\beta$ would change their signs if the orientation were reversed, but $b$ would have to be replaced by $-s-b$, so that choosing the new normalization would not lead to an essential simplification for the purpose of reorientation.

## 8. The Nonorientable Fibered Spaces

As in the orientable case we first determine the classes. First assume the orbit surface $f$ is orientable. Then the genus of $f$ is $>0$, since otherwise $F$ is orientable (see §6 and §7). We show: *For each orientable orbit surface of genus $p > 0$ there is exactly one class of nonorientable spaces.* The claim is true for $p = 1$. For if $a$ and $b$ are two conjugate simple closed curves on a torus, then $a$, say, has value $-1$. We can assume that then $b$ has value $-1$; otherwise we replace $b$ by $ab$. Now suppose the claim is true for genus $p-1$ ($\geqslant 1$). We prove it for $p$ by showing that on a surface of genus $p > 1$ there is a handle on which all curves have value $+1$. Cutting off this handle we get a punctured surface of genus $p-1$ having some curves of value $-1$ which is unique by the induction hypothesis. To show the existence of such a handle choose a system of curves which cuts the surface into a fundamental polygon with boundary $a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_p b_p a_p^{-1} b_p^{-1}$. If there is a pair $a_i, b_i$ of value $+1$, we are done. Otherwise $a_1$, say, has value $-1$. Assume $b_1$ has value $+1$ (otherwise replace $b_1$ by $a_1 b_1$). There is a curve $a_j$ or $b_j$ ($j > 1$) of value $-1$; thus one of the curves $a_1 a_j$ or $a_1 b_j^{-1}$ has value $+1$ and spans together with $b_1$ a handle with each curve of value $+1$.

Since the class is unique we can choose (on a surface of genus $p \geqslant 1$) a canonical system all whose curves have value $-1$.

If the orbit surface $f$ is nonorientable of genus $k$ we represent it as a sphere with $k$ cross-caps $x_1, \ldots, x_k$ (see Fig. 12). Then $a_i$ is a curve which intersects the cross-cap in one point; i.e., $a_i$ is orientation reversing. Then $H_1(f) = \{a_1, \ldots, a_k : 2a_1 + \cdots + 2a_k \sim 0\}$. The valuation of $f$ is therefore determined by the valuation of $a_1, a_2, \ldots, a_k$. If all the $a_i$ have value $-1$, $F$ is orientable. Thus at least one $a_i$ has value $+1$.

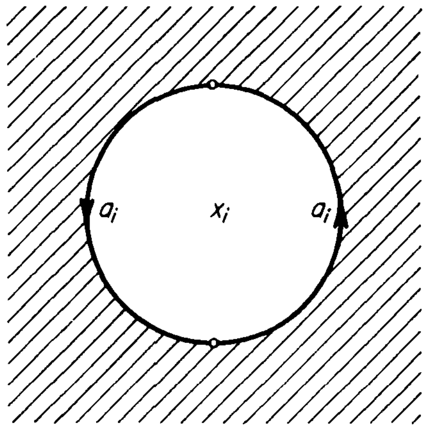


FIG. 12

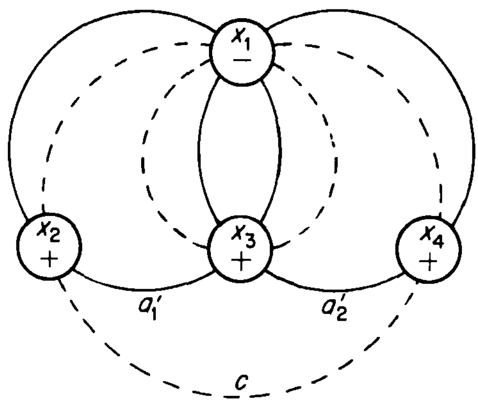


FIG. 13

This leads to the following cases:

Case (a) $a_i$ has value $+1$ for each $i$. Then $F \approx f \times S^1$ and $\overline{F}_0 \approx$ (punctured $f$) $\times S^1$.

Case (b) $k_1$ of the $a_i$ have value $+1$, $k_2 = k - k_1$ have value $-1$ ($k_1 > 0$, $k_2 > 0$). Suppose $f \neq P^2$ ($k = 1$) and $f \neq$ Klein bottle ($k = 2$). We claim that we can always assume that $k_1 = 1$ or $= 2$. This is clear for $k = 3$. Suppose $k > 3$ and $k_1 \neq 1$, $k_1 \neq 2$. There exist at least three $a_i$, say $a_2, a_3, a_4$ of value $+1$ and one, say $a_1$, of value $-1$. Let $l$ be a curve which separates the cross-caps $x_1, x_2, x_3, x_4$ from the others. $l$ separates $f$ into $\varphi$ and $\psi$, where $\varphi$ is a sphere with the cross-caps $x_1, \ldots, x_4$ and one boundary $l$. On $\varphi$ there are two disjoint simple closed curves $a_1' \sim a_1 + a_2 + a_3$ and $a_2' \sim a_1 + a_3 + a_4$ of value $-1$. There is a simple closed orientation reversing curve $c$, disjoint to $a_1' \cup a_2'$, (see Fig. 13), such that the surface $\overline{\varphi}$, obtained from $\varphi$ by cutting along $a_1'$ and $a_2'$, is nonorientable. We can represent $\overline{\varphi}$ as a sphere with two cross caps and three boundary curves $l$, $a_1'^2, a_2'^2$. $\varphi$ is obtained from $\bar{\varphi}$ by identifying diametrical points of $a_1'^2$ and $a_2'^2$. Gluing back $\psi$ and $l$, we have a new representation of $f$ as sphere with $k$ cross-caps. Since $a_1'$ and $a_2'$ now have value $-1$, the number of negative cross-caps has increased at least by 1. Continuing, we get $k_1 = 1$ or $k_1 = 2$.

We now show that the latter two valuations are distinct. Let $d$ be a curve on $f$ such that

$$d \sim \sum \gamma_i a_i \nsim 0 \qquad \text{and} \qquad 2d \sim \sum 2\gamma_i a_i \sim 0. \tag{1}$$

(For example, $d$ can be chosen to be a simple closed curve that intersects each cross-cap exactly once. In this case, cutting $f$ along $d$ we obtain an orientable surface with one or two holes, depending on whether $k$ is odd or even. $d$ is called an *orientation producing* simple closed curve.) Since $2d \sim 0$ is a consequence of $2a_1 + \cdots + 2a_k \sim 0$, $\sum 2\gamma_i a_i$ differs from $\sum 2a_i$ only by a factor and all the $\gamma_i$ are equal and odd since otherwise $d \sim 0$. Hence $d$ has value $(-1)^{k_2}$. Thus the valuations of $f$ with even $k_2$ are different from those with odd $k_2$; in particular the valuations $k_1 = 1$ and $k_1 = 2$ yield different valuations of $f$. The investigation of all classes of fibered spaces is complete.

THEOREM 7. *For each orientable orbit surface $f$ of genus $p$ there is exactly one class of orientable fibered spaces, and if $p > 0$, exactly one class of nonorientable fibered spaces. For each nonorientable orbit surface $f$ of genus $k$ there is exactly one class of orientable fibered spaces, and if $k > 2$, exactly three classes of nonorientable fibered spaces; for $k = 1$ there is one class, for $k = 2$ there are two classes.*

The following table lists the different classes. O, N refer to orientability and nonorientability of $F$, and o, n to the orbit surface $f$, whose genus must be given in order for the class to be determined. Recall that a closed curve $w$ of $f$ is given the value $+1$ if the fiber orientation is preserved along $w$; otherwise $w$ gets the value $-1$, and note that the class and therefore the classifying space $\bar{F}_0$ is uniquely determined by the valuation of all the curves of $f$.

| | |
|---|---|
| Oo | All curves have value $+1$; $\bar{F}_0 \approx$ (punctured $f$) $\times S^1$; |
| On | All one-sided curves have value $-1$; |
| No | There are curves of value $-1$; |
| Nn I | All curves have value $+1$; $\bar{F}_0 \approx$ (punctured $f$) $\times S^1$; |
| Nn II | There are one-sided curves of value $-1$ and of value $+1$; each orientation producing simple closed curve has value $-1$; |
| Nn III | There are one-sided curves of value $-1$ and of value $+1$; each orientation producing simple closed curve has value $+1$. |

For $p = 0$ there is only the class Oo, for k = 1 only On and Nn I, for $k = 2$ only On, Nn I, Nn II. $\bar{F}_0$ can now be constructed as in §6.

We now characterize the nonorientable fibered spaces $F$ by invariants. Let $C_1$ be an exceptional fiber in $F$, $\Omega_1$ a fiber neighborhood of $C_1$, $\Pi_1$ the boundary of $\Omega_1$, $M_1$ a meridian on $\Pi_1$, $Q$ an arbitrary crossing curve on $\Pi_1$, and $H$ a fiber; then

$$M_1 \sim \alpha Q + \beta H \qquad (\text{on } \Pi_1). \tag{2}$$

Using formulas (1) and (4) of §1,

$$H \sim \varepsilon_1 H_1, \qquad Q \sim \varepsilon_4 Q_1 + y H_1. \tag{3}$$

we can choose a new crossing curve $Q_1$ and fiber $H_1$ such that

$$M_1 \sim \alpha_1 Q_1 + \beta_1 H_1 \tag{4}$$

$$\text{with } \alpha_1 > 1, \qquad 0 < \beta_1 \leqslant \tfrac{1}{2}\alpha_1.$$

For the first requirement determines $\varepsilon_4$. Choosing $y$ suitably we reduce $\beta_1$ to $[-\frac{1}{2}\alpha_1, \frac{1}{2}\alpha_1]$ and finally we choose $\varepsilon_1$. There is no orientation on $\Pi_1$ determined by $F$ since $F$ is nonorientable; hence $\varepsilon_1$ and $\varepsilon_4$ can be chosen independently (cf. §7 in the orientable case). $\alpha_1$, $\beta_1$ are uniquely determined by $\Omega_1$ and hence by $C_1$. The same holds, if $\alpha_1 > 2$, for $Q_1$ and $H_1$, up to simultaneously changing their orientation, which is permitted since the orientation of $M_1$ is not given by $\Omega_1$. But for $\alpha_1 = 2$ there is besides $Q_1, H_1$ another system

$$Q_1' \sim Q_1 + H_1, \qquad H_1' \sim -H_1 \tag{5}$$

in which $M_1$ also appears in normal form (4):

$$M_1 \sim 2Q_1' + H_1'. \tag{6}$$

If $\alpha_1 > 2$, we drill out $\Omega_1$ and replace it by an ordinary torus seal $V_1$ having $Q_1$ as meridian and do the same for all exceptional fibers of multiplicity $> 2$. This determines uniquely a nonorientable fibered space $F_s$, which has only $s \geqslant 0$ exceptional fibers of multiplicity 2. To investigate $F_s$ further, we need

LEMMA 7. *A nonorientable fibered space $\bar{F}$ with boundary which is obtained from a (closed) fibered space by drilling out finitely many exceptional fibers admits a fiber preserving autohomeomorphism keeping the boundary tori pointwise fixed except for one, $\bar{\Pi}$. On $\bar{\Pi}$ a given crossing curve $Q$ is mapped to a crossing curve of the form*

$$Q' \sim +(Q + 2zH) \qquad \text{or} \qquad Q' \sim -(Q + 2zH), \tag{7}$$

*where $z$ is an arbitrary integer and $H$ is an oriented fiber on $\bar{\Pi}$. Furthermore, one can choose the homeomorphism orientation preserving or reversing on $\Pi$.*[17]

*Proof.* (a) Let $z = 0$. To find an orientation reversing homeomorphism we glue on $\bar{\Pi}$ an ordinary fibered solid torus $V$ having $Q$ as meridian and get a space $\bar{F} + V$. The required map will be the end result of a fiber preserving deformation of $\bar{F} + V$. Choose on $\bar{F} + V$ a simple closed curve $W$ from an interior point $P$ of $V$ and disjoint to the exceptional fibers which is

[17] The theorem does not claim that we can choose the sign in (7) arbitrarily.

orientation reversing. Deform $\bar{F} + V$ (fiber preservingly) so that $P$ runs along $W$ and at the end $V$ is mapped to itself (see §7). Then $V$ and hence $\bar{\Pi}$ is mapped to itself under an orientation reversing homeomorphism which maps $Q$ to $Q' \sim +Q$ or $Q' \sim -Q$, depending on whether the fiber orientation is changed along the curve $W$. Finally, remove $V$ to get the desired map of $\bar{F}$.

(b) By (a) there is a fiber preserving map of $\bar{F}$ mapping $Q + zH$ to $\pm(Q + zH)$ and orientation reversing on $\bar{\Pi}$. Here $Q$ is mapped to $Q' \sim \pm(Q + 2zH)$. To get such an orientation preserving map, follow this map by a homeomorphism of $\bar{F}$ sending $Q'$ to $\pm Q'$ and reversing orientation on $\bar{\Pi}$.

We use the lemma to show that $F_s$ is uniquely determined by the class and $s$, if $s > 0$. Drill out the $s$ exceptional fibers. The resulting $\bar{F}_s$ is determined by the class of $F_s$ (= class of $F$) and by $s$, because $\bar{F}_s = \bar{F}_0$ (for $s = 1$) or $\bar{F}_s = \bar{F}_0$ (drilled out $(s-1)$ times) (see §6). From $\bar{F}_s$ we obtain $F_s$ by closing with $s$ solid tori of multiplicity $\mu = 2$. This closing is independent of how the torus seal $\Omega$ is sewn (fiber preservingly) onto the boundary $\Pi$ of $\bar{F}_s$. For if $Q$ is a crossing curve, $H$ a fiber of $\bar{\Pi}$, and $M$ a meridian of $\Omega$, then

$$M \sim 2Q + yH \qquad (\text{on } \bar{\Pi}).$$

We show that the result is independent of $y$. Since $M$ is a simple closed curve, $y$ is odd. If $y \equiv 1 \pmod 4$, there is a fiber preserving map of $\bar{F}_s$ keeping all boundary components fixed except for $\bar{\Pi}$ and such that $\bar{\Pi}$ is mapped orientation preservingly and $Q$ is mapped to

$$Q' \sim \pm\{Q + 2[(1-y)/4]H\};$$

hence $M$ is mapped to

$$M' \sim 2Q' + yH' \sim \pm(2Q + H)$$

(Lemma 7). If $y \equiv -1 \pmod 4$ we choose a fiber preserving map of $\bar{F}_s$ which is orientation reversing on $\bar{\Pi}$ and which sends $Q$ to

$$Q' \sim \pm\{Q + 2[(1+y)/4]H\},$$

hence $M$ to

$$M' \sim \pm(2Q + H).$$

Thus instead of

$$M \sim 2Q + yH$$

we can choose $M' \sim \pm(2Q + H)$ as meridian of the torus seal. Therefore $F_s$ depends only on $\bar{F}_0$ and on $s$.

If $s = 0$, we obtain $F_0$ from $\bar{F}_0$ by closing with an ordinary solid torus having a crossing curve $Q$ on $\Pi_0$ as meridian. On $\Pi_0$ there are exactly two essentially distinct crossing curves. For by Lemma 7, $Q$ can be mapped to $Q' \sim \pm(Q + 2zH)$ by a fiber preserving map of $\bar{F}_0$.

Therefore, if $Q_0$ is a crossing curve of $\Pi_0$, for example $Q_0 = \bar{f}_0 \cap \Pi_0$, where $\bar{f}_0$ is the orbit surface embedded in $\bar{F}_0$, we have only the two cases: $Q \sim Q_0$ or $Q \sim Q_0 + H$. If $\bar{f}_0$ can be embedded into $\bar{F}_0$ so that $\bar{f}_0 \cap \Pi_0 = Q_0$, then $\bar{f}_0$ cannot be embedded into $\bar{F}_0$ so that $\bar{f}_0 \cap \Pi_0 = Q_0 + H$, and vice versa (see §6). Therefore the two cross curves $Q_0$ and $Q_0 + H$ are essentially different, i.e., there is no fiber preserving map of $\bar{F}_0$ to itself which sends $Q_0$ to $\pm(Q_0 + H)$. Therefore the fibered spaces $F_0$ and $F_0'$ obtained from $\bar{F}_0$ by taking $Q_0$ and $Q_0 + H$, respectively, as meridian $Q$ of the torus seal are different. For a fiber preserving map $F_0 \to F_0'$ could be so deformed that the torus seals and hence the meridians of $F_0$ and $F_0'$ correspond; hence there would be a fiber preserving map of $\bar{F}_0$ sending $Q_0$ to $\pm(Q_0 + H)$. The two distinct spaces $F_0$ and $F_0'$ are therefore determined by $\bar{F}_0$ and by the number $b = 0$ or $= 1$.

Now suppose we know $F_s$ ($s \geqslant 0$). Then $F$ is uniquely determined by

$$\alpha_i, \beta_i \qquad (\alpha_i > 2, 0 < \beta_i < \tfrac{1}{2}\alpha_i), \quad i = s+1, \ldots, r.$$

For, drilling out $r - s$ arbitrary fibers from $F_s$, there is a unique (unoriented) crossing curve $Q_i$ on each boundary torus $\Pi_i$, namely, the meridian of the drilled-out solid torus. Choosing an oriented fiber $H_i$ on $\Pi_i$, the meridian $M_i$ of the new torus seal is determined by

$$M_i \sim \alpha_i Q_i + \beta_i H_i,$$

by Eq. (4). But, since the orientation of $Q_i$ and $H_i$ is arbitrary, we obtain besides $M_i$ another possible meridian

$$M_i' \sim \alpha_i Q_i - \beta_i H_i.$$

By Lemma 7 there is a fiber preserving map of the bounded space which keeps $\Pi_j$ pointwise fixed ($j \neq i$) and maps $\Pi_i$ under an orientation reversing map to itself such that $Q_i \to \pm Q_i$. Then $M_i \to \pm M_i' \sim \pm(\alpha_i Q_i - \beta_i H_i)$. Hence it does not matter which of $M_i$ or $M_i'$ is chosen as meridian of the torus seal. Thus $F$ is uniquely determined by its class and the numbers $\alpha_i$, $\beta_i$, $s$, and $b$. Analogously to Theorem 5 we formulate the result in:

THEOREM 8. *A nonorientable fibered space $F$ is uniquely determined by a system of invariants*

$$(\mathrm{No}; p \mid b; \alpha_1, \beta_1; \ldots; \alpha_s, \beta_s; \alpha_{s+1}, \beta_{s+1}; \ldots; \alpha_r, \beta_r)$$

*or*

$$(\mathrm{Nn\,I}; k \mid b; \alpha_1, \beta_1; \ldots; \alpha_s, \beta_s; \alpha_{s+1}, \beta_{s+1}; \ldots; \alpha_r, \beta_r)$$

*or*

$$(\mathrm{Nn\,II}; k \mid b; \alpha_1, \beta_1; \ldots; \alpha_s, \beta_s; \alpha_{s+1}, \beta_{s+1}; \ldots; \alpha_r, \beta_r)$$

*or*

$$(\mathrm{Nn\,III}; k \mid b; \alpha_1, \beta_1; \ldots; \alpha_s, \beta_s; a_{s+1}, \beta_{s+1}; \ldots; \alpha_r, \beta_r).$$

*Here* N *means that $F$ is nonorientable*; o (*resp.* n) *means that the orbit surface is orientable* (*resp. nonorientable*). *The numbers $\alpha_i$, $\beta_i$ determine the exceptional fibers. $\alpha_i = 2$, $\beta_i = 1$ for $i \leqslant s$ and $\alpha_i > 2$, $0 < \beta_i < \frac{1}{2}\alpha_i$ for $i > s$. $b$ is of any significance only if $s = 0$. In this case $b = 0$ or $= 1$ and determines the closing of the classifying space $\bar{F}_0$ to $F_0$. If $s > 0$, then $F$ is already uniquely determined without specifying $b$, and $b$ is omitted.*

EXAMPLE. Let $F$ be a nonorientable fibered space with one exceptional fiber of multiplicity 3, with $\bar{F}_0$ determined by Nn I; $k$. Here $\bar{F}_0 \approx$ (punctured nonorientable surface of genus $k$) $\times S^1$. We obtain the two different fibered spaces:

$$(\text{Nn I}; k \mid 0; 3, 1) \qquad \text{and} \qquad (\text{Nn I}; k \mid 1; 3, 1).$$

But adding an exceptional fiber of multiplicity 2, both spaces go over into the same space

$$(\text{Nn I}; k \mid -; 2, 1; 3, 1).$$

## 9. Covering Spaces

1. Let $\tilde{F}$ be a (unbranched) covering of $F$ (i.e., there is a covering map $p$ of $\tilde{F}$ onto $F$ such that for each point $P$ of $F$ and each $P_i$ of $p^{-1}(P)$ there exist neighborhoods $U(P)$, $U(P_i)$ such that $p \mid U(P_i): U(P_i) \to U(P)$ is a homeomorphism).

Let $F$ be a fibered space, $H$ a fiber. Let $\tilde{H}$ be a component of $p^{-1}(H)$. Then $\tilde{H} \approx S^1$ or $R^1$. Let $S$ be the collection of all the curves $\tilde{H}$, for all fibers $H$ of $F$. When is $S$ a fibering of $\tilde{F}$?

2. Let $\Omega_C$ be a fiber neighborhood of a fiber $C$ of $F$ and let $\tilde{\Omega}_{\tilde{C}}$ be a component of $p^{-1}(\Omega_C)$. Then $\tilde{\Omega}_{\tilde{C}}$ consists of curves of $S$ and contains the fiber $\tilde{C}$ [which is a component of $p^{-1}(C)$] in its interior. $\tilde{\Omega}_{\tilde{C}}$ is determined by $\Omega_C$ and an integer $\sigma$ (including $\infty$) which denotes the multiplicity of the covering $\tilde{\Omega}_{\tilde{C}} \to \Omega_C$. Thus $\tilde{C} \to C$ is a $\sigma$-fold covering.

3. If $\sigma < \infty$, then all the curves of $\tilde{\Omega}_{\tilde{C}}$ are closed; if $\sigma = \infty$, they are all open. Thus each curve of $S$ has a neighborhood which consists entirely of closed or of open curves of $S$. Hence $\tilde{F}$ is the union of two disjoint open sets, the sets of closed and open curves of $S$. Since $\tilde{F}$ is connected one of these is the empty set. Hence, *$S$ cannot contain closed and open curves at the same time. If* (*all*) *the curves of $S$ are closed, then $S$ is a fibering of $\tilde{F}$*, since a finite covering of a fiber neighborhood, $\Omega_C$ is again a fibered solid torus.

4. From now on we assume that $S$ is a fibering of $\tilde{F}$. Since the covering $\tilde{\Omega}_{\tilde{C}} \to \Omega_C$ is completely determined by the integer $\sigma$, we can compute the invariants $\tilde{\mu}, \tilde{\nu}$ of $\tilde{\Omega}_{\tilde{C}}$ from the invariants $\mu, \nu$ of $\Omega_C$ and from $\sigma$. Cutting $\tilde{\Omega}_{\tilde{C}}$ into a fibered cylinder, we have to identify the top and bottom disks under a

rotation through

$$2\pi \frac{\tilde{\nu}}{\tilde{\mu}} = 2\pi \frac{\nu}{\mu} \sigma = 2\pi\nu \frac{\sigma/(\mu,\sigma)}{\mu/(\mu,\sigma)} ;$$

$(\mu,\sigma)$ = gcd of $\mu$ and $\sigma$. Therefore, by definition of the characteristic numbers (§1),

$$\tilde{\mu} = \frac{\mu}{(\mu,\sigma)}, \qquad \tilde{\nu} \equiv \pm\nu \frac{\sigma}{(\mu,\sigma)} \pmod{\tilde{\mu}}. \tag{1}$$

Thus in the cylinder $\tilde{\Omega}_{\tilde{C}}$ there are $\tilde{\mu} = \mu/(\mu,\sigma)$ parallel lines, which form one ordinary fiber of $\tilde{\Omega}_{\tilde{C}}$. Thus each ordinary fiber of $\Omega_C$ is covered by $(\mu,\sigma)$ ordinary fibers of $\tilde{\Omega}_{\tilde{C}}$, but the middle fiber $C$ is covered only by one fiber $\tilde{C}$ of $\tilde{\Omega}_{\tilde{C}}$. Therefore $p:\tilde{F}\to F$ induces a continuous map $\bar{p}$ of the orbit surface $\tilde{f}$ onto $f$. If $c$ and $\tilde{c}$ are the points corresponding to the fibers $C$ and $\tilde{C}$, respectively, then if $(\mu,\sigma) > 1$, the covering of $f$ by $\tilde{f}$ is branched over $c$ of branch index $(\mu,\sigma)$. The index of the branching always divides the multiplicity of the exceptional fiber $C$. Hence only exceptional points can occur as branch points.

5. Since $\tilde{\mu} \leqslant \mu$ by (1), the covering $\tilde{C}$ of $C$ is always an ordinary fiber if $C$ is ordinary. But if $C$ is an exceptional fiber ($\mu > 1$), then $\tilde{C}$ may or may not be exceptional. For example, identify two congruently fibered solid tori with an $\alpha$-fold exceptional fiber along their boundary so that congruent points are identified. The result is a fibered space $F$ with invariants $(O,\mathrm{o};\ 0 \mid -1;\ \alpha,\beta;\alpha,\alpha-\beta)$ which is homeomorphic to $S^2 \times S^1$. Taking the $\alpha$-fold covering of each of the solid tori and identifying equivalent points, we get an $\alpha$-fold covering $\tilde{F}\to F$ without exceptional fibers. For the invariants in (1) are $\mu = \alpha$, $\sigma = \alpha$; hence $\tilde{\mu} = 1$ for both (exceptional) fibers.

If $\tilde{H}$ and $\tilde{H}'$ are two fibers of $\tilde{F}$ which cover two *ordinary* fibers $H$ and $H'$, $\rho$ and $\rho'$ times, respectively, then $\rho = \rho'$. For, join $\tilde{H}$ and $\tilde{H}'$ in $\tilde{F}$ by a path whose projection in $F$ does not meet exceptional fibers. Since in a (sufficiently small) neighborhood of an ordinary fiber the multiplicity of the covering is not changed, it remains constant along the entire path.

6. The universal covering space $\hat{F}$ of $F$ is a fibered space if and only if for a fiber $H$ of $F$ a component $\hat{H}$ of $p^{-1}(H)$ is closed (by 3). Then $H$ is covered $\rho$ times by $\hat{H}$, $\rho < \infty$. Since $\hat{H} \simeq 0$ in $\hat{F}$ (simply connected), $H^{\rho} \simeq 0$ in $F$. Therefore, *$\hat{F}$ is a fibered space if and only if a finite multiple of the fiber of $F$ is homotopic to* 0 *in* $F$. Clearly, if this holds for a single fiber $H$, it holds for all fibers of $F$.

7. Let $F$ be a nonorientable fibered space and $\tilde{F}$ the 2-fold orientable covering of $F$. Since any fiber $H$ of $F$ is orientation preserving, $H$ lifts to two closed curves $\tilde{H}$ and $\tilde{H}'$. Hence $\tilde{H}$ is closed and $\tilde{F}$ is a fibered space, and $\sigma = 1$. Therefore $p \mid \tilde{\Omega}_{\tilde{H}}:\tilde{\Omega}_{\tilde{H}} \to \Omega_{H}$ is a fiber preserving homeomorphism. Let $T:\tilde{F}\to\tilde{F}$ be the fiber preserving involution (without fixed

points) which is the nontrivial covering transformation. $T$ reverses the orientation of $\tilde{F}$ and induces a fixed point-free involution of $\tilde{f}$.

For example, let $F$ be the space

$$(\text{No}; p \mid b; \alpha_1, \beta_1; \ldots; \alpha_r, \beta_r). \tag{2}$$

$\tilde{F}$ has $2r$ exceptional fibers; if $H$ is an exceptional fiber with invariants $\alpha_1, \beta_1$, then $H$ is covered by two exceptional fibers $\tilde{H}$ and $\tilde{H}'$ with invariants $\alpha_1, \beta_1$ and $\alpha_1, \alpha_1 - \beta_1$, respectively (by Theorem 6). For the fiber preserving involution of $\tilde{F}$ maps $\tilde{H}$ to $\tilde{H}'$ and reverses the orientation of $\tilde{F}$. Since furthermore $\tilde{f}$ is an (unbranched) 2-fold covering of $f$, $\tilde{f}$ is orientable of genus $2p-1$; hence $\tilde{F}$ is the space

$$(\text{O}, \text{o}; 2p-1 \mid \tilde{b}; \alpha_1, \beta_1; \ldots; \alpha_r, \beta_r; \alpha_1, \alpha_1 - \beta_1; \ldots; \alpha_r, \alpha_r - \beta_r). \tag{3}$$

Since $\tilde{F}$ admits an orientation reversing fiber preserving homeomorphism, the invariants are the same if the orientation of $\tilde{F}$ is reversed. By Theorem 6, $\tilde{F}$ has the invariants

$$(\text{O}, \text{o}; 2p-1 \mid -2r - \tilde{b}; \alpha_1, \beta_1; \ldots; \alpha_r, \beta_r; \alpha_1, \alpha_1 - \beta_1; \ldots; \alpha_r, \alpha_r - \beta_r). \tag{4}$$

For (3) and (4) to be equal we must have that $\tilde{b} = -2r - \tilde{b}$, hence $\tilde{b} = -r$, independent of $b$. Similarly for the other cases. Result:

Let $\tilde{F}$ be the orientable 2-sheeted covering of $F$.

$$\begin{cases} F(\text{No}; p \mid b; \alpha_1, \beta_1; \ldots; \alpha_r, \beta_r) \\ \tilde{F}(\text{Oo}; 2p-1 \mid -r; \alpha_1, \beta_1; \ldots; \alpha_r, \beta_r; \alpha_1, \alpha_1 - \beta_1; \ldots; \alpha_r, \alpha_r, \alpha_r - \beta_r), \end{cases}$$

$$\begin{cases} F(\text{Nn I}; k \mid b; \alpha_1, \beta_1; \ldots; \alpha_r, \beta_r) \\ \tilde{F}(\text{Oo}; k-1 \mid -r; \alpha_1, \beta_1; \ldots; \alpha_r, \beta_r; \alpha_1, \alpha_1 - \beta_1; \ldots; \alpha_r, \alpha_r - \beta_r), \end{cases}$$

$$\begin{cases} F(\text{Nn II}; k \mid b; \alpha_1, \beta_1; \ldots; \alpha_r, \beta_r) \\ \tilde{F}(\text{On}; 2k-2 \mid -r; \alpha_1, \beta_1; \ldots; \alpha_r, \beta_r; \alpha_r, \beta_r; \alpha_1, \alpha_1 - \beta_1; \ldots; \alpha_r, \alpha_r - \beta_r), \end{cases}$$

$$\begin{cases} F(\text{Nn III}; k \mid b; \alpha_1, \beta_1; \ldots; \alpha_r, \beta_r) \\ \tilde{F}(\text{On}; 2k-2 \mid -r; \alpha_1, \beta_1; \ldots; \alpha_r, \beta_r; \alpha_1, \alpha_1 - \beta_1; \ldots; \alpha_r, \alpha_r - \beta_r). \end{cases}$$

In the two latter cases the orbit surface $\tilde{f}$ is nonorientable since there are one-sided curves on $f$ along which the fiber orientation is reversed, i.e., which are orientation preserving in $F$.

8. Let $F$ be a fibered space with orbit surface $f$. Let $\tilde{f}$ be an (unbranched) covering of $f$, $\tilde{p}$ a point over a point $p$ of $f$, and $P$ a point of $F$ which maps to $p$. Let $\tilde{F} = \{(P, \tilde{p})\}$. A neighborhood of a point $(P_0, \tilde{p}_0)$ consists of all points $(P, \tilde{p})$ where $P$ lies in a neighborhood of $P_0$ (in $F$) and $\tilde{p}$ in a neighborhood of $\tilde{p}_0$. Defining $g(P, \tilde{p}) = P$, we see that $g: \tilde{F} \to F$ is a covering of $F$. The

multiplicity of this covering is the multiplicity of the covering $\tilde{f} \to f$. If a point $P$ of $F$ runs along a fiber $H$, then $(P, \tilde{p})$ for fixed $\tilde{p}$ runs along a curve $\tilde{H}$ which lies one-to-one over $H$. Hence $\tilde{F}$ is a fibered space by 3 above and a fiber neighborhood $\Omega_{\tilde{H}}$ of $\tilde{F}$ is mapped onto $\Omega_H$ under a fiber preserving homeomorphism.

For example, let $F$ be the orientable space $(\mathrm{On}; 1 \mid b; \alpha_1, \beta_1; \ldots; \alpha_r, \beta_r)$ with orbit surface the projective plane. Let $\tilde{f}$ be the 2-sphere. Then $\tilde{F}$ is orientable, hence of class (Oo; 0). Orienting $\tilde{F}$ so that $g: \tilde{F} \to F$ is orientation preserving, the fiber neighborhoods $\Omega_{\tilde{H}}$ and $\Omega_{\tilde{H}}'$ map to the same $\Omega_H$ preserving orientations, and therefore to the exceptional fiber with invariants $\alpha, \beta$ correspond in $\tilde{F}$ two exceptional fibers both with invariants $\alpha, \beta$. Drilling out the exceptional fibers of $F$ and filling in ordinary solid tori and doing the same thing in $\tilde{F}$, we obtain $F_0$ and $\tilde{F}_0$ without exceptional fibers and $\tilde{F}_0$ is a 2-fold covering of $F_0$. We find that $\tilde{b} = 2b$; hence $\tilde{F}$ is

$$(\mathrm{Oo}; 0 \mid 2b; \alpha_1, \beta_1; \ldots; \alpha_r, \beta_r; \alpha_1, \beta_1; \ldots; \alpha_r, \beta_r).$$

## 10. Fundamental Groups of Fibered Spaces

We cut the classifying space $\bar{F}_0$ of a fibered space $F$ into a fibered prism with a drill hole, as in §6 but so that the drill hole touches the prism along an edge $H$. Similarly we drill out the $r$ ordinary tori $V_1, \ldots, V_r$ (which have to be replaced by exceptional tori) so that they touch $H$. Then the $r+1$ boundary tori $\Pi_0, \Pi_1, \ldots, \Pi_r$ intersect the bottom surface in the cross curves $Q_0, Q_1, \ldots, Q_r$. (See Fig. 14 for $p = 2$ and $r = 2$).

We obtain the fundamental group of this space $\bar{\bar{F}}_0 = \bar{F}_0 - \mathrm{int}(V_1 \cup \cdots \cup V_r)$ by running around the 2-cells. Then for an orientable orbit surface of genus $p \geqslant 0$ we have[18]

$$\begin{aligned} \pi_1\left(\bar{\bar{F}}_0\right) = \{ & A_1, B_1, \ldots, A_p, B_p, Q_0, Q_1, \ldots, Q_r, H : \\ & A_i H A_i^{-1} = H^{\varepsilon_i}, B_i H B_i^{-1} = H^{\varepsilon_i'} \ (i = 1, \ldots, p; \varepsilon_i, \varepsilon_i' = \pm 1), \\ & Q_0 Q_1 \cdots Q_r = A_1 B_1 A_1^{-1} B_1^{-1} \cdots A_p B_p A_p^{-1} B_p^{-1}, \\ & Q_j H Q_j^{-1} = H \ (j = 0, 1, \ldots, r) \}. \end{aligned} \tag{1}$$

Here $\varepsilon_i$ $(\varepsilon_i') = \pm 1$ or $-1$ depending on whether the fiber orientation is preserved or reversed along $A_i$ $(B_i)$.

For $p = 0$ we get the relations

$$\begin{aligned} & Q_0 Q_1 \cdots Q_r = 1, \\ & Q_j H Q_j^{-1} = H \qquad (j = 0, 1, \ldots, r). \end{aligned} \tag{2}$$

[18] Cf. H. Seifert, Konstruction dreidimensionaler geschl. Räume, *Ber. Sächs. Akad. Wiss.* **83** (1931), 33. The auxiliary paths and therefore the relations of the first type are redundant, since $\bar{\bar{F}}_0$ contains only one vertex.

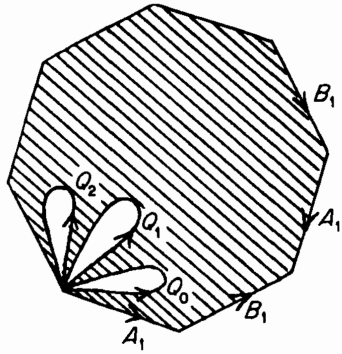


FIG. 14

For a nonorientable orbit surface of genus $k$ we get

$$\begin{aligned}\pi_1\left(\overline{\overline{F}}_0\right) = \{ & A_1, \ldots, A_k, Q_0, Q_1, \ldots, Q_k, H : \\ & A_i H A_i^{-1} = H^{\varepsilon_i} \, (i = 1,2, \ldots, k; \varepsilon_i = \pm 1), \\ & Q_0 Q_1 \cdots Q_r = A_1^2 \cdots A_k^2, \\ & Q_j H Q_j^{-1} = H, \, (j = 0, 1, \ldots, r)\}. \end{aligned} \tag{3}$$

$\pi_1(F)$ is obtained from $\pi_1(\overline{\overline{F}}_0)$ by adding $r+1$ relations which correspond to the $r+1$ torus seals of the boundary tori $\Pi_0, \Pi_1, \ldots, \Pi_r$. They are

$$Q_0 H^b = Q_1^{\alpha_1} H^{\beta_1} = \cdots = Q_r^{\alpha_r} H^{\beta_r} = 1. \tag{4}$$

For example, $Q_1^{\alpha_1} H^{\beta_1} = 1$ means that the meridian $M_1 \sim \alpha_1 Q_1 + \beta_1 H_1$ of the torus seal belonging to $\Pi_1$ is null homotopic in the torus seal. For example, the fundamental group of the space (Oo; $0 \mid$ b; $\alpha_1, \beta_1, \ldots, \alpha_r, \beta_r$) has the relations

$$\begin{aligned} & Q_o H^b = Q_1^{\alpha_1} H^{\beta_1} = \cdots = Q_r^{\alpha_r} H^{\beta_r} = Q_0 Q_1 \cdots Q_r = 1, \\ & Q_j H Q_j^{-1} = H \qquad (j = 0, 1, \ldots, r). \end{aligned} \tag{5}$$

Adding the relations $Q_0 = Q_1 = \cdots = Q_r = H = 1$ we obtain from $\pi_1(F)$ the fundamental group $\pi_1(f)$ of the orbit surface $f$. Geometrically this can be seen as follows: The mapping of $F \to f$ induces a homomorphism[19] of $\pi_1(F)$ onto $\pi_1(f)$ and therefore $\pi_1(f)$ is a quotient group of $\pi_1(F)$. Similarly $H_1(f)$ is a quotient group of $H_1(F)$, and this is also true for open fibered spaces. (We shall use this fact in §14.)

Among the closed 3-dimensional manifolds the ones which occur as fundamental regions (Diskontinuitätsbereiche) of 3-dimensional spherical groups of motions, and thus have finite fundamental groups, have been thoroughly investigated. Therefore we are interested in the question whether

[19] A homomorphism is sometimes called a "one- or multiple-to-one isomorphism".

the fibered spaces give us new manifolds of finite fundamental group, or if they are already included among the fundamental regions. In DB II (see footnote 1) we shall show that the fibered spaces with finite fundamental group coincide with the fundamental regions of fixedpoint-free spherical groups of motions. A necessary condition for the finiteness of the fundamental group of $F$ is that the fundamental group of the orbit space $f$ be finite since the latter is a quotient of the former. Hence $f$ is a 2-sphere or projective plane.

If $f$ is a 2-sphere, then (5) are the relations of the fundamental group of $F$. Adding $H = 1$, we obtain the factor group

$$\{\bar{Q}_0, \bar{Q}_1, \ldots, \bar{Q}_r : \bar{Q}_1^{\alpha_1} = \cdots = \bar{Q}_r^{\alpha_r} = \bar{Q}_1 \cdots \bar{Q}_r = 1\}. \tag{6}$$

For $r \geqslant 3$ this is a polygon net group. Taking an $r$-gon with angles $\pi/\alpha_1, \ldots, \pi/\alpha_r$ on the 2-sphere, the Euclidean plane, or the hyperbolic plane, depending on whether

$$\sum_{i=1}^{r} \frac{1}{\alpha_i} >, =, \text{or} < r-2, \tag{7}$$

and reflecting it successively on its sides, we obtain a polygon net which covers the sphere, or the Euclidean or hyperbolic plane, with alternating congruent and mirror imaged (black and white) $r$-gons. It admits a group of orientation preserving covering translations which has as fundamental region a double polygon, i.e., a white and adjacent black $r$-gon. This group is the above factor group (6).[20] Since for $r > 3$ this polygon cannot lie on the 2-sphere so as to cover it, it follows that (6) and hence (5) is infinite. For $r = 3$ the group (6) is finite only if it is a Platonian group, i.e., if $\alpha_1, \alpha_2, \alpha_3$ is one of the triples $(2,2,n), (2,3,3), (2,3,4), (2,3,5)$ ($n \geqslant 2$). It can be shown (DB II, §7) that for these triples the group (5) is finite. If $r \leqslant 2$, then (5) is cyclic (finite or infinite).

If $f$ is the projective plane, then $F$ is the space

$$(\mathrm{On}; 1 \mid b; \alpha_1, \beta_1; \ldots; \alpha_r, \beta_r) \tag{8}$$

since a nonorientable (closed) 3-manifold has infinite fundamental group. This follows also since the first Betti number of the fundamental groups of fibered spaces is $> 0$.[21] The space (8) has a 2-fold orientable covering (§9), namely,

$$(\mathrm{Oo}; 0 \mid 2b; \alpha_1, \beta_1; \alpha_1, \beta_1; \ldots; \alpha_r, \beta_r; \alpha_r, \beta_r).$$

This space has infinite fundamental group unless $r = 1$. Therefore follows

THEOREM 9. *A fibered space $F$ with finite fundamental group has the projective plane or the 2-sphere as orbit surface. In the first case $F$ has at most*

[20] Cf. W. Threlfall, Gruppenbilder *Abh. Sächs. Akad. Wiss.* **41** No. 6 (1932).

[21] Poincaré has introduced $P_1 = p_1 + 1$ as Betti number. We follow H. Weyl.

*one exceptional fiber, in the latter case $F$ has at most three exceptional fibers. If $F$ has three exceptional fibers, they have to be of multiplicity* $(2,2,n)$, $(2,3,3)$, $(2,3,4)$, *or* $(2,3,5)$.

When are two fibered spaces homeomorphic but not homeomorphic under a fiber preserving map?

THEOREM 10. *Suppose $F$ and $F'$ have the 2-sphere as orbit surface and have at least three exceptional fibers of multiplicities $\alpha_1, \ldots, \alpha_r$ and $\alpha'_1, \ldots, \alpha'_{r'}$, respectively. If $F$ is homeomorphic to $F'$ (not necessarily under a fiber preserving map) then the tuples $\alpha_1, \ldots, \alpha_r$ and $\alpha'_1, \ldots, \alpha'_{r'}$ must be equal (up to order).*

*Proof.* For $r = 3$, the center of (5) is the subgroup $\{H\}$ generated by $H$. For if the center were bigger than $\{H\}$, then (6) would have a nontrivial center. This is not the case if (6) is a group of the Euclidean or hyperbolic plane. If (6) is a Platonian group it has a nontrivial center only if it is a dihedral group whose order is a multiple of 4. It can be shown that in this case the center of (5) is not bigger than $\{H\}$ (DB II, §6). Hence (6) is the quotient of (5) by its center. If $F \approx F'$, then $\pi_1(F) \cong \pi_1(F')$ and $\pi_1(F)/\{H\} \simeq \pi_1(F')/\{H'\}$. But two polygon net groups (6) are isomorphic if and only if the polygons have the same number of vertices and the same angles, which proves the theorem. To see this, we can assume that none of the polygon net groups is a Platonian group, for such a group has necessarily the vertex number 3 and the triples of Theorem 9. The elements $\overline{Q}_1, \ldots, \overline{Q}_r$ of (6) are rotations about the $r$ vertices of a polygon $\Pi$ through $2\pi/\alpha_1, \ldots, 2\pi/\alpha_r$. Since an element of finite order of (6) is (as a transformation of a metric plane) necessarily a rotation about a fixed point, i.e., about a vertex of the polygon net, it follows that each nontrivial element of finite order of (6) is conjugate to a rotation about a vertex of $\Pi$, i.e., to a power $\overline{Q}_i^{\gamma_i}$ ($\gamma_i = 1, \ldots, \alpha_i - 1$). But two such powers $\overline{Q}_i^{\gamma_i}$ and $\overline{Q}_j^{\gamma_j}$ are never conjugates (as can be seen from the geometry). Therefore the numbers $\alpha_1, \ldots, \alpha_r$ determine uniquely the number of conjugate classes of elements of finite order and conversely one can easily verify that the numbers $\alpha_1, \ldots, \alpha_r$ are determined by the number of conjugate classes of elements of given finite order.

## 11. Fiberings of the 3-Sphere (Complete List)

In §3 we described fiberings of $S^3$ with two exceptional fibers of orders $m, n$ where $(m,n) = 1$. We now show that these are the only fiberings of $S^3$. More generally, we look at all simply connected (closed) fibered spaces.

Let $F$ be a fibered space with $\pi_1(F) = 1$. Then $f \approx S^2$ and $F$ is

$$(\mathrm{Oo}; 0 \mid b; \alpha_1, \beta_1; \ldots; \alpha_r, \beta_r).$$

A necessary condition for $\pi_1(F)$ to be finite is that $r \leqslant 3$ (by Theorem 9). For

$r = 3$ the quotient group (6) of $\pi_1(F)$, where $F$ is as in Theorem 9, is a Platonian group, and hence not trivial. Therefore if $\pi_1(F) = 1$, then $r \leqslant 2$.

For $r = 0$, $\pi_1(F) = \{Q_0, H: Q_0 H^b = 1 = Q_0\} = \{H: H^b = 1\}$. Hence $b = \pm 1$. Therefore (Oo; $1 \mid 1$) or (Oo; $0 \mid -1$) are the only simply connected fibered spaces without exceptional fibers. They differ only in their orientation (by Theorem 6) and are the fibering of $S^3$ by circles since this is free from exceptional fibers.

For $r = 1$, $b\alpha_1 + \beta_1 = \pm 1$ is necessary and sufficient for $\pi_1(F) = 1$. Now $\alpha_1$ ($\geqslant 2$) is arbitrary. For $b$ and $\beta_1$ there are then two solutions, $b = 0$, $\beta_1 = 1$ and $b = -1$, $\beta_1 = \alpha_1 - 1$. The two spaces (Oo; $0 \mid 0; \alpha_1, 1$) and (Oo; $0 \mid -1; \alpha_1, \alpha_1 - 1$) differ only in their orientation (Theorem 6), and therefore there is a unique simply connected fibered space (up to orientation) having a single exceptional fiber of order $\alpha_1$. This space is therefore the trace curve fibering of $S^3$ with the values $m = 1$, $n = \alpha_1$.

For $r = 2$, $\pi_1(F)$ is cyclic of order $|b\alpha_1\alpha_2 + \beta_1\alpha_2 + \beta_2\alpha_1|$. The equation

$$b\alpha_1\alpha_2 + \beta_1\alpha_2 + \beta_2\alpha_1 = \pm 1$$

has a solution only if $(\alpha_1, \alpha_2) = 1$. But for any given coprime $\alpha_1, \alpha_2$ ($\geqslant 2$) there are exactly two solutions for $b$, $\beta_1$, $\beta_2$, for which $0 < \beta_1 < \alpha_1$ and $0 < \beta_2 < \alpha_2$. The corresponding spaces differ only in their orientation. This will be proved in §12 for an arbitrary $r$. Therefore there is only one fibering (up to orientation) for any two given coprime exceptional fibers, which therefore has to agree with that of §3. This proves

THEOREM 11. *A closed simply connected fibered space is $S^3$. Any fibering of $S^3$ is uniquely determimed by two positive coprime integers $m$ and $n$. For $m = n = 1$ there are no exceptional fibers; if only one of $m$ (or $n$) is 1 there is one exceptional fiber of order $n$ (or $m$). If $m$ and $n$ are different from 1, they are the orders of the two exceptional fibers. All fiberings of $S^3$ agree with those of* §3.

The ordinary fibers for $m \neq 1$, $n \neq 1$ are torus knots which wind $m$ times around the $z$-axis and $n$ times around the unit circle in the conformal space. For $m = 2$, $n = 3$ they are trefoil knots.

## 12. The Fibered Poincaré Spaces

We now determine which fibered spaces are Poincaré spaces, that is, which have trivial first homology group[22] and which are not homeomorphic to $S^3$. By §10 if $H_1(F) = 1$, then $H_1(f) = 1$; hence $f \approx S^2$ and $F$ is (Oo; $0 \mid b$; $\alpha_1, \beta_1; \ldots; \alpha_r, \beta_r$). $H_1(F)$ is the Abelianized $\pi_1(F)$ and has the $r + 2$ generators

$$Q_0, Q_1, \ldots, Q_r, H$$

[22] Cf. DB I, p. 51.

and in addition to being commutative has the relations

$$Q_0H^b = Q_1^{\alpha_1}H^{\beta_1} = \cdots = Q_r^{\alpha_r}H^{\beta_r} = Q_0Q_1\cdots Q_r = 1.$$

In additive notation,

$$\begin{array}{lllll} Q_0 & & + bH & = & 0 \\ & \alpha_1 Q_1 & + \beta_1 H & = & 0 \\ & & & \vdots & \\ & & \alpha_r Q_r + \beta_r H & = & 0 \\ Q_0 & + Q_1 + \cdots + Q_r & & = & 0 \end{array} \tag{1}$$

We obtain equivalent relations and generators for $H_1(F)$ by transforming the generators and relations by unimodular substitutions. In this way we can transform the coefficient matrix into a normal form which has all entires 0 except possibly in the main diagonal, where the entries are the invariant factors of the original matrix. If $H_1(F) = 1$, then in the normal form all the elements in the main diagonal are 1 (otherwise we would have a nontrivial relation $k_iQ_i = 1$). That is, the Betti number $= 0$ and there are no torsion coefficients. Since the given matrix is square the two conditions are equivalent to

$$\Delta = \begin{vmatrix} 1 & 0 & \cdots & 0 & b \\ 0 & \alpha_1 & \cdots & 0 & \beta_1 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & \alpha_r & \beta_r \\ 1 & 1 & \cdots & 1 & 0 \end{vmatrix} = \pm 1. \tag{2}$$

Computing $\Delta$ we get the equation

$$\Delta = b\alpha_1\cdots\alpha_r + \beta_1\alpha_2\cdots\alpha_r + \alpha_1\beta_2\alpha_3\cdots\alpha_r + \cdots + \alpha_1\alpha_2\cdots\alpha_{r-1}\beta_r$$
$$= \varepsilon \qquad (\varepsilon = \pm 1). \tag{3}$$

If we reverse the orientation of $F$, i.e., if we consider $(\mathrm{O}, \mathrm{o}; 0 \mid -r-b; \alpha_1, \alpha_1-\beta_1; \ldots; \alpha_r, \alpha_r-\beta_r)$, we would get a determinant $\Delta' = -\Delta$. Therefore we can assume that $\varepsilon = \pm 1$. This determines the orientation of $F$. To solve (3) with $\varepsilon = +1$, we let $\alpha_1, \ldots, \alpha_r$ be given ($\alpha_i \geqslant 2$) and try to solve for $b$, $\beta_1, \ldots, \beta_r$. For $r = 0$ and $r = 1$ we get $b = 1$ and $b\alpha_1 + \beta_1 = 1$, which was discussed in §11. Thus assume $r \geqslant 2$. There exists no solution of (3) if two of the $\alpha_i$ have a common divisor. Hence assume the $\alpha_i$ are pairwise coprime. Then

$$\gcd(\alpha_1\cdots\alpha_r, \alpha_2\cdots\alpha_r, \alpha_1\alpha_3\cdots\alpha_r, \cdots, \alpha_1\alpha_2\cdots\alpha_{r-1}) = 1.$$

Hence there exists a solution $b, \beta_1, \ldots, \beta_r$ and $(\beta_i, \alpha_i) = 1$; otherwise the left-hand side of (3) would have a common factor $\neq 1$. But $\beta_i$ need not satisfy

$$0 < \beta_i < \alpha_i.$$

But this condition can be satisfied by replacing $\beta_i$ by $\beta_i + x_i\alpha_i$ and at the same time $b$ by $b - x_i$, which also satisfies (3). This normalized solution is unique for if $b', \beta_1', \ldots, \beta_r'$ is any other normalized solution of (3), then

$$(b - b')\alpha_1 \cdots \alpha_r + (\beta_1 - \beta_1')\alpha_2 \cdots \alpha_r + \cdots = 0.$$

This implies $\beta_i - \beta_i' \equiv 0 \pmod{\alpha_i}$, hence $\beta_i = \beta_i'$.

This solution of (3) completes the proof of Theorem 11. Hence for $r = 2$ the fibered spaces with trivial first homology group are homeomorphic to $S^3$. For $r > 2$ they are Poincaré spaces since by Theorem 11 a fibration of $S^3$ has at most two exceptional fibers. Thus follows

THEOREM 12. *A fibered Poincaré space ($\neq S^3$) has at least three exceptional fibers; their multiplicities $\alpha_1, \ldots, \alpha_r$ are pairwise coprime. Conversely, for any $r \geqslant 3$ pairwise coprime integers $\geqslant 2$, there exists a unique fibered Poincaré space having $r$ exceptional fibers with the given multiplicities. Two fibered Poincaré spaces are homeomorphic if and only if they are homeomorphic under a fiber preserving map; i.e., a Poincaré space admits at most one fibering. The only fibered Poincaré space with finite fundamental group is the dodecahedral space.*[23]

It remains to prove the two latter claims. If two fibered Poincaré spaces are homeomorphic, they must have the same multiplicities for the exceptional fibers, by Theorem 10. But these determine already the fibering of a Poincaré space.

By Theorem 9, a fibered Poincaré space with finite fundamental group can have only three exceptional fibers with the multiplicities 2, 3, 5 because this is the only triple in Theorem 9 with pairwise coprime integers. This space has by (3) the invariants

$$(\mathrm{Oo}; 0 \mid -1; 5, 1; 2, 1; 3, 1)$$

and its fundamental group has relations

$$Q_0H^{-1} = Q_1^5H = Q_2^2H = Q_3^3H = Q_0Q_1Q_2Q_3 = 1.$$

(These relations imply that $H$ commutes with the $Q_i$). Eliminating $H$, we obtain the presentation of the binary icosahedral group[24] of order 120:

$$Q_1^5 = Q_2^2 = Q_3^3 = Q_1Q_2Q_3.$$

In DB II, §7, it is shown that this is the dodecahedral space by exhibiting a fibering of the dodecahedral space.

[23] Cf. DB I, §12.

[24] Cf. Aufgabe 84 in *Jahresber. Deutsch Math.-Verein.* **41** (1936), 6.

## 13. Constructing Poincaré Spaces from Torus Knots

M. Dehn[25] described a method for constructing Poincaré spaces as follows: Let $A$ be the complement of a regular neighborhood of a knot $C$ in $S^3$, and let $\Pi = \partial A$. Then $H_1(M)$ is the free cyclic group generated by a meridian $M$ on $\Pi$. If $B$ is a simple closed curve on $\Pi$ intersecting $M$ in one point, $B \sim xM$ (in $A$) and we can assume that $x = 0$ by replacing (if necessary) $B$ by $B - xM$. Then $B$ is uniquely determined by requiring that $M \cap B$ be a point and $B \sim 0$ in $A$ (up to orientation and deformation on $\Pi$). Closing $A$ with a torus seal $V'$ having as meridian

$$M' \sim M + qB \qquad (\text{on } \Pi;\ q \neq 0), \tag{1}$$

we get a closed space $R$ with $H_1(R) = 0$.

Now suppose $C$ is a torus knot. Such knots are ordinary fibers of the fiberings of $S^3$, given in §3, which are characterized by two coprime integers $m$ and $n$ ($\geqslant 2$). Drill out an ordinary fiber $C$. Then a fiber $H$ of $\Pi$ can be deformed in $A$ into $n$ times the $z$-axis, and since the $z$-axis is $\sim mM$ in $A$ (with suitable orientation of $M$), we have that $H \sim mnM$ (in $A$). Hence $h - mnM \sim 0$ in $A$, i.e., $H = B$. By (1), $M' \sim M + qB \sim (1 - qmn)M + qH$ on $\Pi$. Since $M$ is a crossing curve on $\Pi$, the torus seal has an exceptional fiber of multiplicity $|qmn - 1|$, for since $m$ and $n > 1$ (otherwise $C$ would be unknotted and we would not get a Poincaré space), $|qmn - 1| > \max(m,n) > 1$. Thus $R$ is the unique Poincaré space (by Theorem 12) with three exceptional fibers of multiplicities $m$, $n$, $|qmn - 1|$. Furthermore, since $|q_1mn - 1| \neq |q_2mn - 1|$, if $q_1 \neq q_2$, two Poincaré spaces obtained from the same torus knot with different $q$'s are not homeomorphic by Theorem 12. Finally, two Poincaré spaces obtained from different torus knots are never homeomorphic. For if a Poincaré space with exceptional fibers $\alpha_1 < \alpha_2 < \alpha_3$ is obtained from a torus knot, then it can only be the knot $m = \alpha_1$, $n = \alpha_2$, since $|qmn - 1| > \max(m,n)$. This implies by the way that two torus knots $m < n$ and $m' < n'$ are topologically equivalent only if $m = m'$, $n = n'$, since only in this case are the Poincare spaces which can be constructed from them the same.

THEOREM 13. *A Poincaré space can be constructed from a torus knot if and only if it can be fibered and the fibering has exactly three exceptional fibers of multiplicites $\alpha_1 < \alpha_2 < \alpha_3$, where $\alpha_1, \alpha_2, \alpha_3$ are pairwise coprime integers ($> 1$) and $\alpha_3 = |q\alpha_1\alpha_2 - 1|$ ($q$ an arbitrary integer). Such a Poincaré space can only be constructed from a unique torus knot in a unique way.*

For example, the Dehn trefoil space constructed from a trefoil knot $m = 2$,

[25] M. Dehn, Über die Topologie des dreidimensionalen Raumes, *Math. Ann.* **69** (1910), 137–168.

$n = 3$, $q = 1$ is homeomorphic with the unique fibered Poincaré space with three exceptional fibers of multiplicities 2, 3, 5. Its fiber invariants are listed in §12.

## 14. Translation Groups of Fibered Spaces

A *translation group* $\mathfrak{G}$ of a fibered space $F$ is a finite group of homeomorphisms $F \to F$ such that each map of $\mathfrak{G}$ maps each fiber $H$ onto itself and preserves orientation of $H$. For an arbitrary fiber $H$ of $F$ let $\mathfrak{H} = \{\varphi \mid H, \varphi \in \mathfrak{G}\}$. We claim that $\mathfrak{H}$ is a finite cyclic group of rotations of a circle. For if $P$ is a point of $H$ and $P', P'', \ldots, P^{(i)} = P$ are its equivalent points such that $P'$ is next to $P$ with respect to the given orientation of $H$, the points $P, P', P'', \ldots$ and the arcs between them are cyclically permuted under a map of $\mathfrak{G}$. In particular, if $P$ is a fixed point, then the arc $\overline{PP'}$ is mapped onto itself keeping $P, P'$ fixed, and since the map has finite order it must be the identity. There is a map in $\mathfrak{H}$ which sends $P$ to $P^{(k)}$ ($k$ arbitrary). Therefore $\mathfrak{H}$ consists of the powers of the map which sends $P$ to $P'$.

*Claim. Every translation group $\mathfrak{G}$ is cyclic.* It suffices to show that a map $S$ of $\mathfrak{G}$ which leaves an ordinary fiber $H$ fixed is the identity, for then $\mathfrak{G}$ is isomorphic to $\mathfrak{H}$, which we know to by cyclic. The maximum of the translations of the points of a fiber $H'$ under $S$ converges to 0 as $H'$ converges to $H$. But this maximal translation cannot be arbitrarily small since $S$ is of finite order. Therefore $S$ is the identity on a fiber neighborhood of $H$. The set of all ordinary fibers which are fixed under $S$ is therefore open. The set of all ordinary fibers which are not pointwise fixed is also open, hence empty since $F$ is connected. But then clearly all the exceptional fibers are also left pointwise fixed under $S$.

The following theorem deals with the existence of translation groups:

THEOREM 14. *A closed fibered space of class* (Oo; $p$) *or* (Nn I; $k$) *admits a translation group of arbitrary order* $g$.

*Proof.* We first show that a fibered solid torus with invariants $\mu, \nu$ admits such a group. Cut the solid torus into a Euclidean cylinder of height 1, and let $z$ be the height of a point $P$; then there is a continuous transformation group of the solid torus such that each point runs along its fiber and the $z$-coordinate changes continuously, $z' = z + t$. Here $z'$ is the coordinate of the image point and $t$ the continuous parameter of the group. $z$ has to be considered mod 1. If $t$ increases continuously from 0, then $t = 1$ is the first value for which the middle fiber is mapped to itself, $t = \mu$ is the first value for which the map is the identity. The cyclic translation group $g$ consists of the transformations belonging to $t = 0, \mu/g, \ldots, \mu(g-1)/g$.

Let $F$ be a fibered space with simultaneously oriented fibers; triangulate $f$

so that each exceptional point lies in the interior of a 2-simplex and each 2-simplex contains at most one exceptional point and so that any two 2-simplexes with exceptional points do not intersect. This corresponds to a decomposition of $F$ into solid tori. We define a cyclic translation group of order $g$ in each of the solid tori with exceptional fibers and on the remaining fibers of $F$ which map to vertices of $f$. As generator $Z$ of $\mathfrak{G}$ we take the translation which rotates the ordinary fibers by as little as possible in positive direction. Let $K$ be a fibered annulus that maps to an edge of the triangulation of $f$. If $K$ lies on an exceptional torus, then $\mathfrak{G}$ is already defined on $K$. If $K$ lies on an ordinary torus, then $\mathfrak{G}$ is already defined on the boundary curves $a$ and $b$ of $K$. It is clear that $\mathfrak{G}$ can be defined on all of $K$ ($Z$ is a rotation of $K$ about $2\pi/g$), since $a \sim b$ in $F$, since the fibers are oriented simultaneously. Now $\mathfrak{G}$ is defined on the boundary $\Pi$ of each ordinary fibered solid torus $V$.

We think of $V$ as being embedded in Euclidean space, symmetric with respect to an axis of rotation and such that each fiber of $V$ is mapped to itself under a rotation about this axis. We choose a fiber preserving autohomeomorphism $A$ of the boundary torus $\Pi$ of $V$ such that $AZA^{-1}:\Pi \to \Pi$ is a rigid rotation about the axis of rotation through an angle of $2\pi/g$. This is always possible since the translation $Z$ restricted to each of the three fibered annuli which form $\Pi$ (and which map to the three edges of a 2-simplex of the triangulation of the orbit surface) is conjugate to a rigid rotation of a Euclidean annulus through an angle of $2\pi/g$. We can choose $A$ such that each class of curves on $\Pi$ is mapped to itself. As shown in §5 we can extend $A$ to a fiber preserving autohomeomorphism of $V$. Therefore $V$ can be mapped homeomorphically to a rotation symmetric solid torus $V'$ in Euclidean space (which has the property that a rotation about the axis of rotation rotates each fiber in itself) such that $Z \mid \Pi$ is then conjugate to a rigid rotation of the boundary torus $\Pi'$ of $V'$ through an angle of $2\pi/g$. This rotation $\Pi'$ can be extended to a rigid rotation of $V'$ through the same angle. This defines a translation $Z$ of order $g$ on the sapce $F$, and proves Theorem 14.

We now show that the orbit space of $F$ under $\mathfrak{G}$ is a fibered space $F'$. First, let $\mathfrak{G}$ act on a solid torus $V$. If $V$ is an ordinary fibered solid torus, then clearly the orbit space of $V$ is again an ordinary solid torus. Suppose $V$ is a torus with invariants $\mu, \nu$. Suppose $\mathfrak{U}$ is a nontrivial subgroup of $\mathfrak{G}$ keeping the exceptional fiber pointwise fixed. $\mathfrak{U}$ is cyclic or order $u$. We claim that there exists a meridian disk of $V$ which is mapped to itself under $\mathfrak{U}$. Cut $V$ into a Euclidean cylinder of height 1 and let $E_0$ be the meridian disk of height $\frac{1}{2}$. Let $E_1, E_2, \ldots, E_{u-1}$ be the images of $E_0$ under $\mathfrak{U}$. We can assume that no $E_i$ intersects the top and bottom disk of the cylinder by choosing $V$ sufficiently small. Each fiber of the cylinder intersects $E_0, E_1, \ldots, E_{u-1}$ in $u$ (not necessarily distinct) points. Choosing the highest such point on each fiber we obtain a meridian disk $E$ of $V$ which is mapped to itself under $\mathfrak{U}$.[26] Therefore we can cut $V$ along $E$ into a cylinder on which $\mathfrak{U}$ acts as a group of rigid rotations about the axis and translations of the fibers in themselves. The orbit space is a cylinder sector of an angle $2\pi/u$, where the two vertical faces have to be identified such that we get a fibered cylinder. In this cylinder, top and bottom disks are identified under a rotation of $2\pi\nu/\mu'$, where $\mu' = \mu/u$, hence $(\mu',\nu) = 1$. Therefore the orbit space of $\mathfrak{U}$ is a fibered solid torus $V'$ with a $(\mu/u)$-fold exceptional fiber.

The translation group $\mathfrak{G}$ of $V$ maps to a translation group $\mathfrak{G}'$ of $V'$, where $\mathfrak{G}'$ has order $v = g/u$ and does not contain a translation $\neq 1$ which keeps the exceptional fiber of $V'$ pointwise fixed. The cylinder corresponding to $V'$ is then divided by the $v-1$ images of the bottom disk into $v$ equivalent parts. In each part, bottom and top disks correspond under a rotation of $2\pi\nu''/\mu''$, $(\mu'',\nu'') = 1$. The orbit space $D$ of $\mathfrak{G}'$ (on $V'$), which is also the orbit space of $\mathfrak{G}$ (on $V$), is a fibered solid torus which is covered by $V'$ (unbranched) $v$ times. Since the fibers of $D$ correspond one-to-one to those of $V'$, the orbit surface of $V'$ covers (unbranched) that of $D$. From §9 we have $(\mu'',v) = 1$ and hence by (1) in §9 $\mu' = \mu''$, i.e., $D$ has a $\mu'$-fold exceptional fiber. Now $(g, \mu) = (uv, u\mu') = u(v, \mu') = u$ and $v = g/u = g/(g, \mu)$. The numbers $u, v$ are therefore determined by the order $g$ of $\mathfrak{G}$ and the multiplicity $\mu$ of the exceptional fiber of $V$.

*Result. The orbit space $D$ of a translation group $\mathfrak{G}$ of order $g$ on a fibered solid torus $V$ with a $\mu$-fold exceptional fiber is a fibered torus with exceptional fiber of multiplicity $\mu/(\mu, g)$. For $(\mu, g) > 1$, the covering $V \to D$ is branched, where the exceptional fiber of $V$ is a branch curve of order $(\mu, g)$.* This implies that the orbit space of $F$ under $\mathfrak{G}$ is a fibered space $F'$, and $F \to F'$ is a branched covering.

We now compute the invariants of $F'$. Let $F$ be the space $(\mathrm{Oo};\ p \mid b;\ \alpha_1, \beta_1; \ldots; \alpha_r, \beta_r)$. Drilling out the exceptional fibers and an ordinary fiber we get $\bar{F} \approx \bar{f} \times S^1$, where $\bar{f}$ is an $(r+1)$ times punctured surface of genus $p$. On the boundary tori $\Pi_0, \Pi_1, \ldots, \Pi_r$ we have the crossing curves $Q_0, Q_1, \ldots, Q_r$. The $Q_i$ and $H_i$ $(H_0, H_1, \ldots, H_r$ simultaneously oriented) determine on $\Pi_i$ orientations opposite to that induced by $F$, and

$$Q_0 + Q_1 + \cdots + Q_r \sim 0 \qquad (\text{in } \bar{F}).$$

[26] To see that $\mathfrak{U}$ maps $E$ to itself, suppose there is a map $B$ in $\mathfrak{U}$ which sends a point $P$ of $E$ to a point $P'$ not on $E$. Then the line segment parallel to the axis of the cylinder $V$ intersects $E$ in a point $Q' \neq P'$. The line segment $P'Q'$ is mapped under $B^{-1}$ to a line segment $PQ$, where $Q$ lies on one of the disks $E_i$. But $P$ is the highest of the $n$ intersections of the line segment through $P$ and the disks $E_1, \ldots, E_{u-1}$ and therefore $PQ$ contains a point $R$ of the top disk of $V$, whose image under $B$ is a point $R'$ on the line segment $P'Q'$. Now if $P$ approaches continuously the axis of the cylinder, $P', Q', R'$ move continuously, and since at last $P'$ and $Q'$ coincide, $R'$ must at some time coincide with $P'$ or $Q'$, i.e., there is a map $B^{-1}$ of $\mathfrak{U}$ that maps a point of a certain $E_i$ into a point of the top disk. This contradicts the choice of the disks $E_i$.

We get $F$ by taking $Q_0 + bH_0, \alpha_1 Q_1 + \beta_1 H_1, \ldots, \alpha_r Q_r + \beta_r H_r$ as meridians of the torus seals $V_i$. The orbit space $\bar{F}'$ of $\mathfrak{G} \mid \bar{F}$ is the product of an $(r+1)$ times punctured surface of genus $p$ and $S^1$. The orientation (and fiber orientation) of $\bar{F}$ carries over to $\bar{F}'$. Let $\check{Q}_0, \check{Q}_1, \ldots, \check{Q}_r$ and $\check{H}_0, \check{H}_1, \ldots, \check{H}_r$ be the images of $Q_0, Q_1, \ldots, Q_r, H_0, H_1, \ldots, H_r$ in $\bar{F}'$. Then $\check{Q}_0, \check{Q}_1, \ldots, \check{Q}_r$ are crossing curves on the boundary tori $\Pi_0', \Pi_1', \ldots, \Pi_r'$ of $\bar{F}'$, whereas $\check{H}_i$ covers a fiber $H_i'$ of $\Pi_i'$ $g$ times: $\check{H}_i = gH_i'$. We have

$$\check{Q}_0 + \check{Q}_1 + \cdots + \check{Q}_r \sim 0 \qquad (\text{in } \bar{F}')$$

and the orientation determined by $\check{Q}_i$ and $H_i'$ on $\Pi_i'$ is opposite to that induced by $\bar{F}'$. The orbit space $F'$ is determined by the meridians $M_i' \sim \check{\alpha}_i \check{Q}_i + \check{\beta}_i H_i'$ and $M_0' \sim \check{Q}_0 + \check{b} H_0'$ of the torus seals $V_i'$. $M_i \simeq 0$ in $V_i$, hence $\check{M}_i \simeq 0$ in $V_i'$. Therefore

$$M_i \sim \alpha_i Q_i + \beta_i H_i \qquad (\text{on } \Pi_i)$$

implies

$$\begin{aligned} \check{M}_i \sim \alpha_i \check{Q}_i + \beta_i \check{H}_i \quad &\sim \alpha_i \check{Q}_i + \beta_i g H_i' &&(\text{on } \Pi_i') \\ &\sim 0 &&(\text{in } V_i'). \end{aligned}$$

Therefore

$$\begin{aligned} M_i' &\sim \frac{\alpha_i}{(\alpha_i, g)} \check{Q}_i + \frac{\beta_i g}{(\alpha_i, g)} H_i' \qquad (\text{in } V_i') \\ &= \check{\alpha}_i \check{Q}_i + \check{\beta}_i H_i' \sim 0 \end{aligned}$$

and since $\check{\alpha}_i$ and $\check{\beta}_i$ are coprime, $M_i'$ is a meridian on $V_i'$.

Similarly $M_0' \sim \check{Q}_0 + bgH_0' \sim \check{Q}_0 + \check{b} H_0'$ is a meridian on $V_0'$. But $\check{b}, \check{\alpha}_i, \check{\beta}_i$ are not yet the sought after fiber invariants of $F'$ since $\check{\beta}_i$ need not satisfy $0 \leqslant \check{\beta}_i < \check{\alpha}_i$. But taking instead of $\check{Q}_1, \ldots, \check{Q}_r$ the crossing curves $Q_1' \sim \check{Q}_1 + x_1 H_1', \ldots, Q_r' \sim \check{Q}_r + x_r H_r'$, and instead of $\check{Q}_0$ the crossing curve $Q_0' \sim \check{Q}_0 - (x_1 + \cdots + x_r) H_0'$, we have the correct homology

$$Q_0' + Q_1' + \cdots + Q_r' \sim 0 \qquad (\text{in } \bar{F}')$$

and the orientation induced by $Q_i'$ and $H_i'$ on $\Pi_i'$ is the same as that from $\check{Q}_i$ and $H_i'$. Now in the new basis curves the meridians $M_i'$ are as follows:

$$M_i' \sim \check{\alpha}_i Q_i' + (\check{\beta}_i - \check{\alpha}_i x_i) H_i' = \alpha_i' Q_i' + \beta_i' H_i' \qquad (i = 1, \ldots, r),$$

$$M_0' \sim Q_0' + (\check{b} + x_1 + \cdots + x_r) H_0' = Q_0' + b' H_0'.$$

Choosing $x_i$ such that $0 \leqslant \beta_i' < \alpha_i'$ and omitting those $\alpha_i', \beta_i'$ for which $\alpha_i' = 1$ $(\beta_i' = 0)$, we obtain the fiber invariants of $F'$.

If $F$ is $(\text{Nn I}; k \mid b; \alpha_1, \beta_1; \ldots; \alpha_r, \beta_r)$ we get a similar result.

EXAMPLE. The trefoil space of Dehn $(\text{Oo}; 0 \mid -1; 2, 1; 3, 1; 5, 1)$ with translation group of order $g = 5$. Now

$$(\alpha_1, g) = 1, \qquad (\alpha_2, g) = 1, \quad (\alpha_3, g) = 5,$$

$$\check{\alpha}_1 = \frac{\alpha_1}{(\alpha_1, g)} = 2, \quad \check{\alpha}_2 = 3 \qquad \check{\alpha}_3 = 1, \qquad \check{b} = bg = -5,$$

$$\check{\beta}_1 = \frac{\beta_1 g}{(\alpha_1, g)} = 5, \quad \check{\beta}_2 = 5, \qquad \check{\beta}_3 = 1,$$

hence $x_1 = 2$, $x_2 = 1$, $x_3 = 1$. Therefore the orbit space $F'$ is the space

$$(\mathrm{Oo}; 0 \mid b'; \alpha_1', \beta_1'; \alpha_2', \beta_2') = (\mathrm{Oo}; 0 \mid -1; 2, 1; 3, 2).$$

$\pi_1(F')$ is of order $\Delta' = b'\alpha_1'\alpha_2' + \beta_1'\alpha_2' + \alpha_1'\beta_2' = 1$. Hence $F' \approx S^3$ and the fibers are trefoil knots. In particular, the 5-fold exceptional fiber of $F$ is mapped to an ordinary fiber of $F'$, a trefoil knot. Therefore, $F$ is a 5-sheeted branched covering of $S^3$ with a trefoil as branch curve.

This result can be generalized. Let $F$ be a Poincaré space $(\mathrm{Oo}; 0 \mid \mathrm{b}; \alpha_1, \beta_1; \ldots; \alpha_r, \beta_r)$. Necessary and sufficient for $F$ to be a Poincaré space is that the determinant

$$\Delta = \begin{vmatrix} 1 & 0 & \cdots & 0 & b \\ 0 & \alpha_1 & \cdots & 0 & \beta_1 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & \alpha_r & \beta_r \\ 1 & 1 & \cdots & 1 & 0 \end{vmatrix} = \pm 1.$$

Now $\bar{F}' \approx \bar{f} \times S^1$, where $\bar{f}$ is a $(r+1)$ times punctured 2-sphere. The generators of $H_1(\bar{F}')$ are $\check{Q}_0, \check{Q}_1, \ldots, \check{Q}_r$ and an arbitrary fiber $H'$ and we have the single relation $\check{Q}_0 + \check{Q}_1 + \cdots + \check{Q}_r \sim 0$. Closing $\bar{F}'$ to $F'$ we get the additional relations

$$\check{Q}_0 + \check{b}H' = \check{\alpha}_1\check{Q}_1 + \check{\beta}_1 H' = \cdots = \check{\alpha}_r\check{Q}_r + \check{\beta}_r H' \sim 0.$$

Here

$$\check{b} = bg, \qquad \check{\alpha}_i = \alpha_i/(\alpha_i, g), \qquad \check{\beta}_i = \beta_i g/(\alpha_i, g).$$

The relation matrix of $H_1(F')$ is therefore

$$\begin{bmatrix} 1 & 0 & \cdots & 0 & \check{b} \\ 0 & \check{\alpha}_1 & \cdots & 0 & \check{\beta}_1 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & \check{\alpha}_r & \check{\beta}_r \\ 1 & 1 & \cdots & 1 & 0 \end{bmatrix}$$

and its determinant $\Delta'$ is

$$\Delta' = \begin{vmatrix} 1 & 0 & \cdots & 0 & bg \\ 0 & \dfrac{\alpha_1}{(\alpha_1, g)} & \cdots & 0 & \dfrac{\beta_1 g}{(\alpha_1, g)} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & \dfrac{\alpha_r}{(\alpha_r, g)} & \dfrac{\beta_r g}{(\alpha_r, g)} \\ 1 & 1 & \cdots & 1 & 0 \end{vmatrix}$$

$$= \frac{g}{(\alpha_j, g)(\alpha_2, g) \cdots (\alpha_r, g)} \Delta$$

$$= \pm \frac{g}{(\alpha_1, g)(\alpha_2, g) \cdots (\alpha_r, g)} .$$

$F'$ is a Poincaré space or $S^3$ only if $\Delta' = \pm 1$. Since $\alpha_1, \alpha_2, \ldots, \alpha_r$ are relatively coprime we have

$$(\alpha_1, g)(\alpha_2, g) \cdots (\alpha_r, g) = (\alpha_1 \alpha_2 \cdots \alpha_r, g)$$

and $\Delta' = \pm 1$ if and only if $g$ divides $\alpha_1 \alpha_2 \cdots \alpha_r$. The multiplicities of the excpetional fibers of $F'$ are the $\check{\alpha}_1, \check{\alpha}_2, \ldots, \check{\alpha}_r$ different from 1. By Theorem 12, the $\check{\alpha}_i$ characterize $F'$. Hence follows

THEOREM 15. *The orbit space $F'$ of a translation group of a fibered space $F$ with invariants*

$$(\mathrm{Oo};\, p \mid b; \alpha_1, \beta_1; \ldots; \alpha_r, \beta_r)$$

*or*

$$(\mathrm{Nn}\ 1; k \mid b; \alpha_1, \beta_1; \ldots; \alpha_r, \beta_r)$$

*is a fibered space of the same class, whose invariants are determined by those of $F$ and the order $g$ or $\mathfrak{G}$. If $F$ is the Poincaré space with $r$ exceptional fibers of multiplicites $\alpha_1, \alpha_2, \ldots, \alpha_r$, then $F'$ is a Poincaré space or $S^3$ if and only if $g \mid \alpha_1 \cdots \alpha_r$. In this case $F'$ is the Poincaré space whose exceptional fibers have as multiplicities the following of the numbers which are $\neq 1$:*

$$\frac{\alpha_1}{(\alpha_1, g)}, \frac{\alpha_2}{(\alpha_2, g)}, \ldots, \frac{\alpha_r}{(\alpha_r, g)} .$$

*The covering of $F'$ by $F$ is branched over the exceptional fibers of $F$ for which $(\alpha_i, g) > 1$ of branching index $(\alpha_i, g)$.*

Specializing, we get

THEOREM 16. *The orbit space $F'$ of a Poincaré space $F$ with $r$ exceptional fibers of orders $\alpha_1, \alpha_2, \ldots, \alpha_r$ under a translation group of order $g = \alpha_1 \alpha_2 \cdots \alpha_i$ is a Poincaré space or $S^3$ with exceptional fibers of orders $\alpha_{i+1}, \alpha_{i+2}, \ldots, \alpha_r$.*

THEOREM 17. *Let $\alpha_1, \alpha_2, \ldots, \alpha_r$ be $r \geqslant 3$ pairwise coprime integers $\geqslant 2$, and let $k_1, k_2, \ldots, k_{r-2}$ be $r-2$ torus knots in $S^3$ (which are ordinary fibers of a fibering of $S^3$) of type $m, n$, where $m$ and $n$ are any two of the numbers $\alpha_1, \ldots, \alpha_r$. Delete these two numbers from the sequence $\alpha_1, \ldots, \alpha_r$ and take a one-to-one correspondence between the remaining $\alpha_i$ and the knots $k_i$. Construct the branched covering of $S^3$ having the knots $k_1, \ldots, k_{r-2}$ as branch curves and having the following property* E: *A curve $\tilde{w}$ of the covering space which lies over a closed curve $w$ of $S^3 - (k_1 \cup \cdots \cup k_{r-2})$ is closed if and only if the linking number $x_i(w, k_i)$ is divisible by the number $\alpha_j$ which corresponds to the knot $k_i$ $(i = 1, \ldots, r-2)$. This covering is $(\alpha_1 \alpha_2 \cdots \alpha_r / mn)$-sheeted and is a Poincaré space which is the same regardless of how one picks out the numbers $m, n$ from $\alpha_1, \alpha_2, \ldots, \alpha_r$.*

*Proof.* Assume $m = \alpha_{r-1}$, $n = \alpha_r$, and $\alpha_i$ corresponds to $k_i$ $(i = 1, \ldots, r-2)$. Letting a translation group of order $g = \alpha_1 \cdots \alpha_{r-2}$ act on the Poincaré space $F$ with $r$ exceptional fibers of multiplicities $\alpha_1, \ldots, \alpha_r$, we obtain as orbit space $F'$ a fibered space with two exceptional fibers $\alpha_{r-1}$ and $\alpha_r$ by the previous theorem. Since a Poincaré space has at least three exceptional fibers (Theorem 12), $F' \approx S^3$ with a fibering having torus knots of type $m = \alpha_{r-1}$, $n = \alpha_r$ as ordinary fibers (§3). By Theorem 15, $F$ is a branched covering of $F'$; the branch curves are the exceptional fibers of orders $\alpha_1, \ldots, \alpha_{r-2}$ which map to ordinary fibers in $F'$, hence to $r-2$ torus knots $k_1, \ldots, k_{r-2}$ or type $m, n$. The branching index is $(\alpha_i, g) = \alpha_i$, i.e., a curve in $F$ winding once around the $i$th branch curve maps to a curve in $F'$ winding $\alpha_i$ times around $k_i$. The covering $F \to F'$ is regular and the covering transformation group is cyclic of order $g = \alpha_1 \cdots \alpha_{r-2}$. Therefore (by the lemma in the Appendix) a curve $\tilde{w}$ of $F$ lying over a curve $w$ of $F' - (k_1 \cup \cdots \cup k_{r-2})$ is closed if and only if for each $i$ the linking number of $w$ and $k_i$ is divisible by $\alpha_i$, and this property E characterizes $F$ uniquely as covering of $F'$. Thus the covering of $S^3$ determined by property E is the Poincaré space with $r$ exceptional fibers of multiplicities $\alpha_1, \ldots, \alpha_r$. By Theorem 12, $F$ is uniquely determined by the numbers $\alpha_1, \ldots, \alpha_r$. Therefore $F$ is independent of the choice of the numbers $m, n$ out of $\alpha_1, \ldots, \alpha_r$.

Theorem 17 is interesting because it deals with the homeomorphism type of certain covering spaces, which can be characterized independently of any fibration. This is so since the requirement that the knots $k_1, \ldots, k_{r-2}$ be ordinary fibers of the fibering of $S^3$ can be replaced by the following: $k_1, \ldots, k_{r-2}$ are pairwise disjoint simple closed curves on a torus which separates $S^3$ into two solid tori, and these curves are not null homotopic in either solid torus. Then it can be shown that there is a fibering of $S^3$ that contains these $r-2$ curves as ordinary fibers.

The special case of Theorem 17 for $r = 3$ deserves special attention.

The *g-fold cyclic covering* of a knot $k$ in $S^3$ is the branched covering with the following property: A Curve $\tilde{w}$ of the covering space which lies over a curve $w$ of $S^3\backslash k$ is closed if and only if the linking number of $w$ and $k$ is a multiple of $g$.[27] The special case can now be formulated as follows:

ADDENDUM TO THEOREM 17. *Let $\alpha_1, \alpha_2, \alpha_3$ be three pairwise coprime numbers $\geqslant 2$. Then the $\alpha_3$-fold cyclic covering of the torus knot of type $m = \alpha_1$, $n = \alpha_2$ is a Poincaré space. The same space is obtained if $\alpha_1, \alpha_2, \alpha_3$ are arbitrarily interchanged.*

For this Poincaré space is the fibered Poincaré space with three exceptional fibers of multiplicities $\alpha_1, \alpha_2, \alpha_3$. Thus the Dehn trefoil space, which was obtained by drilling out and sewing back a trefoil of $S^3$, can be obtained as 5-fold cyclic branched covering of a trefoil or as 3-fold cyclic covering of the torus knot $m = 2$, $n = 5$ or as 2-fold cyclic covering of the torus knot $m = 3$, $n = 5$.

Finally, each fibered Poincaré space $(\mathrm{Oo}; 0 \mid \mathrm{b}; \alpha_1, \beta_1; \ldots; \alpha_r, \beta_r)$ can be obtained as $\alpha_1\alpha_2 \cdots \alpha_r$-fold branched covering of $S^3$. For, letting a translation group of order $g = \alpha_1\alpha_2 \cdots \alpha_r$ act on $F$, we get a fibered space without exceptional fibers which is $S^3$ by Theorems 16 and 12. This fibering of $S^3$ is by unknotted curves any two of which are simply linked. The branch curves in $S^3$ are the images of the $r$ exceptional fibers, i.e., $r$ unknotted and pairwise linked curves in $S^3$, of index $\alpha_1, \alpha_2, \ldots, \alpha_r$, respectively.

## 15. Spaces Which Cannot Be Fibered

Let $F$ be a fibered space (open or closed). Let $H$ be an ordinary fiber, $O$ a point of $H$ and $W$ a closed curve starting and ending at $O$. Translating the fiber $H$ along $W$, $H$ comes back as $H' = H^{\pm 1}$. Thus as elements of the fundamental group, $W^{-1}HW = H^{\pm 1}$. Therefore if a manifold $M$ can be fibered, then $\pi_1(M)$ must contain an element $H$ such that for each element $W$ of $\pi_1(M)$, $W^{-1}HW = H^{\varepsilon(W)}$, where $\varepsilon(W) = \pm 1$. This condition turns out to be nontrivial since we shall show that an ordinary fiber $H$ represents the trivial element of the fundamental group only if the fibered space is $S^3$ or a lens space with a fibration that can be explicitly described.[28] In particular, if the fundamental group is infinite, then $H$ is not trivial.

[27] Another characterization of the cyclic covering is as follows: Cut $S^3$ along a spanning surface of $k$ to get a "sheet" and glue $g$ of those sheets together cyclically. H. Kneser communicated to me that there are in general besides this cyclic covering other $g$-fold coverings of a knot which also have the property that for a small loop linking the knot once the $g$-fold multiple is the first to lift to a closed curve in the covering space. The cyclic coverings play some rôle in knot theory. See K. Reidemeister, *Abh. Math. Sem. Univ. Hamburg* **5** (1927), 7, "Knotentheorie." Berlin (1932).

[28] About lens spaces, see. DB II, §1.

First we prove a preliminary theorem.

THEOREM 18. *An open simply connected space cannot be fibered.*

*Proof.* Suppose $F$ is an open simply connected fibered space with orbit surface $f$. Then $f \approx$ open disk. We distinguish two cases:

(a) Suppose $F$ is without exceptional fibers. Since $\pi_1(F) = 1$, $H$ bounds a singular disk $E$ in $F$. The image on $f$ is a singular disk $e$ which can be covered by an orbit neighborhood $\omega$ since $f$ is open and simply connected. $E$ lies in a neighborhood $\Omega$ corresponding to $\vartheta$, i.e., $H \simeq 0$ in the solid torus $\Omega$, a contradiction.

(b) $F$ has at least one exceptional fiber $C$ of order $\alpha$. Drilling out $C$ we obtain a space $\bar{F}$ with orbit surface $\bar{f}$, a punctured open disk. $H_1(\bar{F})$ is free of rank 1, generated by a meridian $M$ of the drilled-out solid torus which maps $\alpha$ times onto the boundary curve $l$ of $\bar{f}$, $\alpha \geqslant 2$. The map $\bar{F} \to \bar{f}$ induces a homomorphism[29] of $H_1(\bar{F}) \to H_1(\bar{f})$ (onto). Since $H_1(\bar{f})$ is infinite cyclic, $M$ has to map onto a generator $\pm l$ of $H_1(\bar{f})$, but $M \to \alpha l$, $\alpha > 1$, a contradiction.

Theorem 18 implies that $R^3$ can not be fibered. If we project as in §3 a fibering of $S^3$ stereographically in Euclidean space, the latter will be filled with curves which resemble closely a fibration. Only one curve, the $z$-axis is not closed.

Using Theorem 18 we can prove

THEOREM 19. *If in a fibered space $F$ a fiber $H$ or a finite multiple of $H$ is homotopic to* 0, *then $F$ is closed and $\pi_1(F)$ is finite.*

*Proof.* The universal covering $\tilde{F}$ of $F$ is a fibered space [by §9, (6)] which is closed by Theorem 18 (therefore $\tilde{f} \approx S^3$) and therefore the covering $\tilde{F} \to F$ is finite sheeted.

THEOREM 20. *If $F$ is a (closed or open) fibered space in which an ordinary fiber is homotopic to* 0, *then $F$ is a Lens space. Any Lens space admits such a fibering.*

*Proof.* By Theorem 19, $\pi_1(F)$ is finite. We apply Theorem 9. If $f \approx S^2$ and $F$ has three exceptional fibers, then

$$\begin{aligned}\pi_1(F) = \{ Q_0, Q_1, Q_2, Q_3, H : Q_0 H^b = 1 = Q_i^{\alpha_i} H^{\beta_i}\ (i = 1,2,3),\\ Q_0 Q_1 Q_2 Q_3 = 1,\ Q_j H Q_j^{-1} = H\ (j = 0,1,2,3)\}. \qquad (1)\end{aligned}$$

$\alpha_1, \alpha_2, \alpha_3$ is one of the Platonian triples. Eliminating $Q_0$ and adding the relation $H^2 = 1$, we obtain a quotient group with defining relations

$$\check{Q}_i^{\alpha_i} \check{H}^{\delta_i} = \check{Q}_1 \check{Q}_2 \check{Q}_3 \check{H}^{\delta_4} = \check{H}^2 = 1, \qquad \check{Q}_i \check{H} \check{Q}_i^{-1} = \check{H} \qquad (i = 1,2,3). \quad (2)$$

[29] See Footnote 19.

Here $\delta_1, \delta_2, \delta_3, \delta_4 = 0$ or $= 1$ depending on whether $\beta_1$, $\beta_2$, $\beta_3, b$ are even or odd, respectively. Taking new generators, we can always assume that $\delta_1 = \delta_2 = \delta_3 = 1$, $\delta_4 = 0$. For in the Platonian triples $\alpha_1, \alpha_2, \alpha_3$ one exponent, say $\alpha_2 = 2$. Then $\beta_2 = 1$ $(0 < \beta_i < \alpha_i)$; hence $\delta_2 = 1$. But if $\alpha_1$ is odd, $\beta_1$ may be even and $\delta_1 = 0$. In this case take as new generator $Q_1'$ defined by $\check{Q}_1 = Q_1'\check{H}$. The relation $\check{Q}_1^{\alpha_1}\check{H}^{\delta_1} = 1$ becomes $Q_1'^{\alpha_1}\check{H}^{\delta_1+\alpha_1} = 1$ and $\alpha_1 + \delta_1 = \alpha_1$ is odd, hence $\check{H}^{\delta_1+\alpha_1} = \check{H}$. Thus assume $\delta_1 = \delta_2 = \delta_3 = 1$. Now if $\delta_4 = 1$, we define $Q_2'$ by $\check{Q}_2 = Q_2'\check{H}$. Then $\delta_4 = 0$ and since $\alpha_2 = 2$ the other relations are not changed. Therefore

$$Q_1'^{\alpha_1} = Q_2'^{\alpha_2} = Q_3'^{\alpha_3} = \check{H}, \qquad Q_1'Q_2'Q_3' = 1, \check{H}^2 = 1. \tag{3}$$

The groups defined by these relations are (for the Platonian triples) the binary platonian groups. In *Math. Ann.* **104**, 26, it is shown that $\check{H}$ has order 2. Therefore $H$ does not have order 1 in $\pi_1(F)$ and $H \not\simeq 0$ in $F$.

Now suppose $f \approx P^2$, hence $r = 1$ or 0. For $r = 1$, $\pi_1(F)$ has relations

$$\begin{gathered} AHA^{-1}H = 1, \qquad Q_0Q_1 = A^2, \qquad Q_jHQ_j^{-1} = H \qquad (j = 0, 1) \\ Q_0H^b = 1 = Q_1^{\alpha_1}H^{\beta_1}. \end{gathered} \tag{4}$$

Eliminating $Q_0$ and adding the relation $H^2 = 1$, we obtain a quotient group with relations

$$\begin{gathered} \check{A}^2\check{Q}_1^{-1}\check{H}^{\delta_1} = \check{Q}_1^{\alpha_1}H^{\delta_2} = 1, \qquad \check{H}^2 = 1, \\ \check{A}\check{H}\check{A}^{-1} = \check{H}, \qquad \check{Q}_1\check{H}\check{Q}_1^{-1} = \check{H}. \end{gathered}$$

Eliminating $\check{Q}_1$ we obtain the Abelian group

$$\check{H}^2 = 1, \qquad \check{A}^{2\alpha_1}\check{H}^{\delta_3} = 1.$$

$\delta_1, \delta_2, \delta_3$ are 0 or 1. In this Abelian group $\check{H}$ does not have order 1, regardless whether $\delta_3 = 0$ or $= 1$; hence $H \not\simeq 0$ in $F$. If $r = 0$, we have $\alpha_1 = 1$ and obtain the same result.

The remaining case is that $f = S^2$ and $F$ has at most two exceptional fibers. We decompose $f$ into two disks each having at most one exceptional point. This corresponds to a decomposition of $F$ into two solid tori $V_1, V_2$. Hence $F$ is a lens space or $S^2 \times S^1$. In $S^2 \times S^1$ the fiber is not $\simeq 0$ (Theorem 19). For each lens space there are infinitely many distinct fiberings in which each ordinary $H \simeq 0$. For a lens space is determined by a simple closed curve on $\partial V_1 = \Pi_1$ which is identified with a meridian $M_2$ of $V_2$. Thus if $M_1, B_1$ are meridian and longitude on $\Pi_1$, the lens space is determined by the homology

$$M_2 \sim pB_1 + qM_1 \qquad (\text{on } \Pi_1), \tag{5}$$

hence by $p, q$. Here $p \neq 0$; otherwise $M_2 \sim {}^{\pm}M_1$ and $F \approx S^2 \times S^1$. Fiber $V_1$ such that

$$H \sim pB_1 + xM_1 \tag{6}$$

where $x \neq q$, $(x, p) = 1$. By Lemma 6 the fibering of the resulting lens space is uniquely determined by the fibering of $V_1$. Now $H \simeq 0$ since $H \sim M_2 - qM_1 + xM_1$; but $M_1$ and $M_2$ are $\simeq 0$ in the lens space. This completes the proof of Theorem 20.

By Theorems 11 and 18, $S^3$ is the only simply connected 3-manifold that admits a fibration. If, however, the fundamental group is not trivial, we can now state a fibration condition:

THEOREM 21. *If a (open or closed) nonsimply connected manifold $M$ can be fibered, then $\Pi_1(M)$ has an element $H \neq 1$ such that $W^{-1}HW = H^{\varepsilon(W)}$, $\varepsilon(W) = \pm 1$ [for each $W \in \pi_1(M)$].*

For either a fiber $H \simeq 1$ in $\pi_1(M)$, then $M$ is a lens space and $\pi_1(M)$ is cyclic, or $H \neq 1$ in $\pi_1(M)$ and the result follows from the first paragraph of this section.

Using this theorem we can exhibit infinitely many (open or closed) manifolds that cannot be fibered, namely, the connected sum of two manifolds. The connected sum of two manifolds $R_A$ and $R_B$ is obtained by removing from each a 3-ball and gluing together the two resulting boundary 2-spheres, which can be done in two different ways. If $A$ and $B$ are the fundamental groups of $R_A$ and $R_B$, then the fundamental group of the connected sum is the free product $A * B$ of $A$ and $B$.[30] The free product $A * B$ is defined as follows[31]: An element is an arbitrary product of finitely many elements of $A$ and $B$ which are called terms. Each such element which is not the identity element can be reduced to a normal form, in which terms of $A$ and $B$ different from the identity alternate. Two elememts of the free product are equal if and only if their normal forms agree term by term. For example,

$$A_{i_1}B_{j_1}A_{i_2}B_{j_2}\cdots A_{i_r}B_{j_r} = A'_{i_1}B'_{j_1}A'_{i_2}B'_{j_2}\cdots A'_{i_r}B'_{j_r}$$

if and only if

$$A_{i_1} = A'_{i_1}, \qquad B_{j_1} = B'_{j_1}, \qquad \cdots, \qquad B_{j_r} = B'_{j_r}.$$

Two elements are multiplied by composing the terms of the two products. We now use

LEMMA 8. *If $A$ and $B$ are nontrivial groups, then the free product $A * B$ has an element $H$ as in Theorem 21 if and only if both $A$ and $B$ have order 2.*

*Proof.* It follows from the normal form of the elements of $A * B$ that $H \notin A$ and $H \notin B$, since, e.g., composing an element of $A$ with an element $\neq 1$ of $B$ cannot give an element of $A$. But since $H \notin A$, $H$ does not commute with any nontrivial element of $A$, since $aHa^{-1}$ does not have the same normal form as $H$. Therefore, for $a \neq 1 \in A$, $aHa^{-1} = H^{-1}$. For $a' \neq 1 \in A$,

[30] The proof of this claim is on p. 36 of the paper cited in Footnote 18.

[31] See C. Schreider, Die Untergruppen der freien Gruppen. *Abh. Math. Sem. Univ. Hamburg* **5** (1927), 161.

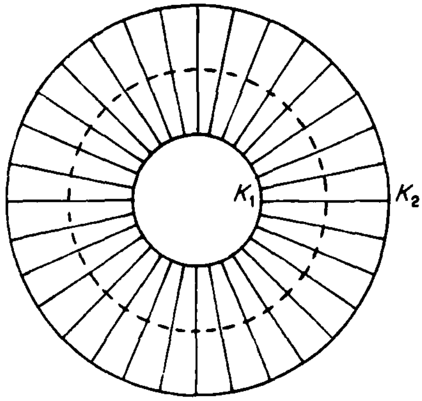


FIG. 15

$a'Ha'^{-1} = H^{-1}$, hence $a'^{-1}aHa^{-1}a' = H$, hence $a^{-1}a' = 1$, Therefore each element $a' \neq 1$ of $A$ is $= a^{-1}$; in particular, $a^{-1} = a$, i.e., $A = \mathbb{Z}_2(a)$. The same holds for $B$.

Theorem 21 now implies

THEOREM 22. *The connected sum of two nonsimply connected* 3*-manifolds can be fibered only if both manifolds have a fundamental group of order* 2.

In the exceptional case the connected sum can be fibered, for example the sum of two projective spaces. $P^3 \# P^3$ is obtained by identifying diametrical points on the boundary spheres $K_1$ and $K_2$ of $S^2 \times I$ (see Fig. 15) since the dotted 2-sphere separates this manifold into two punctured projective spaces. The fibers are the radii of $S^2 \times I$; any two diametrical radii form one fiber. The invariants of the fibering are (On; 1 | 0); $b = 0$ since $P^3 \# P^3$ admits a fiber preserving orientation reversing homeomorphism (reflection on the dotted $S^2$). Therefore by Theorem 6, (On; 1 | b) = (On; 1 | $-b$), hence $b = -b$.

The simplest example of a space that cannot be fibered is $(S^2 \times S^1) \# (S^2 \times S^1)$. We have encountered three possible cases:

(1) $F$ cannot be fibered.

(2) $F$ can be fibered in only one way (Poincaré spaces).

(3) $F$ has infinitely many fiberings ($S^3$). In this example all fibrations have the same orbit surface, namely $S^2$.

We conclude with an example of a space having two fiberings with different orbit surfaces. It is the quaternion space, with fundamental group the quaternion group. It is obtained from a cube by identifying any two opposite faces under a rotation of $\pi/2$. Since the quaternion group, which is generated by $\pm 1, \pm i, \pm j, \pm k$, has an element, namely, $-1$, that commutes with all others, and also another element, e.g., $i$, that commutes with $\pm 1$, and

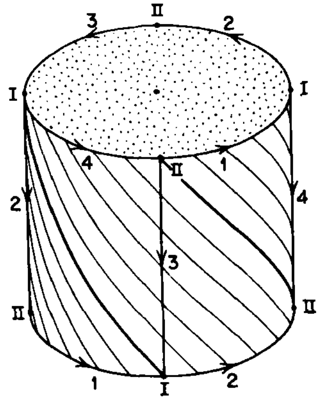


FIG. 16

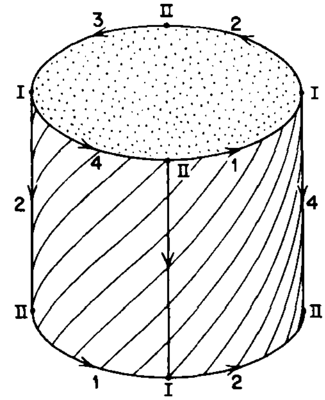


FIG. 17

$\pm i$ and whose conjugate with $\pm j$, $\pm k$ is $-i$, one could conjecture that the space can be fibered in two different ways. This is indeed the case. We deform the cube to a cylinder where bottom and top disks are identified under a (say right-handed) rotation of $\pi/2$, and the lateral surface of the cylinder is divided by four vertical lines into four faces, where each two opposite faces are identified under a right-handed rotation of $\pi/2$ (see Fig. 16). Under the identification the lateral faces are deformed so that a vertical line becomes a quarter circle of the bottom (resp. top) disk.

If we deform the bottom disk of the cylinder under a continuous left rotation of total angle $\pi/2$ into the top disk, then each point of the bottom disk describes a screw line, in particular the center point of the bottom disk. These screw lines form the first fibering of the quaternion space. There are three 2-fold exceptional fibers: the axis and the diagonals of the pairwise corresponding faces.

The second fibering is obtained from the first by reflection on a plane through the axis, i.e., consists of right hand screw lines (see Fig. 17). There are no exceptional fibers.

The two orbit surfaces are distinct, since in the first fibering the fibers can be simultaneously oriented, in the second this is not possible. By Theorem 9 the orbit surface of the first fibering is $S^2$, that of the second is $P^2$. In the first case we can take as orbit surface a semidisk of the bottom disk, where the radii and quarter circles on the boundary have to be identified. In the second case it is the whole bottom disk with diametrical points on the boundary identified.

## Appendix. Branched Coverings

### 1. Definition of Branched Covering

For a Euclidean 3-ball $E$ of radius 1 let $\varphi$ (geographical length), $\vartheta$ (angular height), and $\rho$ (radius) be polar coordinates,

$$0 \leqslant \varphi < 2\pi, \qquad -\pi/2 \leqslant \vartheta \leqslant +\pi/2. \qquad 0 \leqslant \rho \leqslant 1.$$

Denote polar coordinates for a Euclidean ball $\tilde{E}$ by tildes. $\tilde{E}$ is called a *p-fold branched covering of* $E$ if the map of $\tilde{E}$ to $E$ is given by

$$\rho = \tilde{\rho}, \qquad \vartheta = \tilde{\vartheta}, \qquad \varphi \equiv p\tilde{\varphi} \qquad (\mathrm{mod}\ 2\pi) \qquad (p > 1).$$

In both $E$ and $\tilde{E}$, the diameter from south pole to north pole is called the branch curve. If $K$ and $\tilde{K}$ are homeomorphic images of $E$ and $\tilde{E}$, then $\tilde{K}$ is mapped to $K$ via $\tilde{E}$ and $E$. Then $\tilde{K}$ is also called a $p$-fold branched covering of $K$, and the curves in $K$, $\tilde{K}$ which correspond to the branch curves of $E$, $\tilde{E}$, respectively, under the homeomorphisms are called the branch curves of $K$, $\tilde{K}$, respectively. If $\tilde{K}$ maps homeomorphically to $K$, we say that $\tilde{K}$ is an unbranched covering of $K$.

Let $k_1, \ldots, k_r$ be a finite number of simple closed curves, called knots, in a 3-manifold $M$ with the following properties: For each point $P$ of the knot $k_i$ there is a neighborhood $U(P)$ in $M$, disjoint to $k_j$ for $j \neq i$, which can be mapped homeomorphically to the interior of a Euclidean 3-ball so that the image of $k_j \cap U(P)$ is a diameter. $U(P)$ is called a *normal neighborhood* of $P$ and $k_i \cap U(P)$ the *diameter* of $U(P)$. If $P$ does not lie on a knot, we call normal any neighborhood which is homeomorphic to the interior of a 3-ball and which is disjoint from all the knots. An admissible path in $M$ is the image under a continuous map of an oriented line segment such that it is disjoint from the knots except possibly for the endpoint.

Let $\tilde{M}$ be a 3-manifold and $\mathfrak{A}: \tilde{M} \to M$ be a continuous map. We say that the point $\tilde{P}$ of $\tilde{M}$ lies over the point $P$ of $M$ and that $P$ is the projection of $\tilde{P}$ if $\mathfrak{A}(\tilde{P}) = P$. An admissible path in $\tilde{M}$ is a path whose image under $\mathfrak{A}$ is admissible in $M$. Now $\tilde{M}$ is called a branched covering of $M$ with branch curves $k_1, \ldots, k_r$ if the following holds (see also §9):

I. Over each point $P$ of $M$ lies at least one point $\tilde{P}$ of $\tilde{M}$.

II. If $\tilde{P}_1, \tilde{P}_2, \ldots$ are all the points which lie over $P$, there is a normal neighborhood $U(P)$ in $M$ and there are normal neighborhoods $U(\tilde{P}_1)$ $U(\tilde{P}_2), \ldots$ in $\tilde{M}$ which together consist of all points lying over points of $U(P)$ and which have the following properties: (a) If $P$ is a point on a knot $k_j$, then $U(\tilde{P}_i)$ is a branched or unbranched covering of $U(P)$ with $\tilde{k}_j \cap U(P)$ as branch curve; (b) If $P$ does not lie on a knot, then $\mathfrak{A} \mid \mathrm{U}(\tilde{\mathrm{P}}_i): U(\tilde{P}_i) \to U(P)$ is a homeomorphism.

Let $N$ be the open submanifold of $M$ obtained from $M$ by removing all points on the knots; let $\tilde{N}$ be the submanifold $\mathfrak{A}^{-1}(N)$ of $\tilde{M}$. Then we have the following theorems, which we state without proof:

(1) *$\tilde{N}$ is an unbranched covering of $N$* (§9).

(2) *If $P = P(t)$, $0 \leqslant t \leqslant 1$, is an admissible path in $M$ from a point $P(0)$ to a point $P(1)$, and if $\tilde{P}(0)$ is a point over $P(0)$, then there exists a unique lift $\tilde{P}(t)$ in $\tilde{M}$ which starts at $\tilde{P}(0)$ and such that $\tilde{P}(t)$ lies over $P(t)$.*

(3) *If $\tilde{w}$ is a closed curve of $\tilde{N}$ which lies over a contractible curve in $N$, then $\tilde{w}$ is contractible in $\tilde{N}$.*

(4) *If exactly n points lie over some point of N, then exactly n points lie over each point of N (n-fold covering).*

## 2. The Subgroup $\mathfrak{H}$ of the Fundamental Group

Let $\mathfrak{F}, \tilde{\mathfrak{F}}$ be the fundamental group of $N$, $\tilde{N}$, respectively. Choosing the base point $\tilde{O}$ for $\tilde{\mathfrak{F}}$ over the base point $O$ for $\mathfrak{F}$, a homotopy class of (based) loops of $\tilde{N}$ is mapped to such a class of $N$. This induces an isomorphism of $\tilde{\mathfrak{F}}$ onto a subgroup $\mathfrak{H}$ of $\mathfrak{F}$. We call $\mathfrak{H}$ the subgroup of $\mathfrak{F}$ corresponding to the given covering. Note however that $\mathfrak{H}$ depends on the choice of the base point $\tilde{O}$ over $O$; we choose once and for all a fixed $\tilde{O}$ over $O$. (If we would choose another base point over $O$, we would get a subgroup conjugate to $\mathfrak{H}$ in $\mathfrak{F}$.) A based loop in $N$ belongs to $\mathfrak{H}$ if and only if its lift from $\tilde{O}$ is closed in $\tilde{N}$. Decomposing $\mathfrak{F}$ into its cosets of $\mathfrak{H}$,

$$\mathfrak{F} = \mathfrak{H} + \mathfrak{H}F_2 + \mathfrak{H}F_3 + \dots,$$

we get a one-to-one correspondence between these cosets and the points over $O$ as follows: Choose a path $w$ from the coset $\mathfrak{H}F_i$ and lift it from $\tilde{O}$ to $\tilde{w}$. The endpoint of $\tilde{w}$ corresponds to the coset $\mathfrak{H}F_i$. This correspondence is apparently independent of the choice of the path $w$ from $\mathfrak{H}F_i$. In particular, if the covering of $N$ by $\tilde{N}$ is finite sheeted, then the number of sheets equals the index of $\mathfrak{H}$ in $\mathfrak{F}$.

## 3. Unique Determination of $\tilde{M}$ by $\mathfrak{H}$

For the following it is convenient to consider only a particular system of neighborhoods of the covering space. As neighborhoods of a point $P$ of the covering space we consider only those 3-balls which lie concentrically in a normal 3-ball and which cover (branched or unbranched) a normal neighborhood of the image point $P$. This system of neighborhoods (for all points $P$ of $\tilde{M}$) is equivalent to the system of all open sets of $\tilde{M}$.

*If $\tilde{M}_1$ and $\tilde{M}_2$ are two branched covers of $M$ which induce the same subgroup $\mathfrak{H}$ of $\mathfrak{F}$, then they are homeomorphic so that corresponding points have the same image in $M$.* In order to define the homeomorphism $f: \tilde{M}_1 \to \tilde{M}_2$, join a point $\tilde{P}_1 \in \tilde{M}_1$ to $\tilde{O}_1$ by an admissible path $\tilde{\alpha}_1$ and lift the image path $a$ of $\tilde{\alpha}_1$ to a path $\tilde{\alpha}_2$ in $\tilde{M}_2$ from $\tilde{O}_2$. Let $f(\tilde{P}_1)$ be the endpoint of this lift. $f(\tilde{P}_1)$ is uniquely determined by $\tilde{P}_1$ and does not depend on the path $\tilde{\alpha}_1$. For if $\tilde{P}_1$ does not lie over a point on a branch curve, and if $\tilde{b}_1$ is another path joining $\tilde{P}_1$ to $\tilde{O}_1$, then the path $\tilde{a}_1\tilde{b}_1^{-1}$ is a closed curve in $\tilde{M}_1$ and therefore its image in $M$ is contained in the subgroup $\mathfrak{H}$ of $\mathfrak{F}$; since $\tilde{M}_2$ corresponds to the same subgroup $\mathfrak{H}$ it follows that the lift $\tilde{a}_2\tilde{b}_2^{-1}$ is a closed curve in $\tilde{M}_2$ and therefore the endpoint of $\tilde{b}_2$ is the same as that of $\tilde{a}_2$. If $\tilde{P}_1$ lies over a point on a branch curve, we deform the path $\tilde{a}_1\tilde{b}_1^{-1}$ inside an arbitrarily small ball neighborhood $\tilde{U}_1$ of $\tilde{P}_1$ into an admissible path as follows: Choose a point $\tilde{A}_1$ on $\tilde{a}_1$ close to $\tilde{P}_1$ such that the subpath $\tilde{A}_1\tilde{P}_1$ of $\tilde{a}_1$ lies in $\tilde{U}_1$; similarly, choose a point $\tilde{B}_1$ on $\tilde{b}_1$ shortly before $\tilde{P}_1$ and join $\tilde{A}_1$ and $\tilde{B}_1$ by a path $\tilde{v}$ inside $\tilde{U}_1$

which misses the branch curve. The corresponding detachment is done in the ground space $M$. The ball neighborhood $\tilde{U}_1$ is mapped to a normal neighborhood, the points $\tilde{A}_1, \tilde{B}_1$ into two points $a, b$ close to $P$, and the detached ground path belongs to $\mathfrak{H}$ since it is the image of an admissible closed curve in $\tilde{M}_1$. Since we can choose $\tilde{U}_1$ arbitrarily small, we can detach the path $ab^{-1}$ into a curve of $\mathfrak{H}$ in an arbitrarily small normal neighborhood of $P$. Now supposing that $\tilde{a}_2$ and $\tilde{b}_2$ lead from $\tilde{O}_2$ to different endpoints $\tilde{P}_2$ and $\tilde{Q}_2$, we could find disjoint ball neighborhoods $\tilde{U}_2$ and $\tilde{V}_2$ of $\tilde{P}_2$ and $\tilde{Q}_2$. The corresponding normal image neighborhoods $U$ and $V$ of $P$ in $M$ have a neighborhood $W$ in common, inside which we detach the path $ab^{-1}$. Lifting the path $a$ (from $O$ to $A$) to $\tilde{M}_2$, we obtain a path from $\tilde{O}_2$ to a point $\tilde{A}_2$. Running from $A$ along $v$ to $B$, the lift in $\tilde{M}_2$ leads to a point $\tilde{B}_2$ which lies in $\tilde{U}_2$. On the other hand, running from $O$ to $B$ along $b$, the lift in $\tilde{M}_2$ is a path from $\tilde{O}_2$ to a point in $\tilde{V}_2$. But since the detached path $ab^{-1}$ belongs to $\mathfrak{H}$, the latter point has to be $\tilde{B}_2$. Therefore $\tilde{U}_2$ and $\tilde{V}_2$ cannot be disjoint and $\tilde{Q}_2 = \tilde{P}_2$.

This shows that the map $f: \tilde{M}_1 \to \tilde{M}_2$ is well defined and one-to-one. To show that $f$ is a homeomorphism, we have to find for any given neighborhood $\tilde{U}_1$ of $\tilde{P}_1$ a neighborhood $\tilde{U}_2$ of $\tilde{P}_2 = f(\tilde{P}_1)$ such that $f(\tilde{U}_2) \subset \tilde{U}_1$. If $U_1$ is the normal neighborhood of $P$ in $M$ which is (branched or unbranched) covered by $\tilde{U}_1$ and if $a$ is a path from $O$ to $P$ which lifts in $\tilde{M}_1$ to a path from $\tilde{O}_1$ to $\tilde{P}_1$, then each path from $O$ to a point $P'$ of $U_1$, which agrees with $a$ up to a point $A$ shortly before $P$ and from there remains inside $U_1$ lifts in $\tilde{M}_1$ from $\tilde{O}_1$ to a point in $\tilde{U}_1$. Now let $\tilde{U}_2$ be a ball neighborhood which is mapped into a normal subneighborhood $U_2$ of $U_1$. In $\tilde{M}_2$, $a$ lifts to a path from $\tilde{O}_2$ to $\tilde{P}_2$, and we can get to any point $\tilde{P}_2'$ of $\tilde{U}_2$ along a path which agrees with $\tilde{a}_2$ up to a point shortly before $\tilde{P}_2$ and which from there on remains in $\tilde{U}_2$. In the ground space $M$, this path maps to the type of paths from $O$ to a point $P'$, discussed above. This lifts in $\tilde{M}_1$ to a path from $\tilde{O}_1$ to a point in $\tilde{U}_1$. Hence $f: \tilde{M}_1 \to \tilde{M}_2$ is continuous and, since the same arguments apply to the inverse map, $f$ is a homeomorphism.

This shows that the covering $\tilde{M} \to M$ is uniquely determined by the subgroup $\mathfrak{H}$. In the same way one can show that to a given subgroup $\mathfrak{H}$ of finite index there exists a corresponding covering $\tilde{M}$.

#### 4. Regular Coverings*

LEMMA ABOUT BRANCHED COVERINGS OF $S^3$ WITH ABELIAN GROUP OF COVERING TRANSLATIONS. *Let $\tilde{M} \to M = S^3$ be a regular finite sheeted covering branched over the knots $k_1, \ldots, k_x$, with group of covering translations*

* *Translators note*: In this section regular coverings and covering translations are discussed and it is shown that for a regular covering corresponding to the normal subgroup $\mathfrak{H}$ of $\mathfrak{F}$ the group of covering transformations is isomorphic to $\mathfrak{F}/\mathfrak{H}$. A more detailed exposition can be found in Chapter VIII, §57 of "Seifert and Threlfall: A Textbook of Topology."

*Abelian and of order $g = \alpha_1 \cdots \alpha_x$. Assume: For a small loop $C_i$ that links $k_i$ exactly once, the lifts of $C_i^{\alpha_i}$ in $\tilde{M}$ are closed curves.*[32] *Then it follows that a path $\tilde{w}$ of $\tilde{M}$ that covers a path $w$ which misses the knots is closed if and only if for each $i$ the linking number $\chi_i$ of $w$ with $k_i$ is divisible by $\alpha_i$. Since this determines the subgroup $\mathfrak{H}$ of $\mathfrak{F}$ corresponding to $\tilde{M}$ there is by §3 only one covering $\tilde{M}$ with the above property.*

*Proof.* Every loop of $\mathfrak{F}$ lies in a certain coset of $\mathfrak{H}$ in $\mathfrak{F}$. A null homologous loop w of $\mathfrak{F}$ belongs always to $\mathfrak{H}$, since $w$ is a product of commutators which all lie in $\mathfrak{H}$ since $\mathfrak{F}/\mathfrak{H}$ is Abelian. Hence two homologous loops of $\mathfrak{F}$ lie in the same coset. But the homology group of $N$ is the free Abelian group generated by $C_1, \ldots, C_x$. Thus each loop $w$ of $N$ is homologous to a linear combination $\sum_{i=1}^{x} \chi_i C_i$, where $\chi_i$ denotes the uniquely determined linking number of $w$ with $k_i$ (with a suitable orientation of $k_i$). In particular $w \sim O$ in $N$ if and only if all its linking numbers vanish. Therefore loops of $\mathfrak{F}$ with the same linking numbers $\chi_i$ lie in the same coset $\mathfrak{H}$ of $\mathfrak{F}$. The loop $C_i$ need not be based at $O$ and may thus not belong to $\mathfrak{F}$, but joining $O$ to a point of $C_i$ by an admissible path $v_i$ we get a path $c_i = v_i C_i v_i^{-1}$ that belongs to $\mathfrak{F}$, is homologous to $C_i$ in $N$, and whose $\alpha_i$th power belongs to $\mathfrak{H}$. But $C_i^{\alpha_i}$ has linking number $\alpha_i$ with $k_i$ and linking number 0 with the other knots. Therefore those loops of $\mathfrak{F}$ whose linking number $\chi_i$ is divisible by $\alpha_i$ (for each $i$) belong to $\mathfrak{H}$. Two loops $w$ and $w'$ with all $x$ linking numbers congruent, i.e.,

$$\chi_i \equiv \chi_i' \pmod{\alpha_i} \qquad (i = 1, \ldots, x),$$

belong to the same coset of $\mathfrak{H}$ in $\mathfrak{F}$. Since there are only $\alpha_1 \cdots \alpha_x$ incongruent systems of linking numbers, and just as many cosets, all loops of $\mathfrak{F}$ whose linking numbers with the knots $k_1, \ldots, k_x$ are piece by piece congruent make up a coset of $\mathfrak{H}$ in $\mathfrak{F}$. In particular $\mathfrak{H}$ itself consists of all loops whose linking numbers $\chi_1, \ldots, \chi_x$ are divisible by $\alpha_1$ (resp. $\alpha_2, \ldots, \alpha_x$). The theorem therefore is true for all loops based at $O$. But then the theorem holds also for the other loops, since each loop $w$ in $N$ can be deformed without crossing the knots into a loop based at $O$, and this neither changes its linking number with $k_i$ nor its property of being covered by a loop of the covering space.

[32] It suffices to require that at least one lift of $C_i^{\alpha_i}$ is a closed curve; since the covering is regular, it then follows that all other lifts are closed curves.