Homework 4

Topology I, Math 70800, Spring 2017

Instructor: Abhijit Champanerkar

Due: Thursday May 4th before class

Reading [1]

1. Splitting Lemma, page 147.
2. Homology of groups and Proposition 2.45, pages 148 - 149.
3. Discussion and example about last part of the proof of Theorem 2A.1 ($H_1(X)$ and $\pi_1(X)$), pages 167-168.
4. *Alexander Horned Sphere* 2B.2, pages 170-171.
5. (Homological Algebra) Lemmas 3.1 (page 194), 3A.1 ($-\otimes G$ is right exact), 3A.2 and 3A.5 (Properties of Tor), pages 262 - 265.
6. *Cohomology of Spaces* - cohomological analogs of theorems we saw for homology, pages 197 - 204.

Problems

1. (Left exactness of Hom) For an abelian group A, denote by $A^* = \mathrm{Hom(A,G)}$. If $A \to B \to C \to 0$, then show that $A^* \leftarrow B^* \leftarrow C^* \leftarrow 0$ is exact.
2. Show that $\mathrm{Tor(A,B)}$ is always a torsion group.
3. Use the Universal Coefficient Theorem to show that if $H_*(X;\mathbb{Z})$ is finitely generated, then for any coefficient field F, we have $\chi(X) = \sum_n (-1)^n \mathrm{dim}\, H_n(X;F)$.
4. Using the Universal Coefficient Theorems (or using the definition), compute the homology and cohomology groups with the given coefficient groups (assume homology with $\mathbb{Z}$ coefficients):
   (a) M_g, the closed orientable surface of genus g with coefficients $\mathbb{Q}$, $\mathbb{Z}$, $\mathbb{Z}_2$, $\mathbb{Z}_4$.

[1] All section, chapter, page and example numbers refer to the book "Algebraic Topology" by Allen Hatcher freely available at `http://www.math.cornell.edu/~hatcher/AT/ATpage.html`

(b) N_g, the closed non-orientable surface of genus g with coefficients $\mathbb{Q}$, $\mathbb{Z}$, $\mathbb{Z}_2$, $\mathbb{Z}_4$.

(c) $\mathbb{R}P^n$ with coefficients $\mathbb{Q}$, $\mathbb{Z}$, $\mathbb{Z}_3$ and $\mathbb{Z}_4$.

5. Compute the cohomology groups of $X = \{p\}$ using the definition.
6. Show that a degree d map of S^n induces an automorphism of $H^n(S^n; G)$ given by multiplication by d.
7. Show that for any map $f : T \to K$, the induced map $f_* : H_2(T; \mathbb{Z}_2) \to H_2(K; \mathbb{Z}_2)$ is trivial.
8. Let $\mathcal{C}$ be a free chain complex. Show that if $H_i(\mathcal{C})$ is finitely generated for all i, then $H_p(\mathcal{C}; \mathbb{Z}_n) \cong H^p(\mathcal{C}; \mathbb{Z}_n)$.
9. Page 204 - 206: Problems 4, 5, 6, 8, 10

Hand-in: 3, 5, 6, 7, 8