The Surface Subgroup Theorem

and

The Ehrenpreis Conjecture

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Joint with Vladimir Markovic
Theorem (KM) (Ehrenpreis Conjecture)

Let $S$ and $T$ be closed hyperbolic Riemann surfaces. Then $S$ and $T$ have nearby finite covers (in the Teichmüller metric).
Theorem (KM) (Ehrenpreis Conjecture)

Suppose $S$ and $T$ are closed hyperbolic Riemann surfaces, and $k > 1$.

Then we can find finite covers $\hat{S}$ and $\hat{T}$ of $S$ and $T$ and a smooth $k$-quasiconformal map $h: \hat{S} \to \hat{T}$.

$\frac{M}{m} \leq K$
Theorem (KM) (Surface subgroup Conjecture)

Let $M$ be a closed hyperbolic 3-manifold. Then $\pi_1 (M)$ has a surface subgroup. (The isomorphic image of $\pi_1$ of a hyperbolic surface.)
References


The good pants homology and a proof of the Ehrenpreis conjecture. (arXiv: 1101.1330).
2D

Model surface

Finite cover

Given surface

Good immersion (Cover)

3D

Model surface

Finite cover

Given surface

Good immersion

Given 3-manifold
Building Covers of the Circle

degree 2
degree 4
We can join immersed 1-manifolds with boundary to form a cover of a closed 1-manifold (a circle).
We can join immersed orientable $n$-manifolds with boundary to form a cover of a closed orientable $n$-manifold.
We can build more than one cover
We can fail to build a cover
Given a collection $C = \sum n_i \partial \chi_i$ of 1-manifolds immersed in a circle, we can assemble $C$ to form a cover if and only if

$$\partial C = \sum n_i \partial \chi_i = 0$$
We can always build a piecewise immersed 1-manifold in a closed 2-manifold if we use "doubling the trick"
A set of points on $s'$ is "evenly distributed to the scale $\varepsilon$" if every $\varepsilon$-interval of $s'$ has the same number of points, up to a factor of $1 + \varepsilon$.

\[
\frac{103}{98} < 1 + \varepsilon
\]
We can make the piecewise immersed 1-manifold nearly geodesic (small bending) if the segments are "evenly distributed" around each meeting point.
A (hyperbolic) pair of pants is a hyperbolic 2-manifold with geodesic boundary that is diffeomorphic to the following.
We obtain a closed hyperbolic surface when we join pairs of pants.
Hyperbolic Pairs of Pants

\[ \text{L}_3 \rightarrow \text{L}_1 \rightarrow \text{L}_2 \rightarrow \text{double} \rightarrow \text{L}_1, \text{L}_2, \text{L}_3 \]

Half-lengths \( L_1, L_2, L_3 \)
Shear Coordinates for Panted Surfaces

\[ s(\gamma) \in \mathbb{R} / hl(\gamma) \]
Perfect Pants and Perfect Surfaces

$h^l(C) = R$

$hl(C) = R$
$s(C) = 1$

Theorem Any two closed perfect surfaces have a common finite cover.
Good Pants and Good Surfaces

\[ |hl(c) - R| < \varepsilon \]

\[ |s(c) - 1| < \frac{\varepsilon}{R} \]

**Theorem (KM)**

A good closed panted surface is close to a perfect surface.
Theorem (KM)
For all $\varepsilon < \varepsilon_0$, and $R > R_0$, any $(\varepsilon, R)$ good surface is $10^{12}\varepsilon$-close to an $R$-perfect surface in the Teichmüller metric.
Adding the effects of the errors

At most 100 shear errors with
error less than \( \frac{\varepsilon}{R} \) \( \Rightarrow \)
Total error less than \( \varepsilon \).
Good Covers $\implies$ Ehrenpreis
An Immersed Pair of Pants

We can think of the pants as being isometrically immersed and speak of the goodness of pants.
We can identify cuffs of the parametrizing pants whenever the cuffs map to the same geodesic $\gamma$, and the pants are on opposite sides of $\gamma$. 
Easy Theorem

Let \( \mathcal{Q} \) be a finite set of immersed pants, such that for every closed geodesic \( Y \) on \( S \) we have the same number of pants on both sides of \( Y \). (That number can be zero). Then we can assemble \( \mathcal{Q} \) to form a finite cover of \( S \).
Good Pants and Good Curves

We let

\[ \Gamma_{\varepsilon, R} = \left\{ \gamma \mid \gamma \text{ is a closed geodesic on } S \right\} \]

\[ |h\gamma - R| < \varepsilon \]

\[ \Pi_{\varepsilon, R} = \left\{ \pi \mid \pi \text{ is an immersed pair of pants in } S, \text{ and } \right\} \]

\[ \partial \pi \subseteq \Gamma_{\varepsilon, R} \]
Things to Remember

1. We fix $S$ and $\varepsilon$ throughout the discussion.

2. We take $R \geq R_0(\Sigma, \varepsilon)$.
Counting Good Pants and Good Curves

\# \Pi_{\varepsilon, R} \geq e^{3R}

\# \Gamma_{\varepsilon, R} \geq e^{2R} / R

\# \{ p \in \Pi_{\varepsilon, R} \mid \gamma \in \mathbb{G} \} \geq \# \mathbb{R} \cup \mathbb{R} \cup \mathbb{R}

for each \gamma \in \Gamma_{\varepsilon, R}
The square root of a geodesic
Shear Coordinates and $N^4(\sqrt{\gamma})$
The Equidistribution Theorem

For every good curve \( y \) and large \( R \),

\[
\{ \{ x : \rho \mid \rho \text{ is \ an \ integer}, \quad x \in \Omega \} \} 
\leq N^1(\sqrt{R})
\]

is evenly distributed on \( N^1(\sqrt{R}) \) to the scale \( e^{-gR} \) for \( g = g(\delta, \varepsilon) \).

(We assume that \( R > R_0(\delta, \varepsilon) \).)
Counting Connections Between Geodesics

Let $A$ & $B$ be geodesic segments, and take $R_+ > R_- > 0$

Then the number of orthogeodesic connections between $A$ & $B$ with length in $[R_-, R_+]$ is

$$\frac{1}{8 \pi h(x(s))} (e^{R_+} - e^{R_-}) |A \cap B| + O(e^{(1-g(s)) R_+})$$
Equidistributed & Balanced $\Rightarrow$ Good Cover

If there were exactly the same number of pants on both sides of each geodesic $\gamma$, then we would be able to assemble the pants in $\mathbb{R}^k$ to form a good cover of $S$. 
Equidistributed $\Rightarrow$ Nearly balanced

It follows from equidistribution that there are nearly the same number of pants in $T_\pi$ on either side of a good geodesic. $\gamma$
An Interlude in Three Dimensions

In a hyperbolic 3-manifold, $N^3(\mathcal{S})$ (and $N^3(\sqrt{\mathcal{S}})$) is connected so we can use the "doubling trick" to build a nearly geodesic immersed surface (which is therefore essential).

\[ \pi_1(f) \colon \pi_1(S) \to \pi_1(M^3) \text{ injective} \]
A skew pair of pants

\[ \gamma_i = f(C_i) \text{ is a closed geodesic} \]

\[ \eta_i = f(h_0) \text{ is a geodesic segment orthogonal to } f(\gamma_{i+1}) \].
The feet of a skew pair of pants

\[ N^1(\gamma_c) \cong \frac{\ell}{2\pi i} \mathbb{Z} + \text{ht}(\lambda) \mathbb{Z} \]
Two and Three Dimensions

2D

\[ N^4(\gamma) \cong \frac{\mathbb{R}}{\ell(x) \cdot \mathbb{Z}} \times \{ -1, +1 \} \]

\[ \cong \frac{\mathbb{R}^*}{\langle x \mapsto e^{\ell(x)} x \rangle} \]

3D

\[ N^4(\gamma) \cong \frac{\mathbb{C}}{\ell(x) \cdot \mathbb{Z} + 2 \pi i \cdot \mathbb{Z}} \]

\[ \cong \frac{\mathbb{C}^*}{\langle z \mapsto e^{\ell(x)} z \rangle} \]
The meaning of equidistribution

The feet \( \{ \text{feet of } \gamma \in \mathcal{A} \} \)

are \( e^{-2\pi} \) evenly distributed as points on \( N^1(\sqrt{8}) \)

\[ N^1(\sqrt{8}) \rightarrow \mathbb{C} \rightarrow \mathbb{R} \]

\[ \mathbb{C} \rightarrow \mathbb{R} \]
The "doubling trick" revisited

We take

\[ A_\alpha = \{ f \text{foot}\alpha P : x \in \partial P, P \in \mathcal{F}_{\varepsilon, R} \} \]

We can find

\[ \sigma : A_\alpha \rightarrow A_\alpha \text{ a permutation} \]

such that (for all \( x \in A_\alpha \))

\[ |\sigma(x) - x - \varepsilon \pi - 1| < \frac{\varepsilon}{R} \]

and then define

\[ \mathcal{I} : A_\alpha^+ \sqcup A_\alpha^- \rightarrow A_\alpha^+ \sqcup A_\alpha^- \]

\[ \xrightarrow{\sigma^{-1}} \]

\[ \sigma \]
The boundary map on sums of good pants

We define
\[ \partial : \mathcal{Q} \pi \Gamma_{\varepsilon, k} \to \mathcal{Q} \Gamma_{\varepsilon, k} \]
by \[ \partial p = \gamma_1 + \gamma_2 + \gamma_3. \]

For \( \alpha \in \Gamma \mathcal{N} \pi \Gamma_{\varepsilon, k} \),

\[ \partial \alpha = 0 \iff \alpha \text{ has the same number of pants on both sides of every } \gamma. \]

(Note that \( \gamma + \gamma^{-1} = 0 \) in \( \mathcal{Q} \Gamma_{\varepsilon, k} \).)
The Idea of Self-Connection

We find $g : \mathcal{D}^\epsilon, R \rightarrow \mathcal{D}^\epsilon, R$

such that

1. $\partial g(x) = x$ whenever $\alpha = 2\pi$

2. $\| g(x) \|_\infty \leq P(R) e^{-R} \| x \|_\infty$ for all $\alpha$.

(where $P(R) = CR^N$ is a polynomial in $R$)
Then let \( \Pi = \sum_{P \in \Pi_{x, R}} P \), and \( \alpha = \partial \Pi \).

Then \( ||\alpha||_\infty \leq e^{\frac{1}{2} g(r)} \) by equidistribution.

So \( ||g(\alpha)||_\infty \leq P(k) e^{-\frac{1}{2} g} \ll 1 \).

So \( \Pi - g(\alpha) \) is positive and equidistributed.

And \( \partial(\Pi - g(\alpha)) = \partial \Pi - \partial g(\partial \Pi) = \partial \Pi - \partial \Pi = 0 \).

After clearing denominators, we can assemble the pants of \( \Pi - g(\alpha) \) to form a good cover!
The Good Pants Homology

We let \( H_1^{\kappa}(S; \Theta) = \Theta \frac{\Gamma_{\kappa}}{2 \Theta \Pi_{\kappa}} \).

We find a series of identities leading to \( H_1^{\kappa}(S; \Theta) = H_1(S; \Theta) \).
The Good Pants Homology

We let $H_1^{\mathbb{E},k}(S;\mathcal{Q}) = \frac{\mathcal{P}_{\mathbb{E},k}}{\partial \mathcal{P}_{\mathbb{E},k}}$.

We find a series of identities leading to $H_1^{\mathbb{E},k}(S;\mathcal{Q}) = H_1(S;\mathcal{Q})$.

Then we observe that $g: \mathcal{Q}_{\mathbb{E},k} \rightarrow \mathcal{Q}_{\mathbb{E},k}$ has been implicitly defined such that $\partial g(\alpha) = \alpha$ when $\alpha = 0$ in $H_1$. 
The Algebraic Square Lemma

Under conditions of reasonable geometry,

\[ \sum_{i,j=0,1} (-1)^{i+j} [A_i U B_j V] = 0 \]

in \( H^1 \)

(Where \([X]\) denotes the closed freely geodesic homotopic to \( X \), for \( X \in \pi_1(S, x) \).)
\[ \sum (-1)^{i+j} [A_0 U B_1 V] \]
\[ \sum (-1)^{i+j} [A_i U B_j V] = \emptyset \sum (-1)^{i+j} P_{ij} \]
We then define

$$A_T = \frac{1}{2}([TAT^TB] - [TATTB])$$

(which is independent of the choice of $B$ in $H_{\mathbb{C}R}$)

and prove

$$(XY)_T = X_T + Y_T$$

in $H_{\mathbb{C}R}$. 
The Inefficiency of a Closed Piecewise Geodesic

\[ I([A_0 \cdot A_1 \cdot \ldots \cdot A_n \cdot I]) = \sum_{i=0}^{n} \ell(A_i) - \ell([A_0 \cdot \ldots \cdot A_n I]) \]
The Standing Geometric Assumption

$T$ is long

given

bounds on the inefficiency.
The ADCB Lemma

\[[TA\overline{TB}TC\overline{T}D] = [T\overline{A}DTC\overline{T}B]\]

in $H_{\mathbb{E},\mathbb{R}}$
Four - Part Iterization

\[ \begin{bmatrix} T & A \bar{T} & B & T & C & \bar{T} & D \end{bmatrix} = A_T + B_T + C_T + D_T. \]

\[ \begin{bmatrix} T & A \bar{T} & B & T & C & \bar{T} & D \end{bmatrix} = \frac{1}{2} \left( \begin{bmatrix} T & A \bar{T} & B & T & C & \bar{T} & D \end{bmatrix} - \begin{bmatrix} D & T & C & T & B & \bar{T} & A \end{bmatrix} \right) \]
\[ = \frac{1}{2} \left( \begin{bmatrix} T & A \bar{T} & B & T & C & \bar{T} & D \end{bmatrix} - \begin{bmatrix} T & A \bar{T} & B & T & C & \bar{T} & D \end{bmatrix} \right) \]
\[ = \frac{1}{2} \left( \begin{bmatrix} T & A \bar{T} & B & T & C & \bar{T} & D \end{bmatrix} - \begin{bmatrix} T & A \bar{T} & B & T & C & \bar{T} & D \end{bmatrix} \right) \]
\[ = A_T + B_T + C_T + D_T. \]
The Rotation Lemma

\[ \sum (R_i \overline{R}_{i+1})_T + \sum (S_{i+1} S_i)_T = 0 \]
The XY Lemma

\[(XY)_T = X_T + Y_T\]