Kauffman brackets on surfaces

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Joint work with Helen Wong
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here with Grace Tsapsie Hibbard, born March 22, 2013
$S = \text{closed oriented surface of genus } g \geq 0$

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A group homomorphism $\rho : \pi_1(S) \to \text{SL}_2(\mathbb{C})$ defines its **character**

$\mathcal{K}_\rho : \{\text{closed curves in } S\} \to \mathbb{C}$

$K \mapsto \text{Tr} \rho(K)$
A group homomorphism $\rho: \pi_1(S) \to \text{SL}_2(\mathbb{C})$ defines its \textit{character}

$$K = \bigcup_{i=1}^{n} K_i \quad \mapsto \quad (-1)^n \prod_{i=1}^{n} \text{Tr} \, \rho(K_i)$$
Theorem (Helling 1967)

A function $\mathcal{K} : \{\text{closed multicurves in } S\} \longrightarrow \mathbb{C}$ is the character of a group homomorphism $\rho : \pi_1(S) \rightarrow \text{SL}_2(\mathbb{C})$ if and only if:
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\[\text{Diagram with multicurves and homotopy classes}\]
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The Skein Relation just rephrases the classical trace relation of $\text{SL}_2(\mathbb{C})$: 

$$\text{Tr } M \text{ Tr } N = \text{Tr } MN + \text{Tr } MN^{-1}, \quad \forall M, N \in \text{SL}_2(\mathbb{C})$$
Definition

An $\text{SL}_2(\mathbb{C})$–character is a function

$$\mathcal{K}: \{\text{closed multicurves in } S\} \to \mathbb{C}$$

such that:

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Definition

For \( q = e^{2 \pi \mathrm{i} \hbar} \in \mathbb{C} - \{0\} \), a **Kauffman q–bracket** is a function

\[ K : \{\text{framed links in } S \times [0,1]\} \rightarrow \text{End}(E) \]

for a finite-dimensional vector space \( E \), such that:

- **(Isotopy Invariance)** \( K(K) \) depends only on the isotopy class of \( K \) in \( S \times [0,1] \)

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Kauffman brackets on surfaces

Kauffman brackets

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1. When $S = \text{the sphere}$ and $\text{End}(E) = \text{End}(\mathbb{C}) = \mathbb{C}$, the only example is the classical Kauffman bracket ($\cong$ Jones polynomial)

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2. Witten’s interpretation (1987) of the Jones polynomial in the framework of a topological quantum field theory, mathematicalized by Reshetikhin–Turaev, provides a Kauffman $q$–bracket

   $$K_{\text{WRT}} : \{\text{framed links in } S \times [0, 1]\} \longrightarrow \text{End}(E)$$

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**Goal of this talk:** Construct other examples of Kauffman brackets
Kauffman brackets on surfaces

Kauffman brackets

Conceptual motivation

When $q = 1$ and $q^{\frac{1}{2}} = -1$, an irreducible Kauffman 1–bracket is the same thing as an $\text{SL}_2(\mathbb{C})$–character, namely as a point of the character variety

$$\mathcal{R}_{\text{SL}_2(\mathbb{C})}(S) = \{\text{homomorphisms } \rho: \pi_1(S) \to \text{SL}_2(\mathbb{C})\} / / \text{SL}_2(\mathbb{C})$$
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Turaev (1987), Frohman, Bullock, Kania-Bartoszynska, Przytycki, Sikora (around 2000): Interpretation of a Kauffman $q$–bracket as a "point" in a quantization of the character variety $\mathcal{R}_{SL_2(\mathbb{C})}(S)$, namely as a quantum $SL_2(\mathbb{C})$–character.
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From quantum to classical (Bonahon-Wong): When \( q^N = 1 \) with \( N \) odd, every irreducible Kauffman \( q \)–bracket determines a character \( \mathcal{K}_\rho \in \mathcal{R}_{\text{SL}_2(\mathbb{C})}(S) \), called the classical shadow of the Kauffman bracket.
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R_{\text{SL}_2(\mathbb{C})}(S) = \{ \text{homomorphisms } \rho : \pi_1(S) \to \text{SL}_2(\mathbb{C}) \} / / \text{SL}_2(\mathbb{C})
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Today, from classical to quantum: Realize every character \( \mathcal{K}_\rho \in R_{\text{SL}_2(\mathbb{C})}(S) \) as the classical shadow of a Kauffman \( q \)–bracket.
How to construct a group homomorphism $\rho: \pi_1(S) \to \text{SL}_2(\mathbb{C})$?
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Pick a triangulation $\Gamma$ of $S$, with vertex set $V_\Gamma$. 

![Triangulation Diagram]
How to construct a group homomorphism $\rho: \pi_1(S) \to \text{SL}_2(\mathbb{C})$?

Pick a triangulation $\Gamma$ of $S$, with vertex set $V_\Gamma$

Assign a weight $x_i \in \mathbb{C} - \{0\}$ to each edge $e_i$ of $\Gamma$
How to construct a group homomorphism $\rho: \pi_1(S) \to \text{SL}_2(\mathbb{C})$?

Pick a triangulation $\Gamma$ of $S$, with vertex set $V_\Gamma$

Assign a weight $x_i \in \mathbb{C} - \{0\}$ to each edge $e_i$ of $\Gamma$

This defines a pleated surface with shear-bend coordinates $x_i$, and with monodromy $\rho: \pi_1(S - V_\Gamma) \to \text{PSL}_2(\mathbb{C}) = \text{SL}_2(\mathbb{C})/\pm \text{Id}$
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which, after choices of square roots $x_i^{\frac{1}{2}}$ and of a spin structure, defines a homomorphism $\rho: \pi_1(S - V_{\Gamma}) \to \text{SL}_2(\mathbb{C})$
How to construct a group homomorphism $\rho: \pi_1(S) \to \text{SL}_2(\mathbb{C})$?

Pick a triangulation $\Gamma$ of $S$, with vertex set $\mathcal{V}_\Gamma$

Assign a weight $x_i \in \mathbb{C} - \{0\}$ to each edge $e_i$ of $\Gamma$

This defines a pleated surface with shear-bend coordinates $x_i$, and with monodromy $\rho: \pi_1(S - \mathcal{V}_\Gamma) \to \text{PSL}_2(\mathbb{C}) = \text{SL}_2(\mathbb{C})/\pm \text{Id}$

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**Main Point:** The construction is classical and, for a curve $K \subset S - \mathcal{V}_\Gamma$, gives a very explicit formula for $\text{Tr} \rho(K)$
More precisely, if $K$ crosses the edges $e_{i_1}, e_{i_2}, \ldots, e_{i_n}$,

$$\text{Tr } \rho(K) = \pm \text{Tr } \left[ M_1 \begin{pmatrix} x_{i_1}^{\frac{1}{2}} & 0 \\ 0 & x_{i_1}^{-\frac{1}{2}} \end{pmatrix} M_2 \begin{pmatrix} x_{i_2}^{\frac{1}{2}} & 0 \\ 0 & x_{i_2}^{-\frac{1}{2}} \end{pmatrix} \cdots M_n \begin{pmatrix} x_{i_n}^{\frac{1}{2}} & 0 \\ 0 & x_{i_n}^{-\frac{1}{2}} \end{pmatrix} \right]$$

where

$$M_k = \begin{cases} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} & \text{if} \\
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$$= \pm \sum_{\pm \pm \ldots \pm} (0 \text{ or } 1) x_{i_1}^{\pm \frac{1}{2}} x_{i_2}^{\pm \frac{1}{2}} \ldots x_{i_n}^{\pm \frac{1}{2}}$$

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Problem: This defines an $\text{SL}_2(\mathbb{C})$–character on the punctured surface $S - \mathcal{V}_\Gamma$, not necessarily on the closed surface $S$. 
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\[
\text{Tr} \, \rho(K) = \text{Tr} \, \rho(K')?
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Fact

The edge weights $x_i$ define an $\text{SL}_2(\mathbb{C})$–character on the closed surface $S$ if and only if, for every vertex,

$$
\begin{align*}
\frac{1}{2} x_{i_1} \frac{1}{2} x_{i_1} \cdots \frac{1}{2} x_{i_1} &= -1 \\
1 + x_{i_1} + x_{i_1} x_{i_2} + x_{i_1} x_{i_2} x_{i_3} + \cdots + x_{i_1} x_{i_2} \cdots x_{i_{n-1}} &= 0
\end{align*}
$$
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1. Choose a weight $x_i^{\frac{1}{2}} \in \mathbb{C} - \{0\}$ for each edge $e_i$ of the triangulation $\Gamma$
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2. This defines an $SL_2(\mathbb{C})$–character for the punctured surface $S - \mathcal{V}_\Gamma$ by an explicit formula

\[ K_{\rho}(K) = \pm \sum_{\pm\pm\ldots\pm} (0 \text{ or } 1) x_{i_1}^{\pm\frac{1}{2}} x_{i_2}^{\pm\frac{1}{2}} \ldots x_{i_n}^{\pm\frac{1}{2}} \]
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1. Choose a weight $\frac{1}{2} x_i \in \mathbb{C} - \{0\}$ for each edge $e_i$ of the triangulation $\Gamma$

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$$K_\rho(K) = \pm \sum_{\pm \pm \cdots \pm} (0 \text{ or } 1) x_{i_1}^{\pm \frac{1}{2}} x_{i_2}^{\pm \frac{1}{2}} \cdots x_{i_n}^{\pm \frac{1}{2}}$$

3. This character induces a character for the closed surface $S$ if and only if

$$\begin{cases} 
\frac{1}{2} x_{i_1} \cdot \frac{1}{2} x_{i_2} \cdots \frac{1}{2} x_{i_n} = -1 \\
1 + x_{i_1} + x_{i_1} x_{i_2} + x_{i_1} x_{i_2} x_{i_3} + \cdots + x_{i_1} x_{i_2} \cdots x_{i_{n-1}} = 0
\end{cases}$$

for each vertex
Fix $q \in \mathbb{C}$ with $q^N = 1$, $N$ odd
Fix $q \in \mathbb{C}$ with $q^N = 1$, $N$ odd

Want to construct a Kauffman $q$–bracket

$\mathcal{K}: \{\text{framed links in } S \times [0, 1]\} \longrightarrow \text{End}(E)$
Fix $q \in \mathbb{C}$ with $q^N = 1$, $N$ odd

Want to construct a Kauffman $q$–bracket $\mathcal{K}$:

$$\mathcal{K} : \{\text{framed links in } S \times [0, 1]\} \rightarrow \text{End}(E)$$

namely such that:

- (Isotopy Invariance) $\mathcal{K}(K)$ depends only on the isotopy class of $K$ in $S \times [0, 1]$

- (Superposition Rule) $\mathcal{K}(K_1 \cup K_2) = \mathcal{K}(K_1) \circ \mathcal{K}(K_2)$ whenever $K = K_1 \cup K_2$ with $K_1 \subset S \times [0, \frac{1}{2}]$ and $K_2 \subset S \times [\frac{1}{2}, 1]$)

- (Skein Relation) $\mathcal{K}(K_1) = q^{\frac{1}{2}} \mathcal{K}(K_0) + q^{-\frac{1}{2}} \mathcal{K}(K_\infty)$ if $K_1$, $K_0$, $K_\infty$ are the same everywhere, except in a small box where $K_1 = \begin{tikzpicture}[baseline=-.5ex]
\draw (0,0) circle (.3);
\draw (0,.3) circle (.3);
\end{tikzpicture}$, $K_0 = \begin{tikzpicture}[baseline=-.5ex]
\draw (0,0) circle (.3);
\draw (0,.3) circle (.3);
\draw (0,0) -- (0,.3);
\end{tikzpicture}$ and $K_\infty = \begin{tikzpicture}[baseline=-.5ex]
\draw (0,0) circle (.3);
\draw (0,.3) circle (.3);
\draw (0,0) -- (0,.3);
\draw (0,0) -- (0.3,0);
\draw (0,0) -- (-0.3,0);
\end{tikzpicture}$
Summary: Recipe to construct $\text{SL}_2(\mathbb{C})$–characters

1. Choose a weight $x_{i}^{\frac{1}{2}} \in \mathbb{C} - \{0\}$ for each edge $e_i$ of the triangulation $\Gamma$

2. This defines an $\text{SL}_2(\mathbb{C})$–character for the punctured surface $S - \mathcal{V}_\Gamma$ by an explicit formula

$$K_\rho(K) = \pm \sum_{\pm \pm \cdots \pm} (0 \text{ or } 1) x_{i_1}^{\pm \frac{1}{2}} x_{i_2}^{\pm \frac{1}{2}} \cdots x_{i_n}^{\pm \frac{1}{2}}$$

3. This character induces a character for the closed surface $S$ if and only if

$$\left\{\begin{array}{l}
    x_{i_1}^{\frac{1}{2}} x_{i_2}^{\frac{1}{2}} \cdots x_{i_n}^{\frac{1}{2}} = -1 \\
    1 + x_{i_1} + x_{i_1} x_{i_2} + x_{i_1} x_{i_2} x_{i_3} + \cdots + x_{i_1} x_{i_2} \cdots x_{i_{n-1}} = 0
\end{array}\right.$$ 

for each vertex
Step 1. Choose an invertible operator (= matrix) $X_i^{1/2} \in \text{End}(E)$ for each edge $e_i$ of the triangulation $\Gamma$, 
Step 1. Choose an invertible operator (= matrix) $X_i^{\frac{1}{2}} \in \text{End}(E)$ for each edge $e_i$ of the triangulation $\Gamma$, for an appropriate finite-dimensional vector space $E$. 
Step 1. Choose an invertible operator (\(\equiv\) matrix) \(X_i^{1/2} \in \text{End}(E)\) for each edge \(e_i\) of the triangulation \(\Gamma\), for an appropriate finite-dimensional vector space \(E\) and in such a way that

\[
X_i^{1/2} X_j^{1/2} = q X_j^{1/2} X_i^{1/2}
\]

whenever \(e_i e_j\).
Step 1. Choose an invertible operator (= matrix) $X_i^{\frac{1}{2}} \in \text{End}(E)$ for each edge $e_i$ of the triangulation $\Gamma$, for an appropriate finite-dimensional vector space $E$ and in such a way that

$$X_i^{\frac{1}{2}} X_j^{\frac{1}{2}} = qX_j^{\frac{1}{2}} X_i^{\frac{1}{2}}$$

whenever $e_i e_j$. This is the same thing as a representation of the Chekhov-Fock algebra of the triangulation $\Gamma$ (= quantum Teichmüller space of the punctured surface $S - \mathcal{V}_\Gamma$)
Step 1. Choose an invertible operator (= matrix) $X_{i^j}^{1^2} \in \text{End}(E)$ for each edge $e_i$ of the triangulation $\Gamma$, for an appropriate finite-dimensional vector space $E$ and in such a way that

$$X_{i^j}^{1^2} X_{j^i}^{1^2} = q X_{j^i}^{1^2} X_{i^j}^{1^2}$$

whenever $e_i e_j$. This is the same thing as a representation of the Chekhov-Fock algebra of the triangulation $\Gamma$ (= quantum Teichmüller space of the punctured surface $S - \mathcal{V}_\Gamma$).

Proposition (FB + Xiaobo Liu, 2007, relatively easy)

If $q^N = 1$ with $N$ odd, smallest dimensional choices of such operators $X_{i^j}^{1^2} \in \text{End}(E)$ are classified by

- edge weights $x_i \in \mathbb{C}^*$ such that $X_{i^j}^{N^2} = x_i^{1^2} \text{Id}_E$

- choices of $N$–roots for numbers $x_i^{1^2} x_i^{1^2} \ldots x_i^{1^2} \in \mathbb{C}^*$ associated to the vertices
Kauffman brackets on surfaces

Construction of Kauffman brackets

Summary: Recipe to construct $\text{SL}_2(\mathbb{C})$–characters

1. Choose a weight $x_i^{1/2} \in \mathbb{C} - \{0\}$ for each edge $e_i$ of the triangulation $\Gamma$

2. This defines an $\text{SL}_2(\mathbb{C})$–character for the punctured surface $S - \mathcal{V}_\Gamma$ by an explicit formula

$$K_\rho(K) = \pm \sum_{\pm \pm \cdots \pm} (0 \text{ or } 1) x_{i_1}^{\pm 1/2} x_{i_2}^{\pm 1/2} \cdots x_{i_n}^{\pm 1/2}$$

3. This character induces a character for the closed surface $S$ if and only if

$$\begin{cases} x_{i_1}^{1/2} x_{i_2}^{1/2} \cdots x_{i_n}^{1/2} = -1 \\
1 + x_{i_1} + x_{i_1} x_{i_2} + x_{i_1} x_{i_2} x_{i_3} + \cdots + x_{i_1} x_{i_2} \cdots x_{i_{n-1}} = 0 \end{cases}$$

for each vertex
Summary: Recipe to construct $\text{SL}_2(\mathbb{C})$–characters

1. Choose a weight $x_i^{\frac{1}{2}} \in \mathbb{C} - \{0\}$ for each edge $e_i$ of the triangulation $\Gamma$ \checkmark

2. This defines an $\text{SL}_2(\mathbb{C})$–character for the punctured surface $S - \mathcal{V}_\Gamma$ by an explicit formula

$$K_\rho(K) = \pm \sum_{\pm\pm\cdots\pm} (0 \text{ or } 1) x_{i_1}^{\pm\frac{1}{2}} x_{i_2}^{\pm\frac{1}{2}} \cdots x_{i_n}^{\pm\frac{1}{2}}$$

3. This character induces a character for the closed surface $S$ if and only if

$$\left\{ \begin{array}{l}
x_{i_1}^{\frac{1}{2}} x_{i_2}^{\frac{1}{2}} \cdots x_{i_n}^{\frac{1}{2}} = -1 \\
1 + x_{i_1} + x_{i_1} x_{i_2} + x_{i_1} x_{i_2} x_{i_3} + \cdots + x_{i_1} x_{i_2} \cdots x_{i_{n-1}} = 0
\end{array} \right.$$ for each vertex
Step 2.

Theorem (FB + Helen Wong, 2011)

Given operators $X_{\frac{1}{2}} \in \text{End}(E)$ associated to the edges of the triangulation $\Gamma$ as in Step 1, there is an explicit formula

$$K(K) = \sum_{\pm \cdots \pm} (0 \text{ or } \pm q^\Box) X_{i_1}^{\pm \frac{1}{2}} X_{i_2}^{\pm \frac{1}{2}} \cdots X_{i_n}^{\pm \frac{1}{2}}$$

that defines a Kauffman $q$–bracket

$$K: \{\text{framed links in } (S - V_\Gamma) \times [0, 1]\} \longrightarrow \text{End}(E)$$

for the punctured surface $S - V_\Gamma$. 
Step 2.

**Theorem (FB + Helen Wong, 2011)**

Given operators $X_i^{\frac{1}{2}} \in \text{End}(E)$ associated to the edges of the triangulation $\Gamma$ as in Step 1, there is an explicit formula

$$K(K) = \sum_{\pm \pm \ldots \pm} (0 \text{ or } \pm q^{\square}) X_{i_1}^{\pm \frac{1}{2}} X_{i_2}^{\pm \frac{1}{2}} \ldots X_{i_n}^{\pm \frac{1}{2}}$$

that defines a Kauffman q–bracket

$$K: \{\text{framed links in } (S - \mathcal{V}_\Gamma) \times [0, 1]\} \longrightarrow \text{End}(E)$$

for the punctured surface $S - \mathcal{V}_\Gamma$

**Remark** Much harder. Need to worry about the order in which to multiply the operators $X_i^{\frac{1}{2}} \in \text{End}(E)$, which requires the introduction of correction factors $q^{\square}$ related to the classical Kauffman bracket in $\mathbb{R}^3$. 
Step 2.

Theorem (FB + Helen Wong, 2011)

Given operators $X_i^{\frac{1}{2}} \in \text{End}(E)$ associated to the edges of the triangulation $\Gamma$ as in Step 1, there is an explicit formula

$$\mathcal{K}(K) = \sum_{\pm \pm \cdots \pm} (0 \text{ or } \pm q^{\Box}) X_{i_1}^{\pm \frac{1}{2}} X_{i_2}^{\pm \frac{1}{2}} \cdots X_{i_n}^{\pm \frac{1}{2}}$$

that defines a Kauffman $q$–bracket

$$\mathcal{K}: \{\text{framed links in } (S - \mathcal{V}_\Gamma) \times [0, 1]\} \rightarrow \text{End}(E)$$

for the punctured surface $S - \mathcal{V}_\Gamma$

Remark Much harder. Need to worry about the order in which to multiply the operators $X_i^{\frac{1}{2}} \in \text{End}(E)$, which requires the introduction of correction factors $q^{\Box}$ related to the classical Kauffman bracket in $\mathbb{R}^3$.

FB + Qingtao Chen, 2013 More conceptual approach based on the representation theory of the quantum group $U_q(\mathfrak{sl}_2)$
Summary: Recipe to construct $\text{SL}_2(\mathbb{C})$–characters

1. Choose a weight $x_i^{\frac{1}{2}} \in \mathbb{C} - \{0\}$ for each edge $e_i$ of the triangulation $\Gamma$.

2. This defines an $\text{SL}_2(\mathbb{C})$–character for the punctured surface $S - \mathcal{V}_\Gamma$ by an explicit formula

$$K_\rho(K) = \pm \sum_{\pm \pm \cdots \pm} (0 \text{ or } 1) x_{i_1}^{\frac{1}{2}} x_{i_2}^{\frac{1}{2}} \cdots x_{i_n}^{\frac{1}{2}}$$

3. This character induces a character for the closed surface $S$ if and only if

$$\begin{cases} 
  x_{i_1}^{\frac{1}{2}} x_{i_2}^{\frac{1}{2}} \cdots x_{i_n}^{\frac{1}{2}} = -1 \\
  1 + x_{i_1} + x_{i_1} x_{i_2} + x_{i_1} x_{i_2} x_{i_3} + \cdots + x_{i_1} x_{i_2} \cdots x_{i_{n-1}} = 0 
\end{cases}$$

for each vertex.
Summary: Recipe to construct $\text{SL}_2(\mathbb{C})$–characters

1. Choose a weight $x_i^{1/2} \in \mathbb{C} - \{0\}$ for each edge $e_i$ of the triangulation $\Gamma$.

2. This defines an $\text{SL}_2(\mathbb{C})$–character for the punctured surface $S - V_\Gamma$ by an explicit formula:
   
   $$ \mathcal{K}_\rho(K) = \pm \sum_{\pm \pm \cdots \pm} (0 \text{ or } 1)x_i^{1/2}x_j^{1/2}\cdots x_n^{1/2} \checkmark $$

3. This character induces a character for the closed surface $S$ if and only if
   
   $$ \begin{cases} 
   x_{i_1}^{1/2}x_{i_2}^{1/2}\cdots x_{i_n}^{1/2} = -1 \\
   1 + x_{i_1} + x_{i_1}x_{i_2} + x_{i_1}x_{i_2}x_{i_3} + \cdots + x_{i_1}x_{i_2}\cdots x_{i_{n-1}} = 0 
   \end{cases} $$

for each vertex.
Problem: This defines a Kauffman bracket

\[ \mathcal{K}: \{\text{framed links in } (S - \mathcal{V}_\Gamma) \times [0,1]\} \longrightarrow \text{End}(E) \]

on the punctured surface \( S - \mathcal{V}_\Gamma \), not necessarily on the closed surface \( S \)
**Problem:** This defines a Kauffman bracket

\[ \mathcal{K} : \{\text{framed links in } (S - \mathcal{V}_\Gamma) \times [0, 1]\} \longrightarrow \text{End}(E) \]

on the *punctured* surface \( S - \mathcal{V}_\Gamma \), not necessarily on the *closed* surface \( S \).
Problem: This defines a Kauffman bracket

\[ \mathcal{K}: \{\text{framed links in } (S - \mathcal{V}_\Gamma) \times [0, 1]\} \longrightarrow \text{End}(E) \]

on the *punctured* surface \( S - \mathcal{V}_\Gamma \), not necessarily on the *closed* surface \( S \)
Summary: Recipe to construct $SL_2(\mathbb{C})$–characters

1. Choose a weight $x_i^{\frac{1}{2}} \in \mathbb{C} - \{0\}$ for each edge $e_i$ of the triangulation $\Gamma$.

2. This defines an $SL_2(\mathbb{C})$–character for the punctured surface $S - \mathcal{V}_\Gamma$ by an explicit formula

$$K_{\rho}(K) = \pm \sum_{\pm \pm \cdots \pm} (0 \text{ or } 1) x_i^{\pm \frac{1}{2}} x_i^{\pm \frac{1}{2}} \cdots x_i^{\pm \frac{1}{2}}$$

3. This character induces a character for the closed surface $S$ if and only if

$$\begin{cases} 
  x_i^{\frac{1}{2}} x_i^{\frac{1}{2}} \cdots x_i^{\frac{1}{2}} = -1 \\
  1 + x_i + x_i x_i x_i + x_i x_i x_i x_i + \cdots + x_i x_i x_i \cdots x_{i_{n-1}} = 0
\end{cases}$$

for each vertex.
In Step 1, we associated to the edges of the triangulation $\Gamma$ operators $X_i^{\frac{1}{2}} \in \text{End}(E)$ such that $X_i^{\frac{N}{2}} = x_i^{\frac{1}{2}} \text{Id}_E$
In Step 1, we associated to the edges of the triangulation $\Gamma$ operators $X_{i}^{\frac{1}{2}} \in \text{End}(E)$ such that $X_{i}^{\frac{N}{2}} = x_{i}^{\frac{1}{2}} \text{Id}_{E}$

**Step 3a.** If $x_{i_{1}}^{\frac{1}{2}} x_{i_{2}}^{\frac{1}{2}} \ldots x_{i_{n}}^{\frac{1}{2}} = -1$ at a vertex, the corresponding operators $X_{i}^{\frac{1}{2}} \in \text{End}(E)$ can be chosen so that

$$X_{i_{1}}^{\frac{1}{2}} X_{i_{2}}^{\frac{1}{2}} \ldots X_{i_{n}}^{\frac{1}{2}} = -q^{\frac{n+2}{4}} \text{Id}_{E}$$
Summary: Recipe to construct $SL_2(\mathbb{C})$–characters

1. Choose a weight $x_{i}^{\frac{1}{2}} \in \mathbb{C} - \{0\}$ for each edge $e_{i}$ of the triangulation $\Gamma$.

2. This defines an $SL_2(\mathbb{C})$–character for the punctured surface $S - \mathcal{V}_{\Gamma}$ by an explicit formula

$$K_{\rho}(K) = \sum_{\pm \pm \cdots \pm} (0 \text{ or } 1) x_{i_{1}}^{\frac{1}{2}} x_{i_{2}}^{\frac{1}{2}} \cdots x_{i_{n}}^{\frac{1}{2}}$$

3. This character induces a character for the closed surface $S$ if and only if

$$\begin{cases} 
    x_{i_{1}}^{\frac{1}{2}} x_{i_{2}}^{\frac{1}{2}} \cdots x_{i_{n}}^{\frac{1}{2}} = -1 \\
    1 + x_{i_{1}} + x_{i_{1}} x_{i_{2}} + x_{i_{1}} x_{i_{2}} x_{i_{3}} + \cdots + x_{i_{1}} x_{i_{2}} \cdots x_{i_{n-1}} = 0
\end{cases}$$

for each vertex.
Summary: Recipe to construct $SL_2(\mathbb{C})$–characters

1. Choose a weight $x_i^{\frac{1}{2}} \in \mathbb{C} - \{0\}$ for each edge $e_i$ of the triangulation $\Gamma$.

2. This defines an $SL_2(\mathbb{C})$–character for the punctured surface $S - \mathcal{V}_\Gamma$ by an explicit formula
   \[
   \mathcal{K}_\rho(K) = \pm \sum_{\pm \cdots \pm} (0 \text{ or } 1) x_i^{\pm \frac{1}{2}} x_j^{\pm \frac{1}{2}} \cdots x_n^{\pm \frac{1}{2}}
   \]

3. This character induces a character for the closed surface $S$ if and only if
   \[
   \left\{
   \begin{array}{l}
   x_i^{\frac{1}{2}} x_j^{\frac{1}{2}} \cdots x_n^{\frac{1}{2}} = -1 \\
   1 + x_i + x_i x_j + x_i x_j x_3 + \cdots + x_i x_j \cdots x_{i_{n-1}} = 0
   \end{array}
   \right.
   \]
   for each vertex.
Step 3b. For a vertex $v = e_i^1 e_i^2 e_i^3 \ldots e_i^n$ of the triangulation $\Gamma$ for the operators $X_{\frac{1}{2}}^{i_j} \in \text{End}(E)$ associated to the edges, consider

$$1 + qX_{i_1} + q^2 X_{i_1}X_{i_2} + q^3 X_{i_1}X_{i_2}X_{i_3} + \cdots + q^{n-1} X_{i_1}X_{i_2} \ldots X_{i_{n-1}}$$
Step 3b. For a vertex $v = e_{i_1} e_{i_2} e_{i_3} \ldots e_{i_n}$ of the triangulation $\Gamma$ for the operators $X_{i_j}^{\frac{1}{2}} \in \text{End}(E)$ associated to the edges, set

$$F_v = \ker \left( 1 + qX_{i_1} + q^2 X_{i_1} X_{i_2} + q^3 X_{i_1} X_{i_2} X_{i_3} + \cdots + q^{n-1} X_{i_1} X_{i_2} \ldots X_{i_{n-1}} \right)$$
Step 3b. For a vertex \( v = e_{i_1} e_{i_2} \ldots e_{i_n} \) of the triangulation \( \Gamma \) for the operators \( X_{i_1}^{1/2} \in \text{End}(E) \) associated to the edges, set

\[
F_v = \ker \left( 1 + qX_{i_1} + q^2 X_{i_1}X_{i_2} + q^3 X_{i_1}X_{i_2}X_{i_3} + \cdots + q^{n-1}X_{i_1}X_{i_2} \cdots X_{i_{n-1}} \right)
\]

and

\[
F = \bigcap_{\text{vertices } v} F_v \subset E
\]
Theorem

1. The linear subspace $F \subseteq E$ is invariant under the image of the Kauffman bracket

   $\mathcal{K}: \{\text{framed links in } (S - \mathcal{V}_\Gamma) \times [0, 1]\} \longrightarrow \text{End}(E)$

   constructed in Step 2
Theorem

1. The linear subspace $F \subset E$ is invariant under the image of the Kauffman bracket

$$\mathcal{K}: \{\text{framed links in } (S - \mathcal{V}_\Gamma) \times [0, 1]\} \longrightarrow \text{End}(E)$$

constructed in Step 2 (but not invariant under the $X_i$!!)
Theorem

1. **The linear subspace** \( F \subset E \) **is invariant under the image of the Kauffman bracket**

\[
\mathcal{K} : \{ \text{framed links in } (S - \mathcal{V}_\Gamma) \times [0, 1] \} \longrightarrow \text{End}(E)
\]

constructed in Step 2 (but not invariant under the \( X_i \)!!)

2. **If** \( K, K' \subset (S - \mathcal{V}_\Gamma) \times [0, 1] \) **are isotopic in** \( S \times [0, 1] \), **then**

\[
\mathcal{K}(K)|_F = \mathcal{K}(K')|_F
\]
Theorem

1. The linear subspace $F \subset E$ is invariant under the image of the Kauffman bracket
   \[ \mathcal{K} : \{ \text{framed links in } (S - V_{\Gamma}) \times [0, 1] \} \rightarrow \text{End}(E) \]
   constructed in Step 2 (but not invariant under the $X_i$ !!)

2. If $K, K' \subset (S - V_{\Gamma}) \times [0, 1]$ are isotopic in $S \times [0, 1]$, then
   \[ \mathcal{K}(K)|_F = \mathcal{K}(K')|_F \]

Corollary

$\mathcal{K}$ induces a Kauffman $q$–bracket
\[ \bar{\mathcal{K}} : \{ \text{framed links in } S \times [0, 1] \} \rightarrow \text{End}(F) \]

for the closed surface $S$
Theorem

\[ \dim F \geq \begin{cases} 
N^{3(g-1)} & \text{if } g \geq 2 \\
N & \text{if } g = 1 \\
1 & \text{if } g = 0 
\end{cases} \]
Theorem

\[ \dim F \geq \begin{cases} 
N^3(g-1) & \text{if } g \geq 2 \\
N & \text{if } g = 1 \\
1 & \text{if } g = 0 
\end{cases} \]

Theorem

*Up to isomorphism, the Kauffman bracket*

\[ \overline{\mathcal{K}} : \{ \text{framed links in } S \times [0, 1] \} \longrightarrow \text{End}(F) \]

*depends only on the (classical) \( SL_2(\mathbb{C}) \)-character \( \mathcal{K}_\rho \in \mathcal{R}_{SL_2(\mathbb{C})}(S) \) associated to the same edge weights \( x_i \in \mathbb{C}^* \). In particular, it is independent of the triangulation \( \Gamma \)
Theorem

\[ \text{dim } F \geq \begin{cases} \mathcal{N}^3(g-1) & \text{if } g \geq 2 \\ \mathcal{N} & \text{if } g = 1 \\ 1 & \text{if } g = 0 \end{cases} \]

with equality for generic (all?) \( K_{\rho} \in \mathcal{R}_{\text{SL}_2(\mathbb{C})}(S) \)

Theorem

Up to isomorphism, the Kauffman bracket

\[ \widetilde{\mathcal{K}} : \{ \text{framed links in } S \times [0, 1] \} \longrightarrow \text{End}(F) \]

depends only on the (classical) \( \text{SL}_2(\mathbb{C}) \)-character \( K_{\rho} \in \mathcal{R}_{\text{SL}_2(\mathbb{C})}(S) \) associated to the same edge weights \( x_i \in \mathbb{C}^* \). In particular, it is independent of the triangulation \( \Gamma \).
Theorem (Bonahon-Wong, 2012)

When $q^N = 1$ with $N$ odd, every irreducible Kauffman $q$–bracket

$$\mathcal{K}: \{\text{framed links in } S \times [0, 1]\} \longrightarrow \text{End}(E)$$

determines a classical character $\mathcal{K}_\rho \in \mathcal{R}_{SL_2(C)}(S)$
Theorem (Bonahon-Wong, 2012)

When $q^N = 1$ with $N$ odd, every irreducible Kauffman $q$–bracket
\[ K : \{\text{framed links in } S \times [0, 1]\} \to \text{End}(E) \]
determines a classical character $K_\rho \in \mathcal{R}_{\text{SL}_2(\mathbb{C})}(S)$
\[ K_\rho : \{\text{closed multicurves in } S\} \to \mathbb{C} \]
Theorem (Bonahon-Wong, 2012)

When \( q^N = 1 \) with \( N \) odd, every irreducible Kauffman \( q \)-bracket \( \mathcal{K} : \{ \text{framed links in } S \times [0,1] \} \rightarrow \text{End}(E) \)
determines a classical character \( \mathcal{K}_\rho \in \mathcal{R}_{\text{SL}_2(\mathbb{C})}(S) \)
\[
\mathcal{K}_\rho : \{ \text{closed multicurves in } S \} \rightarrow \mathbb{C}
\]
by the property that
\[
\mathcal{K}(K) \in \text{End}(E)
\]
for every knot \( K \subset S \times [0,1] \) whose projection to \( S \) has no crossing and whose framing is vertical.
Theorem (Bonahon-Wong, 2012)

When \( q^N = 1 \) with \( N \) odd, every irreducible Kauffman \( q \)-bracket \( \mathcal{K} : \{\text{framed links in } S \times [0, 1]\} \rightarrow \text{End}(E) \) determines a classical character \( \mathcal{K}_\rho \in \mathcal{R}_{\text{SL}_2(\mathbb{C})}(S) \)

\[ \mathcal{K}_\rho : \{\text{closed multicurves in } S\} \rightarrow \mathbb{C} \]

by the property that

\[ T_N(\mathcal{K}(K)) \in \text{End}(E) \]

for every knot \( K \subset S \times [0, 1] \) whose projection to \( S \) has no crossing and whose framing is vertical.

Here, \( T_N(x) \) is the (normalized) \( N \)-th Chebyshev polynomial of the first type defined by \( 2 \cos N \theta = T_N(2 \cos \theta) \)
Theorem (Bonahon-Wong, 2012)

When $q^N = 1$ with $N$ odd, every irreducible Kauffman $q$–bracket $\mathcal{K}: \{\text{framed links in } S \times [0, 1]\} \rightarrow \text{End}(E)$ determines a classical character $\mathcal{K}_\rho \in \mathcal{R}_{\text{SL}_2(\mathbb{C})}(S)$

$$\mathcal{K}_\rho: \{\text{closed multicurves in } S\} \rightarrow \mathbb{C}$$

by the property that

$$T_N(\mathcal{K}(K)) = \mathcal{K}_\rho(K) \text{Id}_E \in \text{End}(E)$$

for every knot $K \subset S \times [0, 1]$ whose projection to $S$ has no crossing and whose framing is vertical.

Here, $T_N(x)$ is the (normalized) $N$–th Chebyshev polynomial of the first type defined by $2 \cos N\theta = T_N(2 \cos \theta)$.
Theorem (Bonahon-Wong, 2012)

When \( q^N = 1 \) with \( N \) odd, every irreducible Kauffman \( q \)-bracket
\[
\mathcal{K} : \{\text{framed links in } S \times [0, 1]\} \to \text{End}(E)
\]
determines a classical character \( \mathcal{K}_\rho \in \mathcal{R}_{\text{SL}_2(\mathbb{C})}(S) \)
\[
\mathcal{K}_\rho : \{\text{closed multicurves in } S\} \to \mathbb{C}
\]
by the property that
\[
T_N(\mathcal{K}(K)) = \mathcal{K}_\rho(K) \text{Id}_E \in \text{End}(E)
\]
for every knot \( K \subset S \times [0, 1] \) whose projection to \( S \) has no crossing and whose framing is vertical.

Here, \( T_N(x) \) is the (normalized) \( N \)-th Chebyshev polynomial of the first type defined by \( 2 \cos N\theta = T_N(2 \cos \theta) \)

This is not the (normalized) \( N \)-th Chebyshev polynomial of the second type \( S_N(x) \) is defined by \( \sin N\theta = S_N(2 \cos \theta) \sin \theta \) which usually occurs in the representation theory of \( SL_2 \) and \( U_q(sl_2) \)
For the Kauffman $q$–bracket that we constructed,

$$\mathcal{K}(K) = \sum_{\pm\pm\ldots\pm} (0 \text{ or } \pm q) X_{i_1}^{\pm \frac{1}{2}} X_{i_2}^{\pm \frac{1}{2}} \ldots X_{i_n}^{\pm \frac{1}{2}}$$

where the matrices $X_{i}^{\frac{1}{2}} \in \text{End}(E)$ are such that

$$X_{i}^{\frac{1}{2}} X_{j}^{\frac{1}{2}} = q^n X_{j}^{\frac{1}{2}} X_{i}^{\frac{1}{2}} \text{ and } X_{i}^{\frac{N}{2}} = x_i \text{Id}_E$$
For the Kauffman $q$–bracket that we constructed,

$$\mathcal{K}(K) = \sum_{\pm \pm \cdots \pm} (0 \text{ or } \pm q^{\square}) X_{i_1}^{\pm \frac{1}{2}} X_{i_2}^{\pm \frac{1}{2}} \cdots X_{i_n}^{\pm \frac{1}{2}}$$

where the matrices $X_{i}^{\frac{1}{2}} \in \text{End}(E)$ are such that

$$X_{i}^{\frac{1}{2}} X_{j}^{\frac{1}{2}} = q^{\square} X_{j}^{\frac{1}{2}} X_{i}^{\frac{1}{2}} \text{ and } X_{i}^{\frac{N}{2}} = x_i \text{ Id}_E$$

$$T_N(\mathcal{K}(K)) = \sum_{-N \leq k_i \leq N} (\text{polynomial in } q^{\pm 1}) X_{i_1}^{\pm \frac{k_1}{2}} X_{i_2}^{\pm \frac{k_2}{2}} \cdots X_{i_n}^{\pm \frac{k_n}{2}}$$

About $N^n$ terms.
For the Kauffman $q$–bracket that we constructed,

$$\mathcal{K}(K) = \sum_{\pm \pm \cdots \pm} (0 \text{ or } \pm q^{\square}) X^\pm_{i_1} X^\pm_{i_2} \cdots X^\pm_{i_n}$$

where the matrices $X^\frac{1}{2} \in \text{End}(E)$ are such that

$$X^\frac{1}{2} X^\frac{1}{2} = q^{\square} X^\frac{1}{2} X^\frac{1}{2} \text{ and } X^\frac{N}{2} = x_i \text{ Id}_E$$

Miraculous cancelations when $q^N = 1!$

$$T_N(\mathcal{K}(K)) = \pm \sum_{\pm \pm \cdots \pm} (0 \text{ or } 1) X^{\pm \frac{N}{2}}_{i_1} X^{\pm \frac{N}{2}}_{i_2} \cdots X^{\pm \frac{N}{2}}_{i_n}$$

At most $2^n$ terms.
For the Kauffman $q$–bracket that we constructed,
\[
\mathcal{K}(K) = \sum_{\pm\pm\ldots\pm} (0 \text{ or } \pm q^{\Box}) X_{i_1}^{\pm \frac{1}{2}} X_{i_2}^{\pm \frac{1}{2}} \ldots X_{i_n}^{\pm \frac{1}{2}}
\]
where the matrices $X_{i}^{\frac{1}{2}} \in \text{End}(E)$ are such that
\[
X_{i}^{\frac{1}{2}} X_{j}^{\frac{1}{2}} = q^{\Box} X_{j}^{\frac{1}{2}} X_{i}^{\frac{1}{2}} \quad \text{and} \quad X_{i}^{\frac{N}{2}} = x_{i} \text{Id}_E
\]
Miraculous cancelations when $q^{N} = 1!$
\[
T_{N}(\mathcal{K}(K)) = \pm \sum_{\pm\pm\ldots\pm} (0 \text{ or } 1) X_{i_1}^{\pm \frac{N}{2}} X_{i_2}^{\pm \frac{N}{2}} \ldots X_{i_n}^{\pm \frac{N}{2}}
\]
\[
= \pm \sum_{\pm\pm\ldots\pm} (0 \text{ or } 1) x_{i_1}^{\pm \frac{1}{2}} x_{i_2}^{\pm \frac{1}{2}} \ldots x_{i_n}^{\pm \frac{1}{2}} \text{Id}_E
\]
At most $2^n$ terms.
For the Kauffman $q$–bracket that we constructed,

$$
\mathcal{K}(K) = \sum_{\pm\pm\ldots\pm} (0 \text{ or } \pm q^{\square}) X_{i_1}^{\pm \frac{1}{2}} X_{i_2}^{\pm \frac{1}{2}} \ldots X_{i_n}^{\pm \frac{1}{2}}
$$

where the matrices $X_i^{\frac{1}{2}} \in \text{End}(E)$ are such that

$$
X_i^{\frac{1}{2}} X_j^{\frac{1}{2}} = q^{\square} X_j^{\frac{1}{2}} X_i^{\frac{1}{2}} \text{ and } X_i^{\frac{N}{2}} = x_i \text{Id}_E
$$

Miraculous cancelations when $q^N = 1$!

$$
T_N(\mathcal{K}(K)) = \pm \sum_{\pm\pm\ldots\pm} (0 \text{ or } 1) X_{i_1}^{\pm \frac{N}{2}} X_{i_2}^{\pm \frac{N}{2}} \ldots X_{i_n}^{\pm \frac{N}{2}}
$$

$$
= \pm \sum_{\pm\pm\ldots\pm} (0 \text{ or } 1) x_{i_1}^{\pm \frac{1}{2}} x_{i_2}^{\pm \frac{1}{2}} \ldots x_{i_n}^{\pm \frac{1}{2}} \text{Id}_E = \mathcal{K}_\rho(K) \text{Id}_E
$$

At most $2^n$ terms.
Corollary

The classical shadow of the Kauffman q–bracket $\mathcal{K}$ that we constructed is the character $\mathcal{K}_\rho \in \mathcal{R}_{\text{SL}_2(\mathbb{C})}(S)$ associated to the same edge weights $x_i$ as $\mathcal{K}$.
Corollary

The classical shadow of the Kauffman q–bracket $\mathcal{K}$ that we constructed is the character $\mathcal{K}_\rho \in \mathcal{R}_{SL_2(\mathbb{C})}(S)$ associated to the same edge weights $x_i$ as $\mathcal{K}$

Current proof of miraculous cancelations
Corollary

The classical shadow of the Kauffman $q$–bracket $\mathcal{K}$ that we constructed is the character $\mathcal{K}_\rho \in \mathcal{R}_{SL_2(C)}(S)$ associated to the same edge weights $x_i$ as $\mathcal{K}$

Current proof of miraculous cancelations
Wishful thinking to guess
Corollary

The classical shadow of the Kauffman q–bracket $\mathcal{K}$ that we constructed is the character $\mathcal{K}_\rho \in \mathcal{R}_{\text{SL}_2(\mathbb{C})}(S)$ associated to the same edge weights $x_i$ as $\mathcal{K}$

Current proof of miraculous cancelations
Wishful thinking to guess
Brute force to check
Corollary

The classical shadow of the Kauffman q–bracket $\mathcal{K}$ that we constructed is the character $\mathcal{K}_{\rho} \in \mathcal{R}_{SL_2(\mathbb{C})}(S)$ associated to the same edge weights $x_i$ as $\mathcal{K}$.

Current proof of miraculous cancelations
Wishful thinking to guess
Brute force to check

Better conjecture/future proof
This should come from a deep fact in the representation theory of $U_q(\mathfrak{sl}_2)$ when $q^N = 1$