# Snowflake Subgroups of CAT(0) Groups

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# Outline

# Coarse Negative Curvature

- Thin triangles.
- $\delta$-hyperbolic metric space.
- Gromov hyperbolic group. Examples.
- Does not depend on finite generating set.

# CAT($k$) inequalities and spaces

- Model spaces. $\mathbb{E}^2$ and $\mathbb{H}^2$.
- Comparison triangles.
- CAT(0) and CAT(-1) inequalities.
- A geodesic metric space is CAT(0) (resp. CAT(-1)) if every geodesic triangle in the space satisfies the CAT(0) (resp. CAT(-1)) inequality.
- $G$ is said to be a CAT($k$) group if it acts geometrically on a CAT($k$) space.
- Examples: $F_n$, $\pi_1(M)$ for $M$ a closed, non-positively curved $n$-manifold, hyperbolic knot groups, ...

# Properties of NPC groups

- Finitely presented
- Solvable word problem
- Dehn function bounded above by a quadratic function
- Solvable conjugacy problem
- $\mathbb{Z}$ subgroups are undistorted.
- Convex subgroups (quasi-convex in case of Gromov hyperbolic groups) of NPC groups will again be NPC.
- . . .

A word about distorted subspaces of $\mathbb{H}^3$.

# Dehn Functions

## Dehn Functions

- Finite presentation $\langle X \mid R \rangle$
- Cayley graph, and Cayley 2-complex
- Word $w \in F(X)$ which represents 1 in $G$ corresponds to a loop in Cayley graph
- Area of a loop
- Dehn Function

$$\delta_{\langle X \mid R \rangle}(n) = \max\{\mathrm{Area}(w) \mid w =_G 1, \, |w|_X \leq n\}$$

- Particular Dehn function depends on presentation, but the coarse equivalence class of Dehn functions is independent of presentation.

**BACK!**

# Distortion of Subgroups

- $M^3$ (closed) hyperbolic 3-manifold fibering over $S^1$.
- $F_n \rtimes \mathbb{Z}$ examples.
- Many examples of highly distorted finitely generated subgroups of NPC groups.
- Fewer examples of highly distorted finitely presented subgroups of NPC groups.
- Even fewer examples of highly distorted finitely presented non-free subgroups of NPC groups.

# Dehn Functions of Subgroups: What's known

- Bieri Doubling Trick Examples. [Baumslag-Bridson-Miller-Short , 1997]
  - subgroups of CAT(0) groups which have exponential Dehn function.
  - subgroups of CAT(0) groups which have polynomial Dehn function of any given degree.
- Kernels of right-angled Artin groups (RAAGs). Polynomial Dehn functions up to $n^4$. [B, Dison, mid 2000's]
- Finitely presented, non-hyperbolic, subgroup of a hyperbolic group. [B, 1999], [Gersten-Short, 2002]
- Groups with distinct homological and homotopical Dehn functions. [Abrams-B-Dani-Young, 2012]

# The Bieri Doubling Trick

- [Stallings, 1963] F.p. group with non-f.g. integral $H_3$.
- [Bieri, 1976] Stallings $< F_2^3$, and generalization.
- [Baumslag-Bridson-Miller-Short, 1997] The Bieri trick and geometric applications.
- **The Doubling Trick.** The double $(N \rtimes \mathbb{Z}) *_N (N \rtimes \mathbb{Z})$ of the group $N \rtimes \mathbb{Z}$ over the fiber $N$ is contained inside of $(N \rtimes \mathbb{Z}) \times F_2$.

$$\langle N, (tu), (tv) \rangle \; < \; \langle N, t \rangle \; \times \; \langle u, v \rangle$$

- **Example.** If $M^3$ is a closed hyperbolic 3-manifold which fibers over $S^1$ with fiber $\Sigma^2$, then the double group

$$\pi_1(M^3) *_{\pi_1(\Sigma^2)} \pi_1(M^3) \; < \; \pi_1(M^3 \times (S^1 \vee S^1))$$

# Main Theorem.

Which power functions can appear as Dehn functions of subgroups of CAT(0) groups?

**Thm.** [B-Forester] The set

$$\{\alpha \in [2, \infty) \,|\, x^\alpha \text{ is a Dehn function of a subgroup of a CAT(0) group}\}$$

is dense in $[2, \infty)$.

# Properties of the Building Blocks

Building blocks are special free-by-cyclic groups.

$$B = F_2 \rtimes_\varphi \mathbb{Z} = \langle x, y \rangle \rtimes_\varphi \langle t \rangle$$

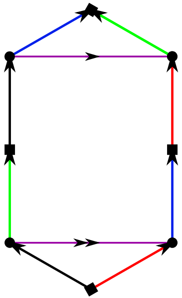
where

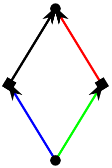
1. $\varphi$ is Anosov, palindromic; and
2. $B = \pi_1(K)$ where $K$ is a NPC 2-complex admitting an isometry $\sigma : K \to K$ satisfying
   - $\sigma^2 = \mathbb{I}_K$
   - $\sigma_*(x) = x^{-1}$, $\sigma_*(y) = y^{-1}$, and $\sigma_*(t) = t$.

# An Explicit Building Block

$\varphi : F_{\{x,y\}} \to F_{\{x,y\}}$ defined by $\varphi(x) = xyx$ and $\varphi(y) = x$.

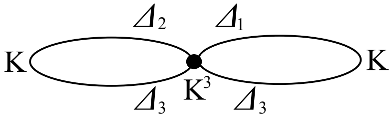
- $F_2 \rtimes \mathbb{Z}$ is $\pi_1$ of a punctured torus bundle.
- Matrix $\longrightarrow$ ideal triangulation $\longrightarrow$ spine [Hatcher-Floyd, 1982].
- Spine has a piecewise Euclidean CAT(0) structure [Tom Brady, 1995].
- Isometry $\sigma$ is given by reflection in vertical axes through 2-cells.





# The CAT(0) Group: Geometry

- The map
$$\Sigma : K^2 \to K^2 : (p_1, p_2) \mapsto (\sigma(p_2), \sigma(p_1))$$
is an isometry, and so its fixed set is a locally convex subspace of $K^2$. This is just the set $\{(p, \sigma(p)) | p \in K\}$ which we denote by $\Delta_K$.
- There are three copies of $\Delta_K$ in $K^3$ (corresponding to the three copies $K^2 \subset K^3$).
- Graph of spaces:




- This is a NPC 6-complex, with fundamental group a CAT(0) group $G$.

# The CAT(0) Group: Graph of Groups Structure

We have building blocks $B_i = \langle x_i, y_i \rangle \rtimes_\varphi \langle t_i \rangle$ and diagonal subgroups (from properties of $\sigma$):

$$\Delta_1 = \langle x_1^{-1}x_2, y_1^{-1}y_2, t_1t_2 \rangle = H_1 \rtimes \langle t_1t_2 \rangle$$

$$\Delta_2 = \langle x_2^{-1}x_3, y_2^{-1}y_3, t_2t_3 \rangle = H_2 \rtimes \langle t_2t_3 \rangle$$

$$\Delta_3 = \langle x_1^{-1}x_3, y_1^{-1}y_3, t_1t_3 \rangle = H_3 \rtimes \langle t_1t_3 \rangle$$

$$G \;=\; \langle B_1 \times B_2 \times B_3, u, v \,|\, u\Delta_3u^{-1} = \Delta_1, v\Delta_3v^{-1} = \Delta_2 \rangle$$

Use Teitze moves to rewrite this as

$$G \;=\; \langle B_1 \times B_2 \times B_3, e, f \,|\, e\Delta_3e^{-1} = \Delta_1, f\Delta_3f^{-1} = \Delta_2 \rangle$$

where $e = (t_1t_2)u$ and $f = (t_2t_3)v$. We have added in the $\varphi$-twisting; compare Bieri trick.

# The Snowflake Subgroup

- The $H_i = \Delta_i \cap F_2^3$ are diagonal copies of the free group of rank 2 in $F_2^3$. So $H_1 = \langle x_1^{-1}x_2, y_1^{-1}y_2 \rangle$ etc.
- Define the snowflake subgroup to be

$$H \;=\; \langle x_1, y_1, x_2, y_2, x_3, y_3, e, f \rangle \;<\; G$$

- By a result of [Bass, 1993] the snowflake group has the following graph of groups description:

$$H \;=\; \langle F_{\{x_1,y_1\}} \times F_{\{x_2,y_2\}} \times F_{\{x_3,y_3\}}, e, f \mid eH_3e^{-1} = H_1, fH_3f^{-1} = H_2 \rangle$$

where conjugation by $e$ and $f$ involve an application of $\varphi$.

# Key relations in the vertex group $(F_2)^3$

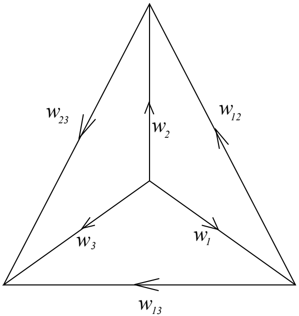
If $w(p,q) \in F_{\{p,q\}}$ is a palindrome, then the relations

$$w(x_1^{-1}x_2, y_1^{-1}y_2)w(x_2^{-1}x_3, y_2^{-1}y_3) \;=\; w(x_1^{-1}x_3, y_1^{-1}y_3)$$

and

$$w(x_2^{-1}x_3, y_2^{-1}y_3)w(x_1^{-1}x_2, y_1^{-1}y_2) \;=\; w(x_1^{-1}x_3, y_1^{-1}y_3)$$

hold in $F_{\{x_1,y_1\}} \times F_{\{x_2,y_2\}} \times F_{\{x_3,y_3\}}$ and have quadratic area.



# Snowflake Diagrams

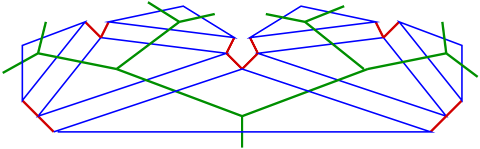


Figure: Half of Snowflake Diagram and Dual Tree

- $|\partial|$ is a multiple of the number of edges in the dual tree; that is, a multiple of $2^n$
- The area is the square of the diameter; that is $|\mathrm{Area}| \geq \lambda^{2n}$
- Thus $|\mathrm{Area}| \geq (2^{\log_2(\lambda)})^{2n} \simeq |\partial|^{2\log_2(\lambda)}$.
- This provides lower bound of $x^{2\log_2(\lambda)}$ for the Dehn function.

# Questions/Projects

- Is there a special cubical version of this construction (for the ambient CAT(0) group)? Interesting Dehn functions for subgroups of RAAGs.
- Are there CAT(0) groups containing finitely presented subgroups with Dehn function greater than exponential?
- $\cdots$