Gonality and Genus of Character Varieties

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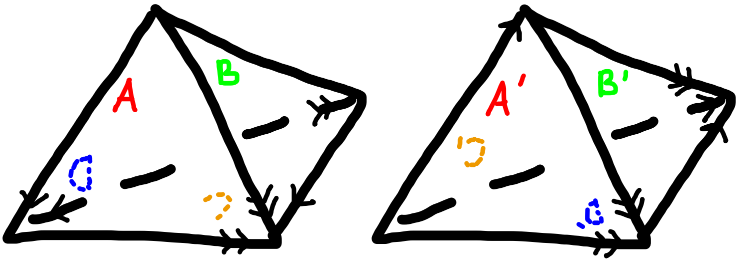
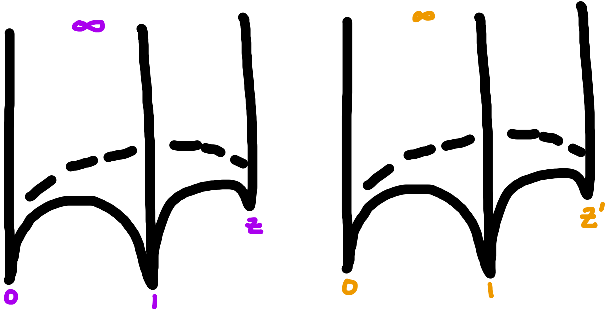
Gluing Varieties



The figure-8 knot complement can be realized as the identification of two (truncated) tetrahedra.

Gluing Varieties

To give these a hyperbolic structure, we consider them as truncated ideal tetrahedra.



In the hyperbolic upper half-plane the shape of the tetrahedron is determined by a complex number (z or z').

Gluing Varieties

The gluing conditions can be expressed algebraically as

$$z(z-1)z'(z'-1)=1$$

This defines a rational curve in $\mathbb{C}^2$, the *gluing variety*.

A point on the gluing variety corresponds to a hyperbolic structure on the figure-8 knot complement.

Character Varieties

Let M be a finite volume hyperbolic 3-manifold.

The character of any representation $\rho : \pi_1(M) \to (\mathrm{P})\mathrm{SL}_2(\mathbb{C})$ is the function $\chi_\rho : \pi_1(M) \to \mathbb{C}$ given by

$$\chi_\rho(\gamma) = \mathrm{trace}(\rho(\gamma)).$$

The character variety

$$X(M) = \{\chi_\rho \mid \rho : \pi_1(M) \to \mathrm{SL}_2(\mathbb{C})\}$$

and is a $\mathbb{C}$-algebraic set defined over $\mathbb{Q}$. It is defined by a finite number of characters.

Character Varieties

By Mostow-Prasad rigidity, $\mathbb{H}^3/\Gamma_1$ is isometric to $\mathbb{H}^3/\Gamma_2$ if and only if Γ_1 is conjugate to Γ_2.

Reducible representations are those $\rho : \pi_1(M) \to \mathrm{SL}_2(\mathbb{C})$ such that up to conjugation

$$\rho(\pi_1(M)) \subset \begin{pmatrix} \star & \star \\ 0 & \star \end{pmatrix}$$

For these, many non-conjugate representations have the same character.

For irreducible representations, each character uniquely corresponds to a representation (up to conjugation)

Character Varieties

In the $PSL_2(\mathbb{C})$ case there are just two characters of discrete and faithful representations – both orientations.

For $SL_2(\mathbb{C})$ there may be multiple lifts of each.

Points in $X(M)$ corresponds to (usually incomplete) hyperbolic structures on M.

This is a finite-to-one correspondence (with lifting and different orientations).

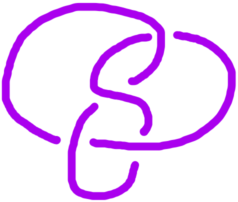
Character Varieties

A component of $X(M)$ is called a canonical component and written $X_0(M)$ if it contains the character of a discrete and faithful representation.

Thurston:

$$\dim_{\mathbb{C}} X_0(M) = \dim_{\mathbb{C}} Y_0(M) = \text{number of cusps of } M$$

Figure-8 knot complement



$$\pi_1 \cong \langle \alpha, \beta : w\alpha = \beta w, w = \alpha^{-1}\beta\alpha\beta^{-1} \rangle$$

X is determined by $x = \chi_\rho(\alpha) = \chi_\rho(\beta)$ and $r = \chi_\rho(\alpha\beta^{-1})$.

Reducible representations are those with $r = 2$.

Up to conjugation an irreducible representation is:

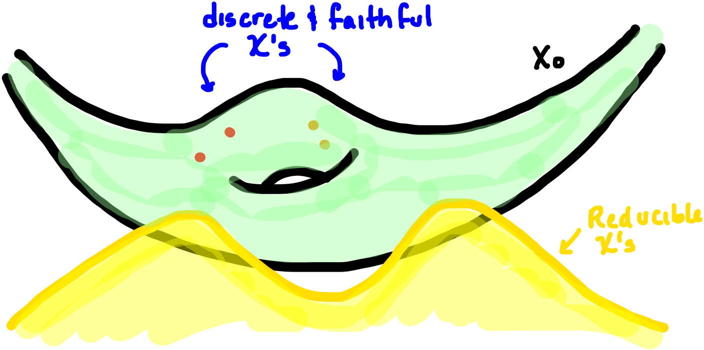
$$\rho(\alpha) = \begin{pmatrix} a & 1 \\ 0 & a^{-1} \end{pmatrix} \qquad \rho(\beta) = \begin{pmatrix} a & 0 \\ 2-r & a^{-1} \end{pmatrix}$$

with $x = a + a^{-1}$.
The relation:

$$\rho(\alpha)\rho(w) - \rho(w)\rho(\beta) = \begin{pmatrix} 0 & \star \\ (r-2)\star & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

where

$$\begin{aligned} \star &= (a^2 + a^{-2})(1-r) + 1 - r + r^2 \\ &= (x^2 - 2)(1-r) + 1 - r + r^2 \\ &= x^2(r-1) - 1 + r - r^2 \end{aligned}$$



X_0 is the vanishing set of the equation $\star = 0$ which is

$$z^2 = r^3 - 2r + 1$$

with the substitution $z = x(r-1)$.

The $\mathrm{PSL}_2(\mathbb{C})$ variety is given by the variables $y = \mathrm{trace}(\rho(\alpha))^2$ and r:

$$Y_0 : y(1-r) + 1 - r + r^2 = 0$$

Propaganda: Why character varieties are great

Key tools in the proof of the cyclic surgery theorem (Culler-Gordon-Luecke-Shalen) and the finite surgery theorem (Boyer-Zhang)

Culler-Shalen used group actions on trees to show that you can 'detect' many surfaces in 3-manifolds by valuations at ideal points of $X(M)$.

Connection to conjectures like the volume conjecture and AJ conjecture through the A-polynomial which is 'almost the same set'.

Character Variety Structural Theorems

Question

How is the topology of M reflected in $X(M)$ what does the geometry of $X(M)$ inform us about the topology of M?

Boyer-Luft-Zhang, Ohtsuki-Riley-Sakuma: For any n, there is a (one cusped finite volume hyperbolic) 3-manifold M such that $X(M)$ has more than n components.

Culler-Shalen: If M is a small knot complement, all components have dimension 1.

For any n there is a (twist) knot complement M such that genus$(X_0(M)) > n$. (Follows from Macasieb-P-van Luijk)

Consider the 'easiest' case, where M is a finite volume hyperbolic 3-manifold with just one cusp. Then $X_0(M)$ and $Y_0(M)$ are $\mathbb{C}$-curves, Riemann Surfaces.

Question

Can every (isomorphism class of smooth projective) curve defined over $\mathbb{Q}$ be (birational to) a character variety?

Consider the 'easiest' case, where M is a finite volume hyperbolic 3-manifold with just one cusp. Then $X_0(M)$ and $Y_0(M)$ are $\mathbb{C}$-curves, Riemann Surfaces.

Question

Can every (isomorphism class of smooth projective) curve defined over $\mathbb{Q}$ be (birational to) a character variety?

Or perhaps easier questions:

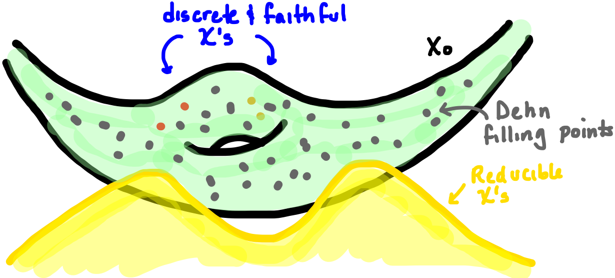
Question

- *What can we say about some of the classical invariants of curves: genus, degree, and* **gonality***?*
- *What can we say about* **families** *of one-cusped manifolds?*

Dehn Filling

Let M be a cusped manifold and $M(r)$ the result of $r = \frac{p}{q}$ filling of a fixed cusp of M. By van Kampen's theorem, $\pi_1(M(r))$ is $\pi_1(M)$ with the extra relation that $m^p l^q = 1$.

Since $\pi_1(M) \twoheadrightarrow \pi_1(M(r))$ we get $X(M(r)) \subset X(M)$



Thurston: If M is (finite volume and) hyperbolic then fixing a cusp to fill, for all but finitely many slopes r, so is $M(r)$.

Gonality Primer

The gonality is

$$\gamma(C) = \min_{\text{degree}(\varphi)} \{\varphi : C \to \mathbb{C} \text{ is a rational map to a dense subset of } C\}$$

Example: The hyperelliptic curve given by $y^2 = f(x)$ (with f separable and degree(f) > 2) has

- genus $\lfloor \frac{\text{degree}(f)-1}{2} \rfloor$.
- gonality 2.

The replacement $w = y^2$ (that is $y \mapsto y^2$) determines the curve $w = f(x)$. The gonality of $w = f(x)$ is one by the map $(x, f(x)) \to x$.

Gonality Primer

There are curves of fixed gonality and arbitrary genus.

The Brill-Noether bound relates the two

$$\text{gonality} \leq \lfloor \frac{\text{genus} + 3}{2} \rfloor.$$

(You can explicitly contract a projection of degree $\lfloor \frac{\text{genus}+3}{2} \rfloor$ to $\mathbb{C}$.)

For a non-singular curve in $\mathbb{P}^2$ of degree ≥ 2, gonality, genus, and degree are all related.

Noether:

$$\text{gonality} = \text{degree} - 1$$

Genus degree formula :

$$\text{genus} = \frac{1}{2}(\text{degree} - 1)(\text{degree} - 2)$$

‘Key Lemma’

Lemma (Gonality Lemma)

Let $g : X \to Y$ be a dominant rational map of projective curves. Then

$$gonality(Y) \leq gonality(X) \leq degree(g) \cdot gonality(Y).$$

If $\varphi : Y \to \mathbb{P}^1$ is a map realizing the gonality then

$$\begin{array}{ccc} X & \xrightarrow{g} & Y \\ & \searrow & \downarrow \varphi \\ & & \mathbb{P}^1 \end{array}$$

the map $\varphi \circ g$ gives one inequality. The other inequality follows by looking at degree as a field extension and showing that the degree of extension defining the gonality of X must be realized by a degree

Results

The *height* of $\frac{p}{q} \in \mathbb{Q} \cup \infty$ (in lowest terms) is $h(\frac{p}{q}) = \max\{|p|, |q|\}$ if $pq \neq 0$ and $h(0) = h(\infty) = 1$.

Theorem (P-Reid)

Let M be a hyperbolic two cusped manifold, and $M(r)$ be a hyperbolic Dehn filling of M. There is a constant c depending only on M and the framing of the filled cusp such that

$$\textit{gonality}\Big(X_0(M(r))\Big) \leq c.$$

Corollary

1) $genus\Big(X_0(M(r))\Big) \leq c \cdot h(r)$
2) $degree\Big(A_0(M(r))\Big) \leq c \cdot h(r)^2$

From the Character Variety to the A-polynomial

We look at the image of X_0 in the A-polynomial variety, A_0.

For a two-cusped M, $A_0(M) \subset \mathbb{C}^4(m_1, l_1, m_2, l_2)$ is a surface where the coordinates m_i and l_i correspond to the meridianal and longitudinal parameters of the i^{th} cusp.

That is, they correspond to to

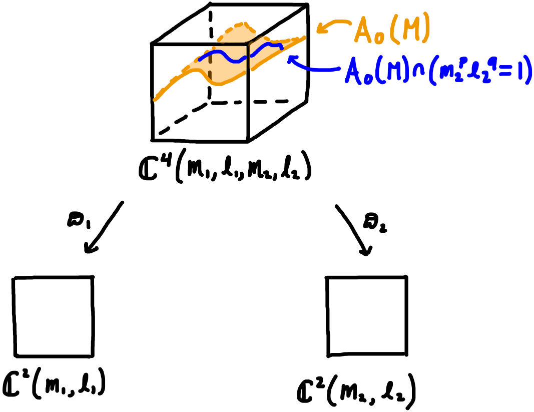
$$\mu_i \mapsto \begin{pmatrix} m_i & * \\ 0 & m_i^{-1} \end{pmatrix}, \quad \lambda_i \mapsto \begin{pmatrix} l_i & * \\ 0 & l_i^{-1} \end{pmatrix}.$$

For $M(\frac{p}{q})$ (filling of the second cusp) $A_0(M(\frac{p}{q}))$ is a curve in $\mathbb{C}(m_1, l_1)$.

Dunfield: $X_0(M(\frac{p}{q})) \to A_0(M(\frac{p}{q}))$ has finite degree depending only on M.

It suffices to bound gonality of $A_0(M(\frac{p}{q}))$.

The Diagram



Key Observation: $A_0(M(\frac{p}{q})) \subset \varpi_1(A_0(M) \cap (m_2^{\pm p} l_2^q = 1))$.

We will look at the projection maps ϖ_1 and ϖ_2.

Geometric Isolation Interlude

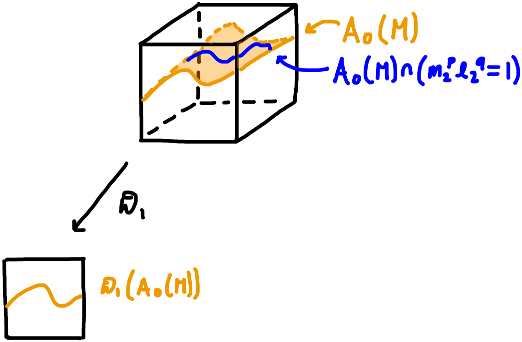
The first cusp of M is geometrically isolated from the second cusp if any deformation induced by Dehn filling of the second cusp while keeping the first cusp complete does not change the Euclidean structure of the first cusp.

They are strongly geometrically isolated if integral Dehn filling of the first cusp and replacing the cusp by a geodesic and then deforming the second cusp does not change the geometry of the geodesic.

If M is a two cusped hyperbolic 3-manifold... the first cusp of M is geometrically isolated from the second cusp of M if and only if $\varpi_1(A_0(M))$ is a curve.

The cusps are strongly geometrically isolated if and only if $A_0(M) = C_1 \times C_2$ where C_i is a curve in $\mathbb{A}^2(m_i, l_i)$.

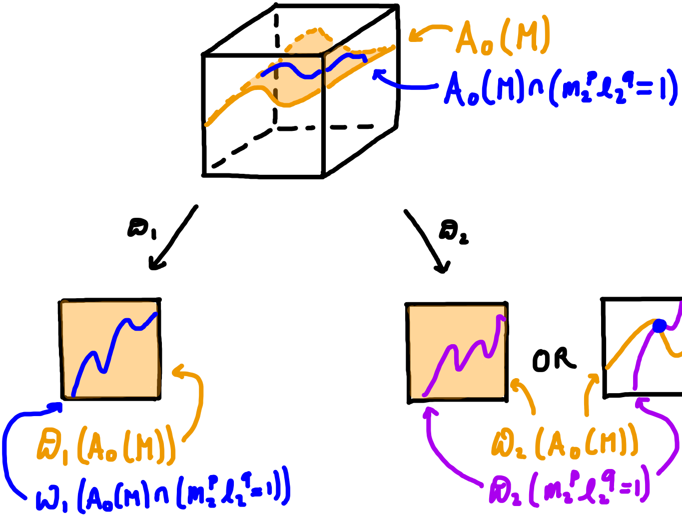
If the first cusp is isolated from the second cusp



The projection $\varpi_1(A_0(M))$ is a curve.

Since $A_0(M(\frac{p}{q})) \subset \varpi_1(A_0(M))$ and is also a curve, all these Dehn fillings give the same A-polynomial – therefore the gonality is bounded.

The generic case - $\varpi_1(A_0(M))$ is dense in $\mathbb{C}$.



The generic case - $\varpi_1(A_0(M))$ is dense in $\mathbb{C}$.

Hironaka resolution, Stein Factorization: the degree d of ϖ_1 is finite.

ϖ_1 looks like birational maps composed with a 'finite' map.

The finite map is defined everywhere and is d-to-1 (or less) on all points. Birational maps may have infinite degree or not be defined on a subset of curves - but otherwise are one-to-one.

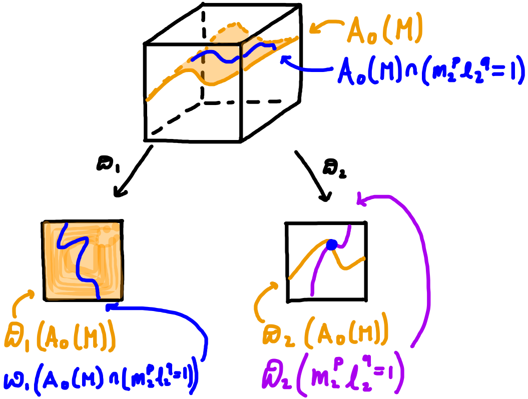
Claim: It suffices to bound the gonality of (all curve components of) $A_0(M) \cap (m_2^p l_2^{\pm q} = 1)$. (Generically this is a union of curves.)

Since

$$A_0(M(\tfrac{p}{q})) \subset \varpi_1(A_0(M) \cap (m_2^p l_2^{\pm q} = 1))$$

it follows since if $A_0(M(\frac{p}{q}))$ isn't one of the 'bad' curves then the degree of ϖ_1 restricted to $\varpi_1(A_0(M) \cap (m_2^p l_2^{\pm q} = 1))$ is bounded by d, independent of p/q.

The first cusp is not isolated from the second, but the second is isolated from the first



If the second cusp is geometrically isolated from the first cusp, then $\varpi_2(A_0(M))$ is a curve.

$\varpi_2(A_0(M) \cap (m_2^p l_2^q = 1))$ is a collection of points

Any irreducible component of $A_0(M) \cap (m_2^p l_2^q = 1)$ is

$$(curve) \times (a_r, b_r)$$

If $A_0(M)$ is cut out by polynomials $\varphi_i(m_1, l_1, m_2, l_2)$ then $A_0(M) \cap (m_2^p l_2^q = 1)$ is cut out by the polynomials $\varphi_i(m_1, l_1, a_r, b_r)$.

$\Rightarrow$ **an upper bound on the degree is independent of a_r and b_r.**

The degree is bounded independent of r.

Genus-Degree theorem

$$genus \leq \frac{1}{2}(degree - 1)(degree - 2)$$

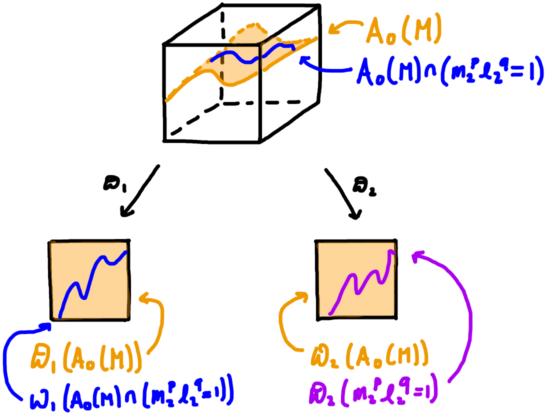
$\Rightarrow$ genus is bounded.

Brill-Noether bound

$$gonality \leq \lfloor \frac{1}{2}(genus + 3) \rfloor$$

$\Rightarrow$ gonality is bounded.

No Geometric Isolation



It suffices to bound the gonality of any curve component of $A_0(M) \cap (m_2^p l_2^{\pm q} = 1)$.

ϖ_2 is the composition of birational maps and a finite map.

If we avoid finitely many curves, using the key lemma, it suffices to bound the gonality of the image $\varpi_2\Big(A_0(M) \cap (m_2^p l_2^{\pm q} = 1)\Big)$.

This is the curves given by $m_2^p l_2^{\pm q} = 1$.

Gonality of $x^p y^q = 1$

Since p and q are relatively prime, there are a and b such that

$$ap + bq = 1.$$

The gonality of $x^p y^q = 1$ is equal to the gonality of its image under the birational map

$$(x, y) \mapsto (x^b y^{-a}, x^p y^q) = (x', y').$$

The image is dense in $y' = 1$ since

$$y' = x^p y^q = 1$$

Core Curve

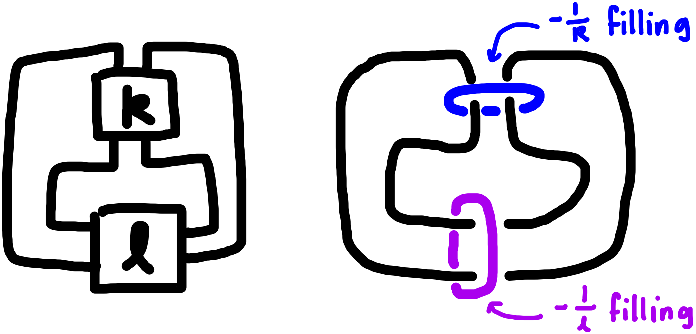
The core curve of the filled second cusp is $\gamma = \mu_2^{-b}\lambda_2^a$, with $ap + qb = 1$.

We parametrize so that $T^{-q} = m_2$ and $T^p = l_2$. (T is basically the distinguished eigenvalue corresponding to $\rho(\gamma)$.)

Our mapping to control the gonality is

$$(m_1, l_1, m_2, l_2) \xrightarrow{\varpi_2} (m_2, l_2) = (T^{-q}, T^p) \longrightarrow (m_2^b l_2^{-a}, m_2^p l_2^q) = (T^{-1}, 1)$$

Example: Double Twist Knots



Character varieties, and their genera, for these knot complements were computed by Macasieb-P-van Luijk.

Example: Double Twist Knots

Let $m = \lfloor \frac{|k|}{2} \rfloor$ and $n = \lfloor \frac{|l|}{2} \rfloor$.

If $k \neq l$:

$$\text{genus}\big(X_0(k,l)\big) = 3mn - m - n - b$$

$$\text{gonality}\big(X_0(k,l)\big) = 2\min\{m,n\}$$

$\Rightarrow$ There are hyperbolic 3-manifolds such that X_0 has arbitrarily large gonality.

If $k = l$:

$$\text{genus}\big(X_0(k,k)\big) = n - 1$$

$$\text{gonality}\big(X_0(k,l)\big) = 2$$

More Consequences

Assume the genus of a Riemann surface is ≥ 2.

Li-Yau : Let λ_1 be the first non-zero eigenvalue of the Laplacian:

$$\lambda_1 \leq \frac{\text{gonality}}{\text{genus} - 1}$$

Hwang-To:

$$\text{injectivity radius} \leq 2\cosh^{-1}(\text{gonality})$$

$$\lambda_1(X_0(k,l)) \leq \frac{2\min\{m,n\}}{3mn - m - n - b}$$

$\Rightarrow$ as $|k|$ or $|l| \to \infty$, $\lambda_1 \to 0$.

(For $n, m > 6$)

$$\text{injectivity radius of } X_0(k,l) \leq 2\cosh^{-1}(\min\{m,n\})$$