An Introduction to the Volume Conjecture and its generalizations, III

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Tohoku University

Workshop on Volume Conjecture and Related Topics in Knot Theory
Indian Institute of Science Education and Research, Pune
21st December, 2018

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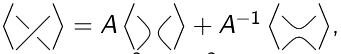
- 
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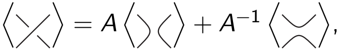
- 
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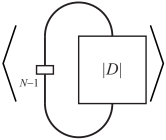

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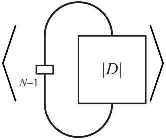

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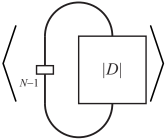

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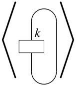
- $\Delta_k := (-1)^k \frac{A^{2(k+1)} - A^{-2(k+1)}}{A^2 - A^{-2}} =$ ,

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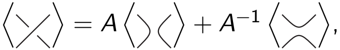
- 
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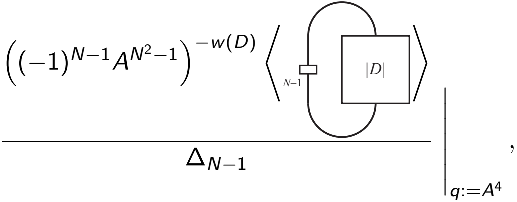
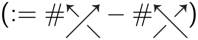
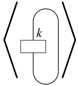
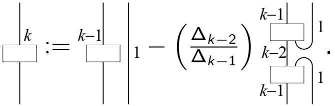


- $w(D)$ is the writhe  of D,
- $\Delta_k := (-1)^k \frac{A^{2(k+1)} - A^{-2(k+1)}}{A^2 - A^{-2}} =$ ,
-  (Jones–Wenzl idempotent)

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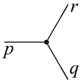
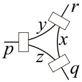
1. 

:= 

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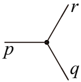
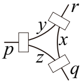
1. 

:= 

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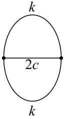
2. $\theta(k, k, 2c) :=$ 

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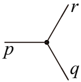
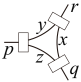
1. 

$:=$ 

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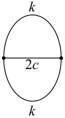
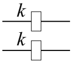
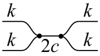
2. $\theta(k, k, 2c) :=$ 

3. 

$= \displaystyle\sum_{0 \leq c \leq k} \frac{\Delta_{2c}}{\theta(k, k, 2c)}$ 

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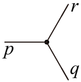
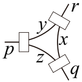
1. 

$:=$ 

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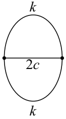
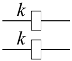
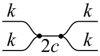
2. $\theta(k, k, 2c) :=$ 

3. 

$= \sum_{0 \leq c \leq k} \frac{\Delta_{2c}}{\theta(k, k, 2c)}$ 

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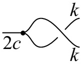
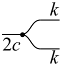
4. 

$= (-1)^{c-k} A^{-2k+2c+2c^2-k^2}$ 

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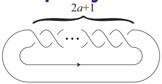
$T(2, 2a+1) :=$ 

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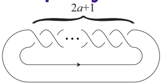
$T(2, 2a+1) :=$ 

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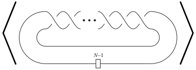

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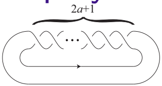
$T(2, 2a+1) :=$ 

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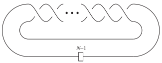
$\Big\langle$ 

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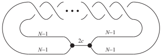
$$= \sum_{c=0}^{N-1} \frac{\Delta_{2c}}{\theta(N-1, N-1, 2c)}$$ $\Big\langle$ 

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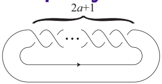
$T(2, 2a+1) :=$ 

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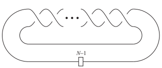
$\Bigg\langle$ 

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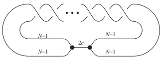
$$= \sum_{c=0}^{N-1} \frac{\Delta_{2c}}{\theta(N-1, N-1, 2c)} \Bigg\langle$$ 

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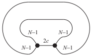
$$= \sum_{c=0}^{N-1} \frac{\Delta_{2c} \left((-1)^{c-N+1} A^{-2(N-1)+2c+2c^2-(N-1)^2} \right)^{2a+1}}{\theta(N-1, N-1, 2c)} \Bigg\langle$$ 

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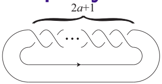
$T(2, 2a+1) :=$ 

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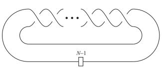
$\Big\langle$ 

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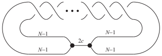
$= \sum_{c=0}^{N-1} \frac{\Delta_{2c}}{\theta(N-1, N-1, 2c)} \Big\langle$ 

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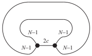
$= \sum_{c=0}^{N-1} \frac{\Delta_{2c}\left((-1)^{c-N+1} A^{-2(N-1)+2c+2c^2-(N-1)^2}\right)^{2a+1}}{\theta(N-1, N-1, 2c)} \Big\langle$ 

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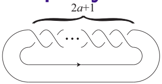
$T(2, 2a+1) :=$ 

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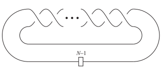
$\Bigg\langle$ 

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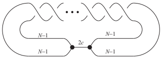
$= \sum_{c=0}^{N-1} \frac{\Delta_{2c}}{\theta(N-1, N-1, 2c)} \Bigg\langle$ 

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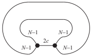
$= \sum_{c=0}^{N-1} \frac{\Delta_{2c}\left((-1)^{c-N+1}A^{-2(N-1)+2c+2c^2-(N-1)^2}\right)^{2a+1}}{\theta(N-1, N-1, 2c)} \Bigg\langle$ 

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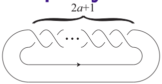
$T(2, 2a+1) :=$ 

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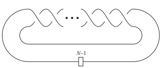
$\Big\langle$ 

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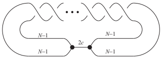
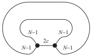
$= \sum_{c=0}^{N-1} \frac{\Delta_{2c}}{\theta(N-1, N-1, 2c)} \Big\langle$  $\Big\rangle$

$= \sum_{c=0}^{N-1} \frac{\Delta_{2c} \left((-1)^{c-N+1} A^{-2(N-1)+2c+2c^2-(N-1)^2} \right)^{2a+1}}{\theta(N-1, N-1, 2c)} \Big\langle$  $\Big\rangle$

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So we have

$$J_N(T(2, 2a+1); q) = \frac{(-1)^{N-1} q^{-(2a+1)(N^2-1)/2}}{q^{N/2} - q^{-N/2}} \times \sum_{c=0}^{N-1} (-1)^c q^{(2a+1)(c^2+c)/2} \left(q^{(2c+1)/2} - q^{-(2c+1)/2} \right).$$

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& J_N(T(2,2a+1); e^{\xi/N}) \\
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& \times (\Sigma_{+} - \Sigma_{-}),
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=&\frac{(-1)^{N-1}\exp\left[\frac{-(2a+1)(N^2-1)\xi}{2N} - \frac{\xi}{N}\left(\frac{2a+1}{8} + \frac{1}{2(2a+1)}\right)\right]}{2\sinh(\xi/2)} \\
&\times(\Sigma_+ - \Sigma_-),
\end{aligned}
$$

where

$$
\Sigma_{\pm} = \sum_{c=0}^{N-1}(-1)^c \exp\left[\frac{(2a+1)\xi}{2N}\left(c + \frac{2a+1\pm 2}{2(2a+1)}\right)^2\right].
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$$\Sigma_\pm=\sqrt{\frac{N}{2(2a+1)\xi\pi}}\times\sum_{c=0}^{N-1}(-1)^c\int_C\exp\left[\frac{-N}{2(2a+1)\xi}x^2+\left(c+\frac{2a+1\pm 2}{2(2a+1)}\right)x\right]dx$$

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$$\begin{aligned}\Sigma_{\pm}=&\sqrt{\frac{N}{2(2a+1)\xi\pi}}\\&\times\sum_{c=0}^{N-1}(-1)^c\int_{C}\exp\left[\frac{-N}{2(2a+1)\xi}x^2+\left(c+\frac{2a+1\pm 2}{2(2a+1)}\right)x\right]dx\\=&\sqrt{\frac{N}{2(2a+1)\xi\pi}}\\&\times\int_{C}\exp\left[\frac{-N}{2(2a+1)\xi}x^2\right]\exp\left(\frac{\pm x}{2a+1}\right)\left(\frac{1-(-1)^Ne^{Nx}}{e^{x/2}+e^{-x/2}}\right)dx.\end{aligned}$$

Asymptotic behavior of the colored Jones polynomial, III

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$$
\begin{aligned}
&\Sigma_{+} - \Sigma_{-} \\
=&\sqrt{\frac{N}{2(2a+1)\xi\pi}} \\
&\times \left(\int_C \frac{\sinh(\frac{x}{2a+1})}{\cosh(\frac{x}{2})} \exp\left[\frac{-N}{2(2a+1)\xi} x^2 \right] dx \right. \\
&\quad \left. - (-1)^N \int_C \frac{\sinh(\frac{x}{2a+1})}{\cosh(\frac{x}{2})} \exp\left[\frac{-N}{2(2a+1)\xi} x^2 + Nx \right] dx \right)
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since the the first integrand is an odd function.

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Integral in $\Sigma_{+}-\Sigma_{-}$

$$=e^{(2a+1)\xi N/2}\int_{C}\frac{\sinh(\frac{x}{2a+1})}{\cosh(\frac{x}{2})}\exp\left[\frac{-N}{2(2a+1)\xi}\bigl(x-(2a+1)\xi\bigr)^{2}\right]dx.$$

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Let $\tilde{C}$ be the line parallel to C that passes through $(2a+1)\xi$. By the residue theorem

$$\boxed{\text{Integral above}}$$
$$= \int_{\tilde{C}} \frac{\sinh(\frac{x}{2a+1})}{\cosh(\frac{x}{2})} \exp\left[\frac{-N}{2(2a+1)\xi}\left(x-(2a+1)\xi\right)^{2}\right] dx$$
$$+ 2\pi\sqrt{-1}\sum_{k} \operatorname{Res}\left(f(x); x=(2k+1)\pi\sqrt{-1}\right),$$

where $f(x) := \frac{\sinh(\frac{x}{2a+1})}{\cosh(\frac{x}{2})} \exp\left[\frac{-N}{2(2a+1)\xi}\left(x-(2a+1)\xi\right)^{2}\right]$ and $(2k+1)\pi\sqrt{-1}$ is between C and $\tilde{C}$.

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&= \int_{\tilde{C}} \frac{\sinh(\frac{x}{2a+1})}{\cosh(\frac{x}{2})} \exp\left[\frac{-N}{2(2a+1)\xi}\bigl(x-(2a+1)\xi\bigr)^2\right] dx \\
&\quad + 2\pi\sqrt{-1}\sum_{k}(-1)^k 2\sin\left(\frac{(2k+1)\pi}{2a+1}\right) \\
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We have

$$\int_{C_\theta} g(\zeta)\exp\left[\frac{-N\zeta^2}{H}\right] d\zeta \underset{N\to\infty}{\sim} \sqrt{\frac{\pi H}{N}}\, g(0),$$

where $f(N) \underset{N\to\infty}{\sim} g(N) \Leftrightarrow f(N)/g(N) \to 1 \ (N \to \infty)$.

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Note that the assumption $\operatorname{Re}\bigl(H^{-1}\exp(2\theta\sqrt{-1})\bigr) > 0$ is to make the integral converge.

Asymptotic behavior of the colored Jones polynomial, VI

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$$\underset{N\to\infty}{\sim} \sqrt{\frac{2(2a+1)\pi\xi}{N}} \frac{\sinh(\xi)}{\cosh(\frac{(2a+1)\xi}{2})}.$$

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$$J_N\big(T(2,2a+1);\exp(\xi/N)\big) \underset{N\to\infty}{\sim} \frac{1}{\Delta\left(T(2,2a+1);\exp(\xi)\right)} + \frac{\sqrt{-\pi}}{2\sinh(\xi/2)}\sum_{k}\exp\left[\frac{N}{\xi}S_k(\xi)\right]\sqrt{\frac{N}{\xi}}\tau_k,$$

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where $\Delta(K;t)$ is the Alexander polynomial of a knot K and

$$S_k(\xi) := \frac{-\bigl((2k+1)\pi\sqrt{-1}-(2a+1)\xi\bigr)^2}{2(2a+1)},$$
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Note that $\Delta(T(2,2a+1);t)=\dfrac{\left(t^{2a+1}-t^{-(2a+1)}\right)\left(t^{1/2}-t^{-1/2}\right)}{\left(t^{(2a+1)/2}-t^{-(2a+1)/2}\right)\left(t-t^{-1}\right)}.$

SL(2; $\mathbb{C}$) Chern–Simons invariant for closed three-manifolds

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$$\mathrm{cs}_M([\rho]) := \frac{1}{8\pi^2} \int_M \mathrm{Tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right) \in \mathbb{C} \pmod{\mathbb{Z}}.$$

Chern–Simons for knot complements

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$$\mathrm{cs}_M\colon X(M) \to \mathrm{Hom}(\pi_1(\partial M), \mathbb{C}) \times \mathbb{C}^*/\sim,$$

$$\sim\colon \begin{cases} (s,t;z) & \sim (s+1,t;z\exp(-8\pi\sqrt{-1}t)), \\ (s,t;z) & \sim (s,t+1;z\exp(8\pi\sqrt{-1}s)), \\ (s,t;z) & \sim (-s,-t;z). \end{cases}$$

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Theorem (P. Kirk & E. Klassen (1993))

$$\frac{z_1}{z_0} = \exp\left[\frac{\sqrt{-1}}{2\pi}\int_0^1 \left(u_t \frac{d\,v_t}{d\,t} - v_t \frac{d\,u_t}{d\,t}\right) dt\right]$$

Calculation of cs_M for $S^3 \setminus T(2, 2a + 1))$, I

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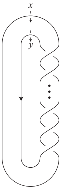

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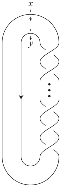

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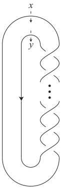

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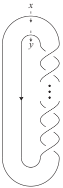

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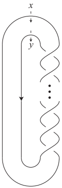

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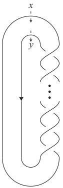

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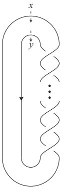

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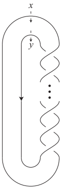

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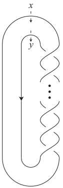

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Calculation of cs_M for $S^3 \setminus T(2, 2a+1)$, II

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Colored Jones and Chern–Simons for $T(2, 2a + 1)$

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$$S_k(2\pi\sqrt{-1}+u) - u\pi\sqrt{-1} - \frac{1}{4}uv_k = \frac{(2k+1)^2\pi^2}{2(2a+1)} + (k-2a-1)u\pi\sqrt{-1} \pmod{\pi^2\mathbb{Z}}.$$

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If we put $l := k - 2a - 1$, then

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Definition of the Reidemeister torsion, I

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Definition of the Reidemeister torsion, II

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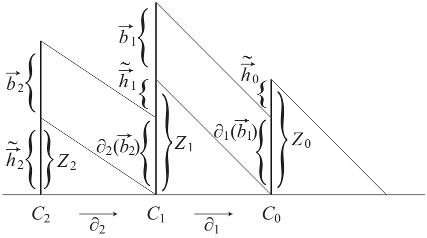

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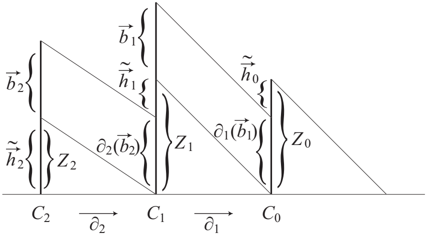

- $\mathbb{T}(\rho) := \pm \prod_{i=0}^{2} \left[\partial_{i+1}(\vec{b}_{i+1}) \cup \tilde{\vec{h}}_i \cup \vec{b}_i \;\middle|\; \vec{c}_i\right]^{(-1)^{i+1}}$, where $[\vec{x} \mid \vec{y}]$ is the determinant of the basis change matrix.

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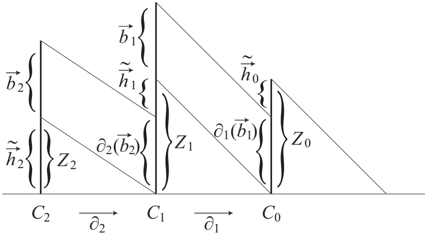

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Under a certain codition $\mathbb{T}(\rho)$ is well-defined once we fix bases for H_*. In the following, our Reidemeister torsion $\mathbb{T}(\rho)$ is defined by using the basis of H_1 associated with the meridian.

Calculation of $\mathbb{T}$ for $S^3 \setminus$, I

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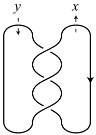

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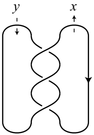

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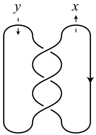

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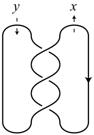

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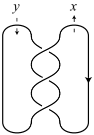

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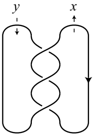

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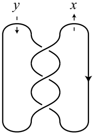

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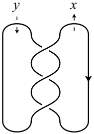

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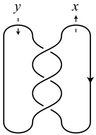
Calculation of $\mathbb{T}$ for $S^3 \setminus$ [knot], I



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- $\vec{c}_0 = \{\tilde{p} \otimes E, \tilde{p} \otimes H, \tilde{p} \otimes F\}$ ($\tilde{p}$: lift of the basepoint p)
- $\frac{\partial r}{\partial x} = 1 + xy - xyxy^{-1}x^{-1}$, $\frac{\partial r}{\partial y} = x - xyxy^{-1} - xyxy^{-1}x^{-1}y^{-1}$.

Calculation of $\mathbb{T}$ for $S^3 \setminus$, II

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So $\dim B_2 = 3 - 1 = 2$, and $\dim B_1 = 6 - 1 - 2 = 3$.

Calculation of $\mathbb{T}$ for $S^3 \setminus$, III

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Calculation of $\mathbb{T}$ for $S^3 \setminus$, III

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By using twisted Alexander polynomial (Y. Yamaguchi, T. Kitano, J. Porti, etc.), we can also prove $\mathbb{T}(\rho_{u,k}) = \pm \dfrac{2a+1}{8\sin^2\left(\frac{(2k+1)\pi}{2(2a+1)}\right)}$.

CJ, CS, and Reidemeister for $T(2, 2a+1)$

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- $S_k(\xi)$: defines the Chern–Simons invariant associated with the meridian and the longitude.

CJ, CS, and Reidemeister for $T(2, 2a+1)$

We have proved:

$$J_N\bigl(T(2,2a+1);\exp(\xi/N)\bigr) \underset{N\to\infty}{\sim} \frac{1}{\Delta\left(T(2,2a+1);\exp(\xi)\right)} + \frac{\sqrt{-\pi}}{2\sinh(\xi/2)}\sum_k \exp\left[\frac{N}{\xi}S_k(\xi)\right]\sqrt{\frac{N}{\xi}}\tau_k,$$

where

- $\Delta(K;t)$: Alexander polynomial,
- $S_k(\xi)$: defines the Chern–Simons invariant associated with the meridian and the longitude.
- τ_k^{-2}: Reidemeister torsion associated with the meridian.

Colored Jones for $u = 0$, $T(2, 2a + 1)$

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Asymptotic behavior of CJ for , I

$\xi := 2\pi\sqrt{-1} + u$ with $u > 0$: small.

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For a complex number γ with $\mathrm{Re}(\gamma) > 0$, define the quantum dilogarithm (L. Faddeev):

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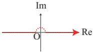
where $C :=$ 

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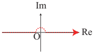
where $C :=$ 

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Note: $S_{\pi/N} = \exp\left(\frac{N}{2\pi\sqrt{-1}}\mathrm{Li}_2(-e^{\sqrt{-1}z}) + O(1/N)\right)$ $(N \to \infty)$.

Asymptotic behavior of CJ for , II

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$$(1 + e^{\sqrt{-1}(\pi - \sqrt{-1}u - 2l\gamma)})S_\gamma(\pi - \sqrt{-1}u - 2l\gamma + \gamma) = S_\gamma(\pi - \sqrt{-1}u - 2l\gamma - \gamma)$$

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$$\begin{aligned}\prod_{l=1}^{k}(1 - e^{(N-l)\xi/N}) &= \prod_{l=1}^{k} \frac{S_\gamma(\pi - \sqrt{-1}u - (2l+1)\gamma))}{S_\gamma(\pi - \sqrt{-1}u - (2l-1)\gamma)} \\ &= \frac{S_\gamma(\pi - \sqrt{-1}u - (2k+1)\gamma)}{S_\gamma(\pi - \sqrt{-1}u - \gamma)}.\end{aligned}$$

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We also have

$$\prod_{l=1}^{k}(1-e^{(N+l)\xi/N}) = \frac{S_\gamma(\pi-\sqrt{-1}u+\gamma)}{S_\gamma(\pi-\sqrt{-1}u+(2k+1)\gamma)}.$$

Asymptotic behavior of CJ for , III

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$J_N($  $;\exp(\xi/N))$

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Putting $g_N(w):=e^{-Nuw}\frac{S_\gamma(\pi-\sqrt{-1}u+\sqrt{-1}\xi w)}{S_\gamma(-\pi-\sqrt{-1}u-\sqrt{-1}\xi w)}$, $J_N($  $;\exp(\xi/N))$ is equal

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Asymptotic behavior of CJ for , III

$$J_N(\,;\, \exp(\xi/N))$$

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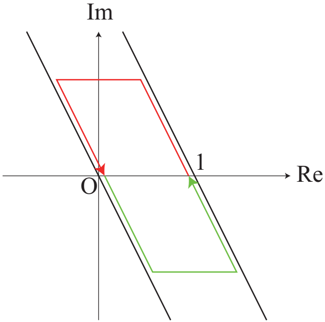
Putting $g_N(w) := e^{-Nuw} \frac{S_\gamma(\pi - \sqrt{-1}u + \sqrt{-1}\xi w)}{S_\gamma(-\pi - \sqrt{-1}u - \sqrt{-1}\xi w)}$, $J_N(\,;\, \exp(\xi/N))$ is equal to $\frac{S_\gamma(-\pi - \sqrt{-1}u + \gamma)}{S_\gamma(\pi - \sqrt{-1}u - \gamma)} \frac{\sqrt{-1}Ne^{u/N}}{2} \int_P \tan(N\pi w) g_N(w)\, dw,$

from the residue theorem, where P is the dotted parallelogram. Note that g_N is defined between the two parallel lines and that the poles of $\tan(N\pi w)$ inside P are $\frac{1}{2N}, \frac{3}{2N}, \dots, \frac{2N-1}{2N}$.



Asymptotic behavior of CJ for , IV

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Since

$$\tan(x+\sqrt{-1}y)=\frac{1}{\sqrt{-1}}\frac{e^{\sqrt{-1}x-y}-e^{-\sqrt{-1}x+y}}{e^{\sqrt{-1}x-y}+e^{-\sqrt{-1}x+y}}\sim\begin{cases}\sqrt{-1} & (y\to\infty),\\ -\sqrt{-1} & (y\to-\infty),\end{cases}$$

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$$G_{\pm,N} := \int_{P_\pm} \tan(N\pi w) g_N(w)\, dw \underset{N\to\infty}{\sim} \pm\sqrt{-1}\int_{P_\pm} g_N(w)\, dw$$

where P_+ (P_-, resp.) is the upper (lower, resp.) half parallelogram.

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Now we have

$$g_N(w)\underset{N\to\infty}{\sim}\exp(N\Phi(w))$$

with

$$\Phi(w):=\frac{1}{\xi}\left(\mathrm{Li}_2(e^{u-\xi w})-\mathrm{Li}_2(e^{u+\xi w})-uw\right).$$

Asymptotic behavior of CJ for , V

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Therefore

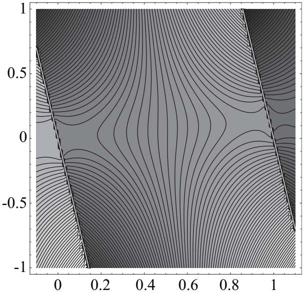
$$G_{\pm,N} \underset{N\to\infty}{\sim} \pm\sqrt{-1}\int_{P_\pm} \exp(N\Phi(w))\,dw$$

and

$$G_{+,N} + G_{-,N} = -2\sqrt{-1}\int_{P_-} \exp(N\Phi(w))\,dw$$

since $\Phi(w)$ is analytic.

Contour plot of Φ



Asymptotic behavior of CJ for , VI

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Theorem (Saddle Point Method)

Assume that

1. $d\,h(z_0)/dz = 0$ *and* $d^2\,h(z_0)/dz^2 \neq 0$.
2. $\operatorname{Im} h(z)$ *is constant for* z *in some neighborhood of* z_0.
3. $\operatorname{Re} h(z)$ *takes its strict maximum along* Γ *at* z_0.

Then

$$\int_\Gamma \exp\bigl(Nh(z)\bigr)\,dz \underset{N\to\infty}{\sim} \frac{\sqrt{2\pi}\exp\bigl(Nh(z_0)\bigr)}{\sqrt{N}\sqrt{-d^2\,h(z_0)/dz^2}}.$$

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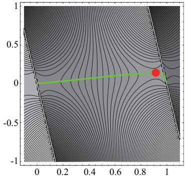
Saddle point w_0 of Φ



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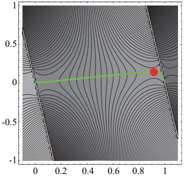

$$\int_{P_-} \exp(N\Phi(w))\,dw$$

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Asymptotic behavior of CJ for , VII

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Theorem (Yokota & HM (2007), HM (2011))

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- $S(u) := \mathrm{Li}_2(e^{u-\varphi(u)}) - \mathrm{Li}_2(e^{u+\varphi(u)}) - u\varphi(u)$. ($\mathrm{Li}_2$: dilogarithm)
- $\varphi(u) := \operatorname{arccosh}(\cosh(u) - 1/2)$.
- $\tau(u) := \sqrt{\dfrac{2}{\sqrt{(e^u+e^{-u}+1)(e^u+e^{-u}-3)}}}$.

Calculation of cs_M for $S^3 \setminus$ , I

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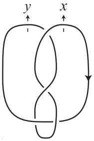

Calculation of cs_M for $S^3 \setminus$ , I

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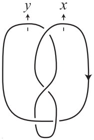
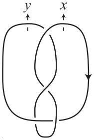


Calculation of cs_M for $S^3 \setminus$, I



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$$\rho\colon \pi_1(S^3 \setminus \;) \to \mathrm{SL}(2;\mathbb{C}),$$
$$x \mapsto \begin{pmatrix} e^{u/2} & 1 \\ 0 & e^{-u/2} \end{pmatrix}, y \mapsto \begin{pmatrix} e^{u/2} & 0 \\ -d(u) & e^{-u/2} \end{pmatrix}$$
$$\left(d(u) := \cosh u - \tfrac{3}{2} + \tfrac{1}{2}\sqrt{(2\cosh u + 1)(2\cosh u - 3)}\right).$$

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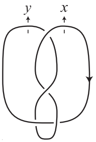

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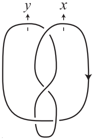

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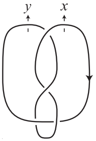
- Put $\tau := \operatorname{arccosh}(3/2)$, so that $d(\tau) = 0$.

Calculation of cs_M for $S^3 \setminus$, I



$$\pi_1(S^3 \setminus \ ) = \langle x, y \mid xy^{-1}x^{-1}yx = yxy^{-1}x^{-1}y \rangle,$$

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$$x \mapsto \begin{pmatrix} e^{u/2} & 1 \\ 0 & e^{-u/2} \end{pmatrix}, y \mapsto \begin{pmatrix} e^{u/2} & 0 \\ -d(u) & e^{-u/2} \end{pmatrix}$$

$$(d(u) := \cosh u - \tfrac{3}{2} + \tfrac{1}{2}\sqrt{(2\cosh u + 1)(2\cosh u - 3)}).$$

We will calculate $\mathrm{cs}_M([\rho])$.

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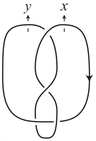

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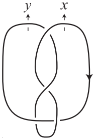

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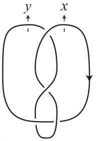

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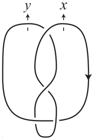

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Therefore

$$\mathrm{CS}_{u,v(u)}([\rho]) = S(u) - u\pi\sqrt{-1} - \frac{1}{4}uv(u).$$

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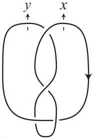

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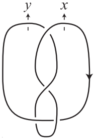

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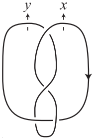

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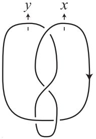

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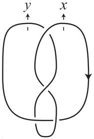

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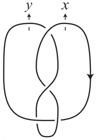

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We will calculate $\mathbb{T}(\rho)$.

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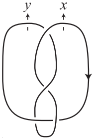

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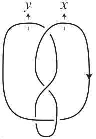

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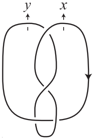

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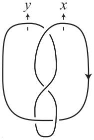

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- $\frac{\partial r}{\partial x} = 1 - xy^{-1}x^{-1} + xy^{-1}x^{-1}y + xy^{-1}x^{-1}yxy^{-1} - xy^{-1}x^{-1}yxy^{-1}xyx^{-1}$,
  $\frac{\partial r}{\partial y} = -xy^{-1} + xy^{-1}x^{-1} - xy^{-1}x^{-1}yxy^{-1} + xy^{-1}x^{-1}yxy^{-1}x - xy^{-1}x^{-1}yxy^{-1}xyx^{-1}y^{-1}$.

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- ∂_1 is given by the 3×6 matrix $\big((X - I_3) \quad (Y - I_3)\big)$.
- $X = \begin{pmatrix} e^{-u} & 2e^{-u/2} & -1 \\ 0 & 1 & -e^{u/2} \\ 0 & 0 & e^{u} \end{pmatrix}$,

  $Y = \begin{pmatrix} e^{-u} & 0 & 0 \\ -e^{-u/2}d(u) & 1 & 0 \\ -d(u)^2 & 2e^{u/2}d(u) & e^{u} \end{pmatrix}$.

$\Rightarrow H_2 = H_1 = \mathbb{C}$ and $H_0 = \{0\}$. So $\dim B_2 = 2$, and $\dim B_1 = 3$

Calculation of $\mathbb{T}$ for $S^3 \setminus$ , III

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- $\tilde{\vec{h}}_2 = \left\{ \begin{pmatrix} 2e^{u/2}(d(u) - e^u + 1) \\ -2d(u) \\ 2e^{-u/2}(e^u - 1)d(u) \end{pmatrix} \right\} = \left\{ [\partial \left(S^3 \setminus \operatorname{Int} N(\ ) \right) \otimes P] \right\}$,

  where $[\partial \left(S^3 \setminus \operatorname{Int} N(\ ) \right)]$ is the fundamental class, and P is invariant under the adjoint actions of the meridian x and the longitude.

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Calculation of $\mathbb{T}$ for $S^3 \setminus$ [figure-eight knot symbol], III

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- *both associated with the holonomy representation (that defines the complete hyperbolic structure for* $S^3 \setminus E$*).*

Similar results holds for hyperboli knots up to six crossings (T. Ohtsuki, Ohtsuki & Y. Yokota). $\Rightarrow$ Refinement of VC (S. Gukov & H. Murakami, Ohtsuki).

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- $\Delta(K;t)$: Alexander polynomial,
- $\{\rho_k\}$: the set of irreducible representations of π_1 to $\mathrm{SL}(2;\mathbb{C})$,
- $S_{\rho_k}(\xi)$: defines the Chern–Simons invariant of ρ_k associated with the meridian and the longitude,
- $\tau_{\rho_k}^{-2}$: Reidemeister torsion of ρ_k associated with the meridian.

Speculation

Speculation

Speculation

K: any knot, $\xi = 2\pi\sqrt{-1} + u$.

$$J_N(K;\exp(\xi/N)) \underset{N\to\infty}{\sim} \frac{1}{\Delta(K;\exp(\xi))} + \frac{c_k}{2\sinh(\xi/2)} \sum_{\rho_k} \tau_k(u) \left(\frac{N}{\xi}\right)^{d_k/2} \exp\left(\frac{N}{\xi} S_k(u)\right),$$

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