Dynamics on free-by-cyclic groups.

Chris Leininger (UIUC)

joint with S. Dowdall and I. Kapovich

August 15, 2013

Outline 1/17

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Motivation from fibered hyperbolic 3–manifolds.

Motivation: Pseudo-Anosov homeomorphisms 2/17

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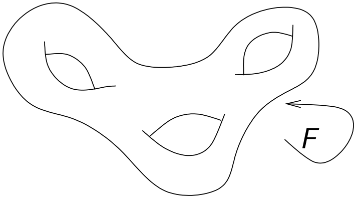

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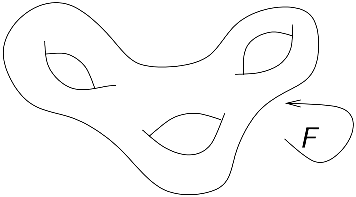

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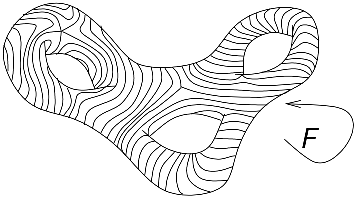

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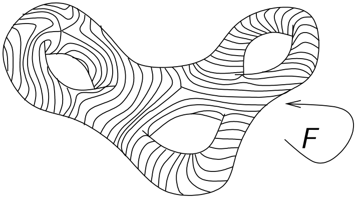

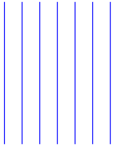

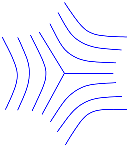

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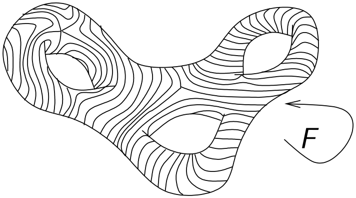

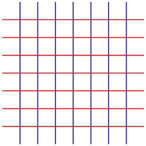

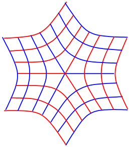

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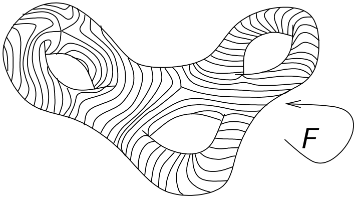

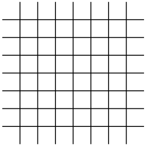

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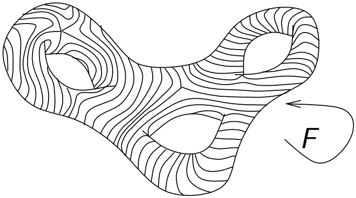

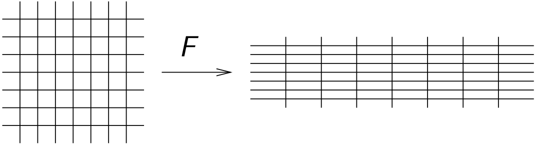

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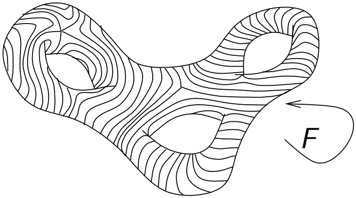

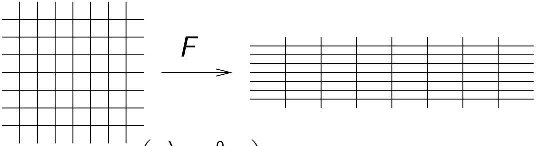

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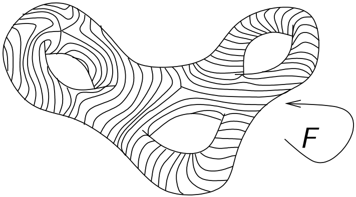

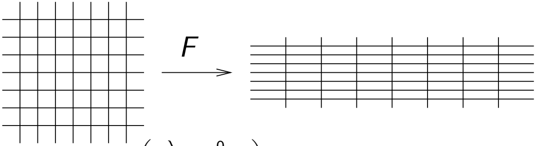

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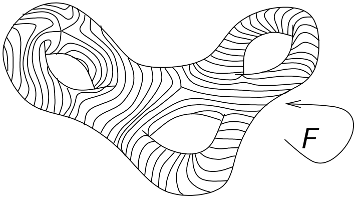

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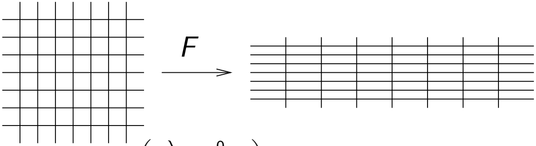

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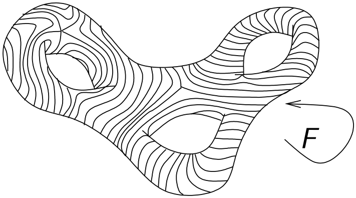

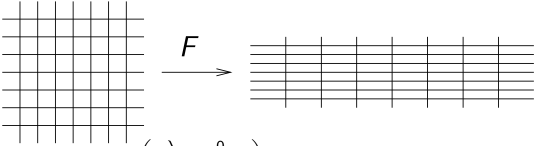

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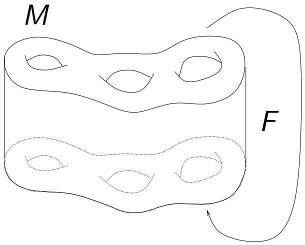

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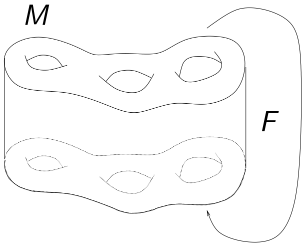

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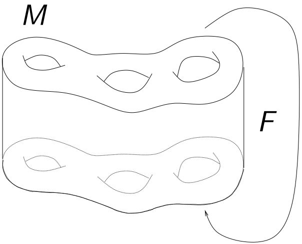

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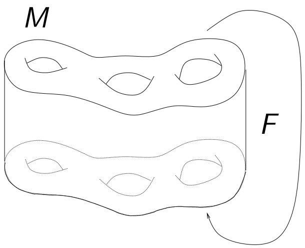

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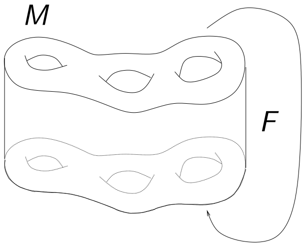

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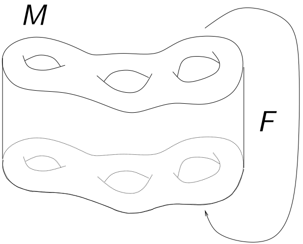

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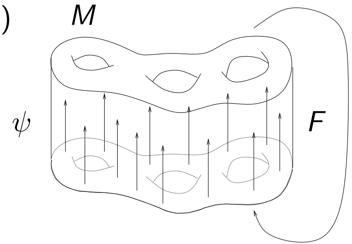

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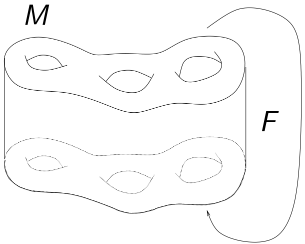

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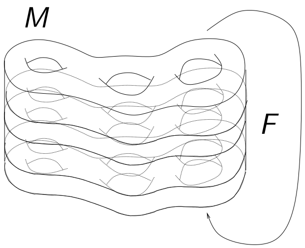

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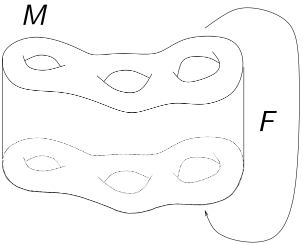

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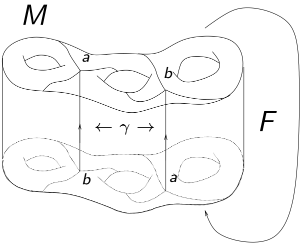

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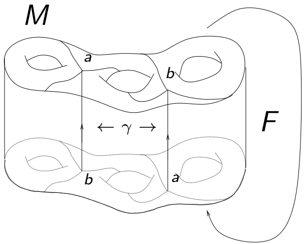

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$$\Rightarrow \qquad \mathrm{PD}(e) = \frac{1}{2} \sum_{\gamma \in \mathcal{O}} (2 - \deg(\gamma))\gamma \in H_1(M)$$

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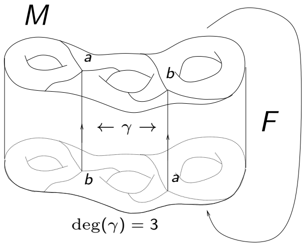

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- $\mathcal{O} = \{\gamma \subset M \mid \gamma$ closed orbit of singularity of $\mathcal{F}_S^{\pm}\}$

$$\Rightarrow \qquad \mathrm{PD}(e) = \frac{1}{2}\sum_{\gamma\in\mathcal{O}}(2-\deg(\gamma))\gamma \in H_1(M)$$

Motivation: Thurston and Fried

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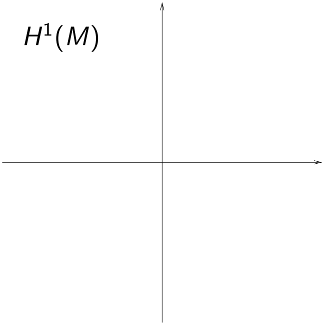

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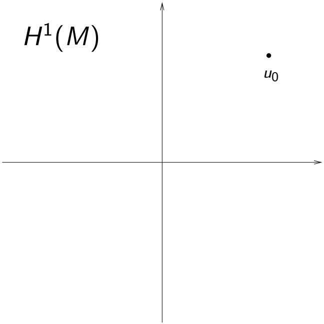

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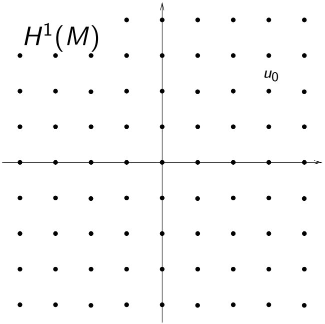

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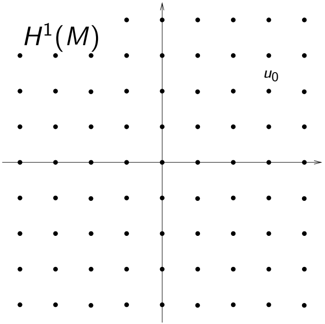

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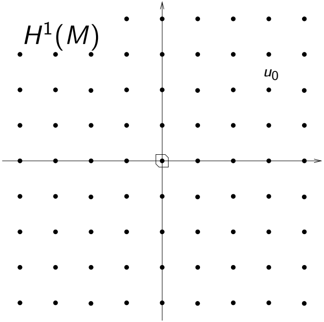

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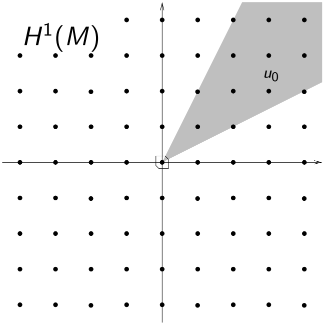

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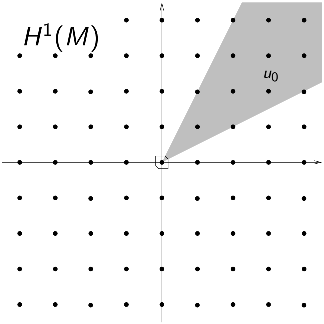

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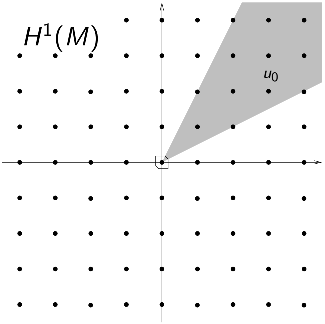

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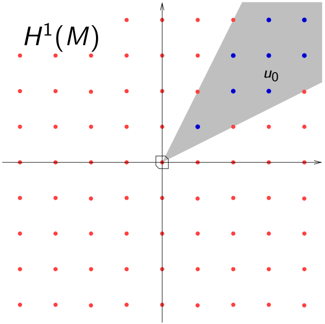

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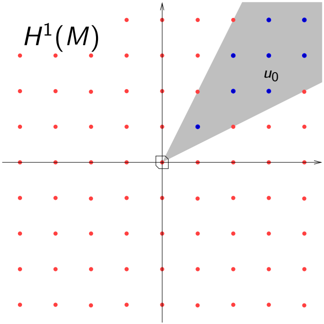

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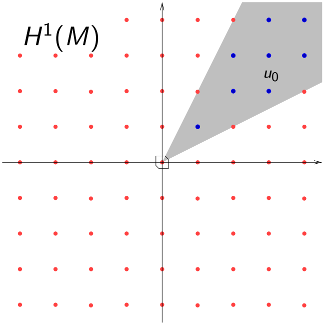

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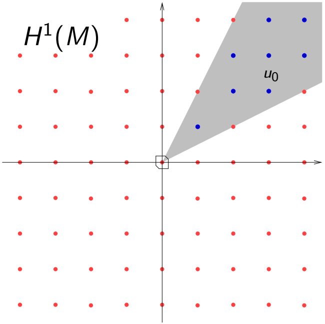

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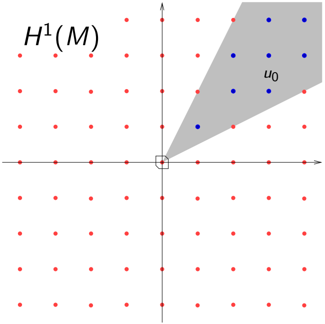

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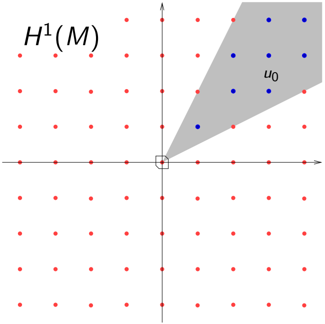

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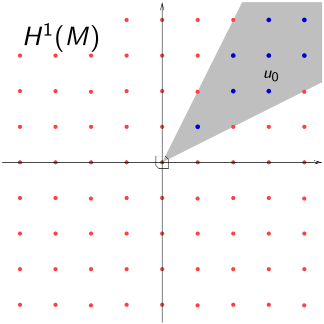

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Motivation: Dilatation asymptotics 5/17

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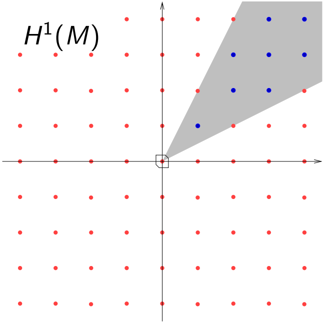

Motivation: Dilatation asymptotics 5/17

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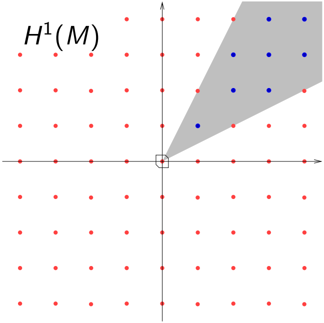

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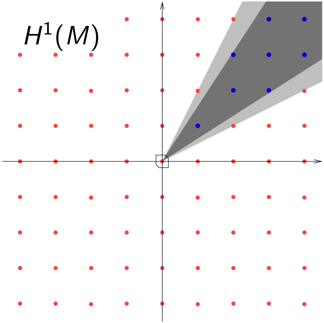

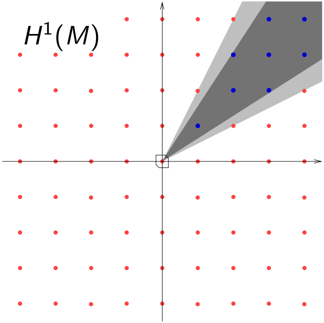

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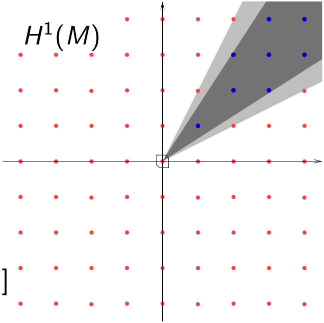

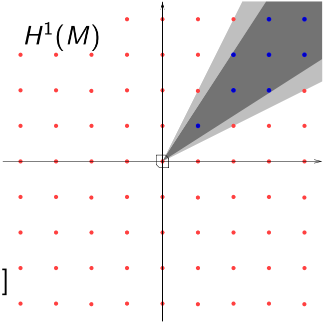

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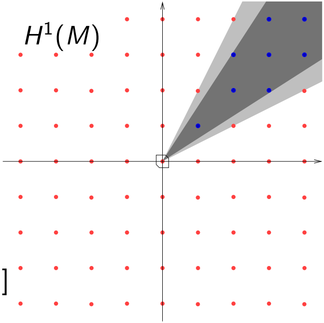

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See also [Agol].

Transition: Group theory 6/17

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Integral $u \in \mathrm{Hom}(\pi_1 M, \mathbb{R}) = H^1(M)$ is induced by a fibration over S^1 if and only if $\ker(u)$ is finitely generated [Stallings]

Atoroidal and fully irreducible 7/17

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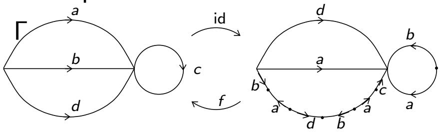

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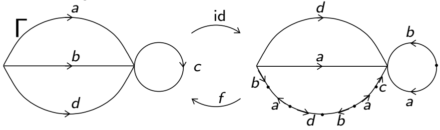

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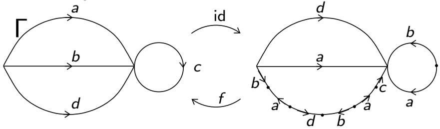

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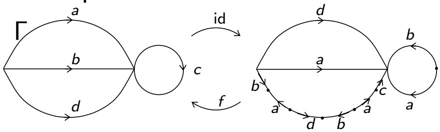

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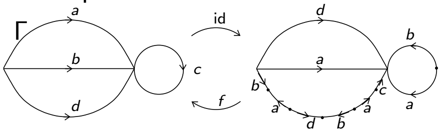

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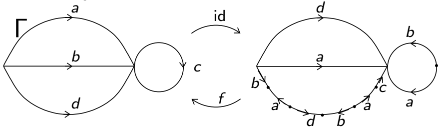

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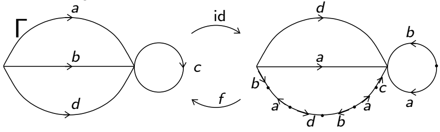

Many other examples [Clay-Pettet]

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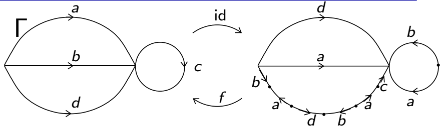
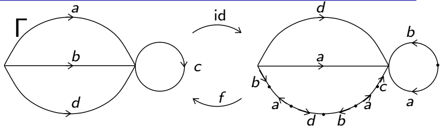
Dynamics and stretch factors 8/17





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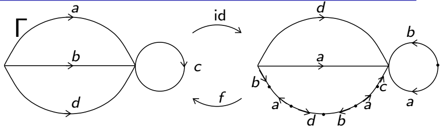

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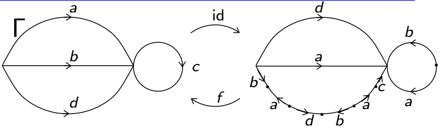

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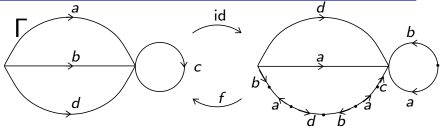

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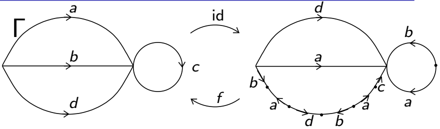

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A model for free-by-cyclic group 9/17

Idea: Dynamics on branched surfaces in 3–manifolds

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Theorem [Dowdall-Kapovich-L] 11/17

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Small stretch factors 13/17

Corollary With the setup as above suppose $K \subset \mathcal{A}$ is compact and

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Theorem [Algom-Kfir–Rafi] All irreducible $\phi \in \mathrm{Out}(\mathbf{F}_N)$ with $\log(\lambda(\phi)) \leq c/N$ (over all $N \geq 2$) are monodromies of "surgeries" on the mapping torus of one of a finite set of graph maps.

Idea of construction and proof. 14/17

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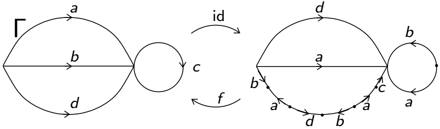

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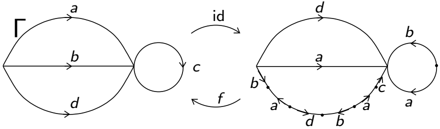

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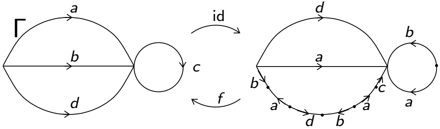

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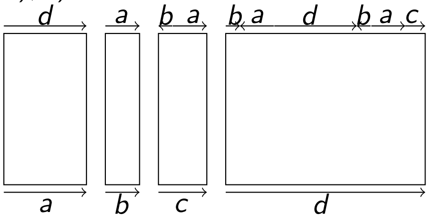

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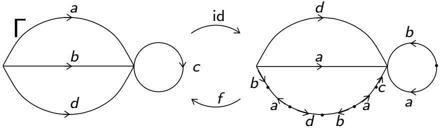

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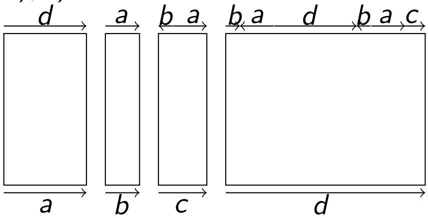

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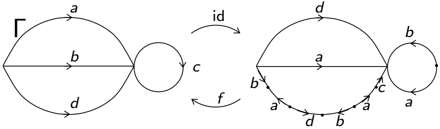

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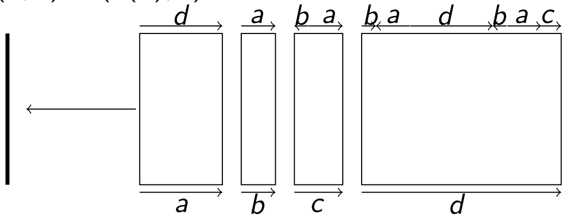

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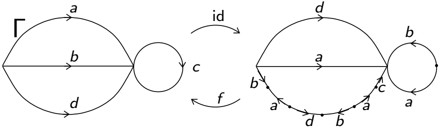

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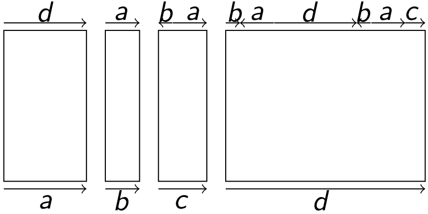

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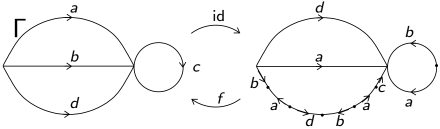

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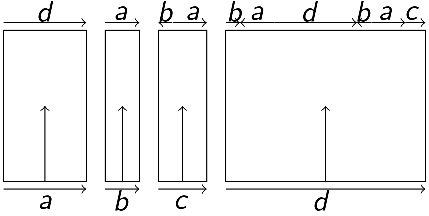

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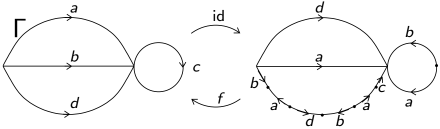

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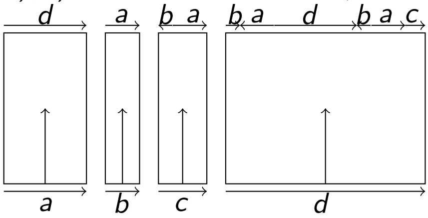

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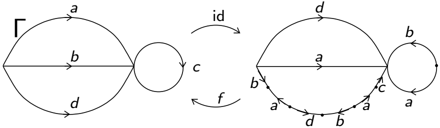

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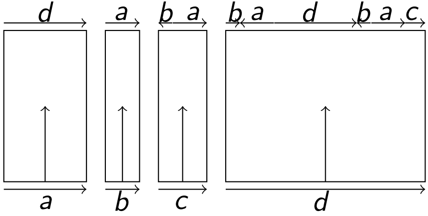

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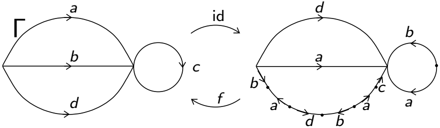

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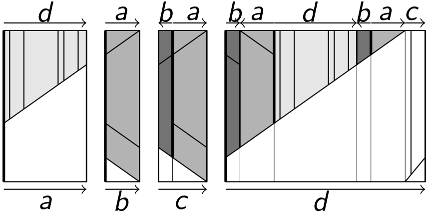

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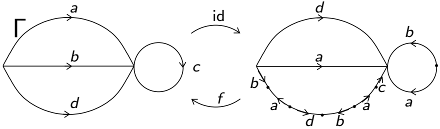

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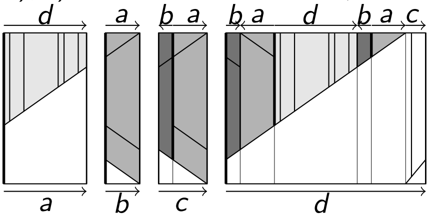

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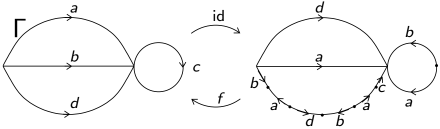

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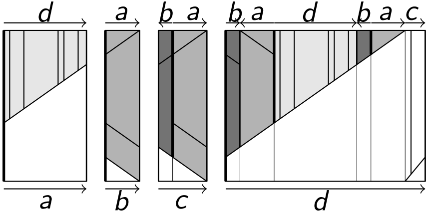

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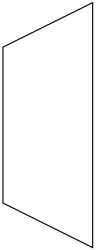

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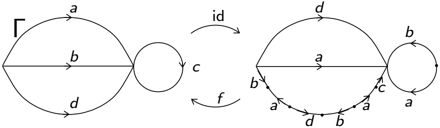

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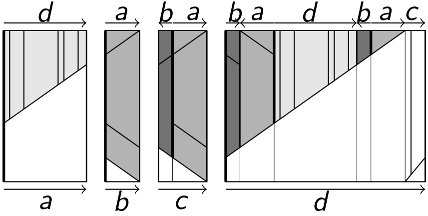

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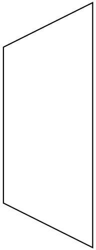

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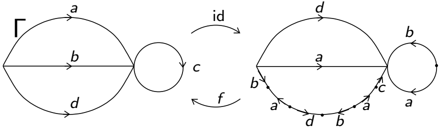

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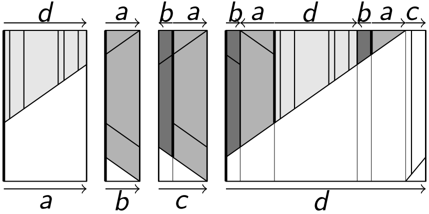

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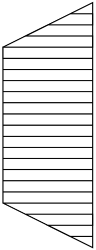

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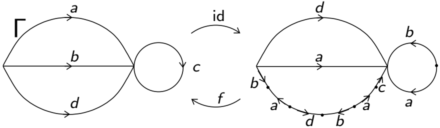

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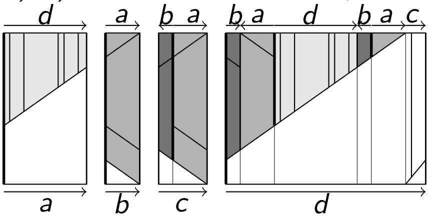

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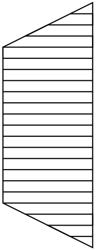

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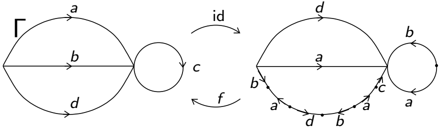

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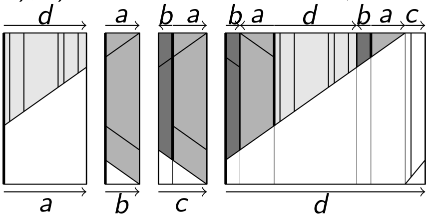

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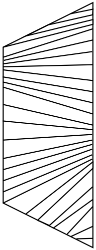

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Train track map 15/17

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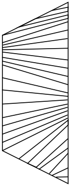

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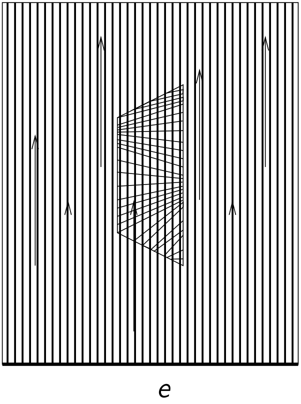

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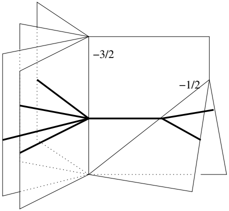

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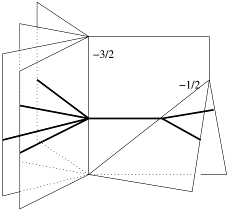

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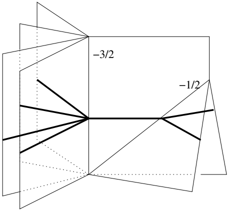

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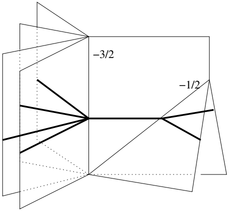

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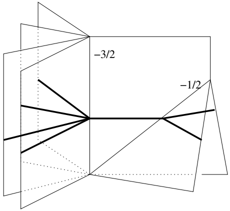

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What's next?... 17/17

Work in progress (w/ Dowdall and Kapovich)

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