Kauffman brackets on surfaces

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University of Southern California

Geometric Topology in New York, August 2013

Joint work with Helen Wong

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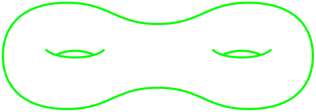
here with Grace Tsapsie Hibbard, born March 22, 2013

S = closed oriented surface of genus $g \geqslant 0$



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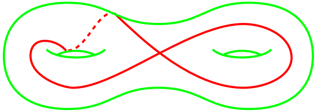

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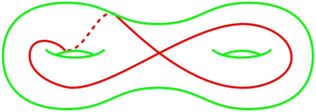

A group homomorphism $\rho \colon \pi_1(S) \to \mathrm{SL}_2(\mathbb{C})$ defines its *character*

$$\mathcal{K}_\rho \colon \{\text{closed curves in } S\} \longrightarrow \mathbb{C}$$
$$K \longmapsto \mathrm{Tr}\, \rho(K)$$

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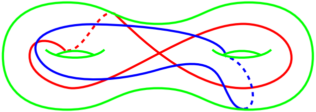

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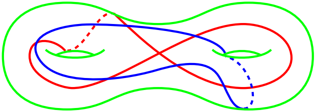
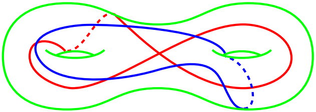
$$\mathcal{K}_\rho\colon \{\text{closed multicurves in } S\} \longrightarrow \mathbb{C}$$

$$K = \bigcup_{i=1}^{n} K_i \qquad \longmapsto (-1)^n \prod_{i=1}^{n} \mathrm{Tr}\, \rho(K_i)$$



Theorem (Helling 1967)

A function $\mathcal{K}\colon \{$*closed multicurves in* $S\} \longrightarrow \mathbb{C}$ *is the character of a group homomorphism* $\rho\colon \pi_1(S) \to \mathrm{SL}_2(\mathbb{C})$ *if and only if:*



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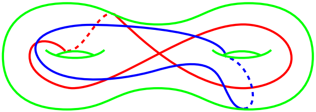

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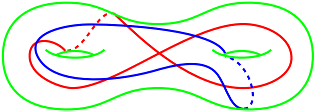

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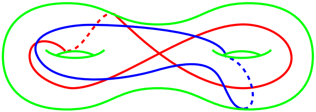

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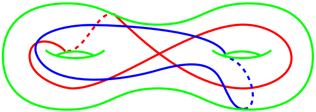
The Skein Relation just rephrases the classical trace relation of $\mathrm{SL}_2(\mathbb{C})$:

$$\mathrm{Tr}\, M \;\mathrm{Tr}\, N = \mathrm{Tr}\, MN + \mathrm{Tr}\, MN^{-1}, \quad \forall M, N \in \mathrm{SL}_2(\mathbb{C})$$



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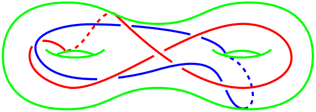

Definition

For $q = \mathrm{e}^{2\pi i \hbar} \in \mathbb{C} - \{0\}$, *a Kauffman q–bracket is a function*

$$\mathcal{K}\colon \{\textit{framed links in } S \times [0,1]\} \longrightarrow \mathrm{End}(E)$$

for a finite-dimensional vector space E*, such that:*

- *(Isotopy Invariance)* $\mathcal{K}(K)$ *depends only on the isotopy class of* K *in* $S \times [0,1]$
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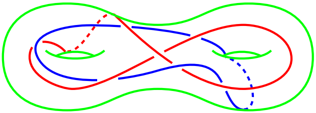

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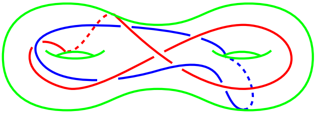

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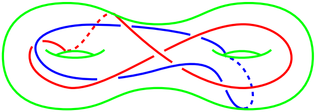

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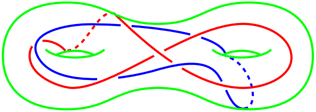

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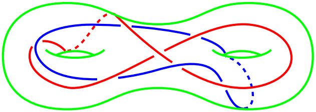

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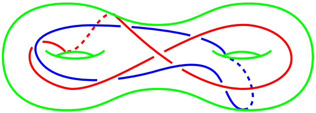

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Goal of this talk: Construct other examples of Kauffman brackets

When $q = 1$ and $q^{\frac{1}{2}} = -1$, an irreducible Kauffman 1–bracket is the same thing as an $\mathrm{SL}_2(\mathbb{C})$–character, namely as a point of the *character variety*

$$\mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S) = \{\text{homomorphisms } \rho\colon \pi_1(S) \to \mathrm{SL}_2(\mathbb{C})\}/\!\!/\mathrm{SL}_2(\mathbb{C})$$

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From quantum to classical (Bonahon-Wong): When $q^N = 1$ with N odd, every irreducible Kauffman q–bracket determines a character $\mathcal{K}_\rho \in \mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$, called the *classical shadow* of the Kauffman bracket.

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Today, from classical to quantum: Realize every character $\mathcal{K}_\rho \in \mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$ as the classical shadow of a Kauffman q–bracket.

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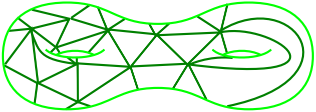

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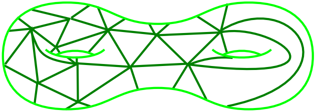

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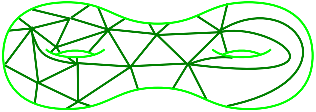

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This defines a pleated surface with shear-bend coordinates x_i, and with monodromy $\rho\colon \pi_1(S - \mathcal{V}_\Gamma) \to \mathrm{PSL}_2(\mathbb{C}) = \mathrm{SL}_2(\mathbb{C})/\pm \mathrm{Id}$

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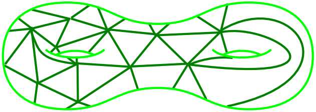

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This defines a pleated surface with shear-bend coordinates x_i, and with monodromy $\rho \colon \pi_1(S - \mathcal{V}_\Gamma) \to \mathrm{PSL}_2(\mathbb{C}) = \mathrm{SL}_2(\mathbb{C})/\pm\mathrm{Id}$

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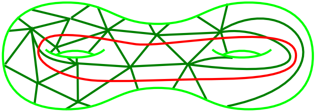
How to construct a group homomorphism $\rho\colon \pi_1(S) \to \mathrm{SL}_2(\mathbb{C})$?

Pick a triangulation Γ of S, with vertex set $\mathcal{V}_\Gamma$



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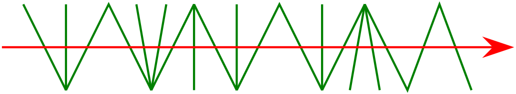
Main Point: The construction is classical and, for a curve $K \subset S - \mathcal{V}_\Gamma$, gives a very explicit formula for $\mathrm{Tr}\,\rho(K)$

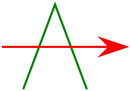
More precisely, if K crosses the edges $e_{i_1}, e_{i_2}, \dots, e_{i_n}$,

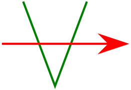


$$\operatorname{Tr} \rho(K) = \pm \operatorname{Tr} \left[M_1 \begin{pmatrix} x_{i_1}^{\frac{1}{2}} & 0 \\ 0 & x_{i_1}^{-\frac{1}{2}} \end{pmatrix} M_2 \begin{pmatrix} x_{i_2}^{\frac{1}{2}} & 0 \\ 0 & x_{i_2}^{-\frac{1}{2}} \end{pmatrix} \dots M_n \begin{pmatrix} x_{i_n}^{\frac{1}{2}} & 0 \\ 0 & x_{i_n}^{-\frac{1}{2}} \end{pmatrix} \right]$$

where

$$M_k = \begin{cases} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} & \text{if} \\ \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} & \text{if} \end{cases}$$





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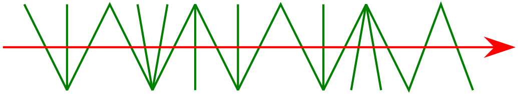

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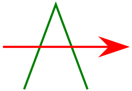

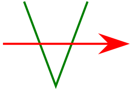

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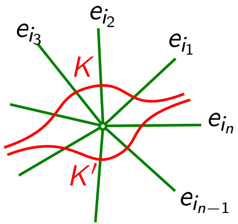

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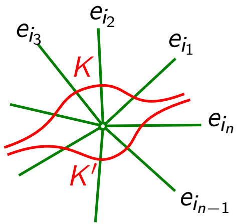

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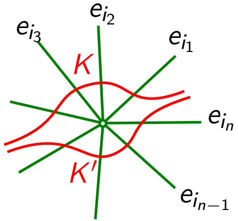

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Fact

The edge weights x_i define an $\mathrm{SL}_2(\mathbb{C})$–character on the closed surface S if and only if, for every vertex,

$$\begin{cases} x_{i_1}^{\frac{1}{2}} x_{i_1}^{\frac{1}{2}} \dots x_{i_1}^{\frac{1}{2}} = -1 \\ 1 + x_{i_1} + x_{i_1}x_{i_2} + x_{i_1}x_{i_2}x_{i_3} + \dots + x_{i_1}x_{i_2}\dots x_{i_{n-1}} = 0 \end{cases}$$

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namely such that:

- (Isotopy Invariance) $\mathcal{K}(K)$ depends only on the isotopy class of K in $S \times [0,1]$
- (Superposition Rule) $\mathcal{K}(K_1 \cup K_2) = \mathcal{K}(K_1) \circ \mathcal{K}(K_2)$ whenever $K = K_1 \cup K_2$ with $K_1 \subset S \times [0, \frac{1}{2}]$ and $K_2 \subset S \times [\frac{1}{2}, 1]$
- (Skein Relation) $\mathcal{K}(K_1) = q^{\frac{1}{2}} \mathcal{K}(K_0) + q^{-\frac{1}{2}} \mathcal{K}(K_\infty)$ if K_1, K_0, K_∞ are the same everywhere, except in a small box where $K_1 =$ , $K_0 =$  and $K_\infty =$ 

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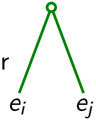

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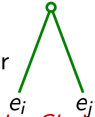

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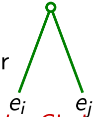

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Proposition (FB + Xiaobo Liu, 2007, relatively easy)

If $q^N = 1$ with N odd, smallest dimensional choices of such operators $X_i^{\frac{1}{2}} \in \mathrm{End}(E)$ are classified by

- *edge weights $x_i \in \mathbb{C}^*$ such that $X_i^{\frac{N}{2}} = x_i^{\frac{1}{2}} \mathrm{Id}_E$*
- *choices of N–roots for numbers $x_{i_1}^{\frac{1}{2}} x_{i_2}^{\frac{1}{2}} \dots x_{i_n}^{\frac{1}{2}} \in \mathbb{C}^*$ associated to the vertices*

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Theorem (FB + Helen Wong, 2011)

Given operators $X_i^{\frac{1}{2}} \in \mathrm{End}(E)$ associated to the edges of the triangulation Γ as in Step 1, there is an explicit formula

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Remark Much harder. Need to worry about the order in which to multiply the operators $X_i^{\frac{1}{2}} \in \mathrm{End}(E)$, which requires the introduction of correction factors $q^{\square}$ related to the classical Kauffman bracket in $\mathbb{R}^3$.

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FB + Qingtao Chen, 2013 More conceptual approach based on the representation theory of the quantum group $\mathrm{U}_q(\mathfrak{sl}_2)$

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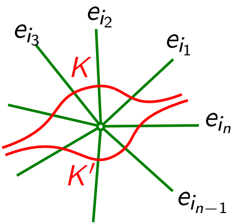

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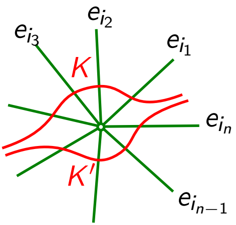

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3. This character induces a character for the *closed* surface S if and only if

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for each vertex

Summary: Recipe to construct $\mathrm{SL}_2(\mathbb{C})$–characters

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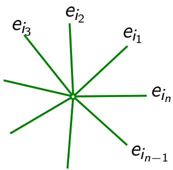
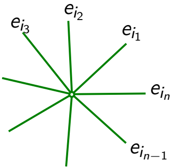
Step 3b. For a vertex $v =$ 

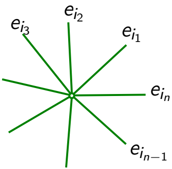
of the triangulation Γ for the operators $X_{i_j}^{\frac{1}{2}} \in \mathrm{End}(E)$ associated to the edges, consider

$$1+qX_{i_1}+q^2X_{i_1}X_{i_2}+q^3X_{i_1}X_{i_2}X_{i_3}+\cdots+q^{n-1}X_{i_1}X_{i_2}\dots X_{i_{n-1}}$$

Step 3b. For a vertex $v =$ 

of the triangulation Γ for the operators $X_{i_j}^{\frac{1}{2}} \in \mathrm{End}(E)$ associated to the edges, set

$$F_v = \ker\left(1 + qX_{i_1} + q^2 X_{i_1} X_{i_2} + q^3 X_{i_1} X_{i_2} X_{i_3} + \cdots + q^{n-1} X_{i_1} X_{i_2} \ldots X_{i_{n-1}}\right)$$

Step 3b. For a vertex $v =$ 

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and

$$F = \bigcap_{\text{vertices v}} F_v \subset E$$

Theorem

1. *The linear subspace $F \subset E$ is invariant under the image of the Kauffman bracket*

$$\mathcal{K} \colon \{\textit{framed links in } (S - \mathcal{V}_\Gamma) \times [0,1]\} \longrightarrow \mathrm{End}(E)$$

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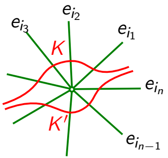

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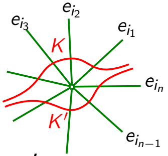

Corollary

$\mathcal{K}$ induces a Kauffman q–bracket
$$\bar{\mathcal{K}}\colon \{\textit{framed links in } S \times [0,1]\} \longrightarrow \mathrm{End}(F)$$
for the closed surface S

Theorem

$$\dim F \geqslant \begin{cases} N^{3(g-1)} & \text{if } g \geqslant 2 \\ N & \text{if } g = 1 \\ 1 & \text{if } g = 0 \end{cases}$$

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depends only on the (classical) $\mathrm{SL}_2(\mathbb{C})$*–character* $\mathcal{K}_\rho \in \mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$ *associated to the same edge weights* $x_i \in \mathbb{C}^*$*. In particular, it is independent of the triangulation* Γ

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with equality for generic (all?) $\mathcal{K}_\rho \in \mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$

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Theorem (Bonahon-Wong, 2012)

When $q^N = 1$ with N odd, every irreducible Kauffman q–bracket

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This is not the (normalized) N–th Chebyshev polynomial of the second type $S_N(x)$ is defined by $\sin N\theta = S_N(2\cos\theta)\sin\theta$ which usually occurs in the representation theory of SL_2 and $\mathrm{U}_q(\mathfrak{sl}_2)$

For the Kauffman q–bracket that we constructed,

$$\mathcal{K}(K) = \sum_{\pm\pm\cdots\pm} (0 \text{ or } \pm q^{\square})\, X_{i_1}^{\pm\frac{1}{2}} X_{i_2}^{\pm\frac{1}{2}} \dots X_{i_n}^{\pm\frac{1}{2}}$$

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About N^n terms.

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Corollary

The classical shadow of the Kauffman q–bracket $\mathcal{K}$ that we constructed is the character $\mathcal{K}_\rho \in \mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$ associated to the same edge weights x_i as $\mathcal{K}$

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Current proof of miraculous cancelations

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Better conjecture/future proof

This should come from a deep fact in the representation theory of $\mathrm{U}_q(\mathfrak{sl}_2)$ when $q^N = 1$