Homework 3

Topology I, Math 70800, Spring 2017
Instructor: Abhijit Champanerkar
Due: Thursday April 6th before class

Reading [1]

1. Read Example 2.23 from page 125 which describes the generator of homology of S^n explicitly.
2. Read the proof for infinite dimensional case of (a) equivalence of simplicial and singular homology (page 130), (b) equivalence of cellular and singular homology (pages 138-139).
3. Read details of the proof of Proposition 2.20 (computation of degree in terms of local degrees) on pages 135-136, and the cellular boundary formula on pages 140-141.
4. Read about the homology of Lens Spaces in Example 2.43 on pages 144-145.
5. Read Proposition 2.33, page 137 about degree of suspensions.

Problems

1. Compute the homology of the following spaces using the Mayer-Vietoris sequence.
    (a) Closed surfaces M_g and N_g (orientable and non-orientable surface of genus g).
    (b) The 2-complex obtained by identifying the sides of a n-sided polygon using the word a^n.
    (c) Prove that $\tilde{H}_n(X) \cong \tilde{H}_n(SX)$ where SX is the suspension of X.
2. Compute the cellular homology of the following spaces after choosing an appropriate cell structure.
    (a) Closed surface M_g and N_g (orientable and non-orientable surface of genus g).

[1] All section, chapter, page and example numbers refer to the book "Algebraic Topology" by Allen Hatcher freely available at `http://www.math.cornell.edu/~hatcher/AT/ATpage.html`

(b) Let X be a closed surface given by the identification of sides of an octagon using the word $abca^{-1}dc^{-1}db$. Compute Identify the surface.

(c) S^n with the cell structure with two cells in each dimension and the $k-$skeleton is S^k (described in the last paragraph on page 7).

(d) $S^1 \times (S^1 \vee S^1)$.

(e) $\mathbb{R}P^n/\mathbb{R}P^m$ for $m < n$.

(f) $\mathbb{R}P^\infty$ (described in the last paragraph on page 7).

3. Let X be a CW complex. Prove that $H_n(X^n)$ is free. (Hint: Look at cellular boundary map from $H_n(X^n, X^{n-1})$ to $H_{n-1}(X^{n-1}, X^{n-2})$.

4. Read about products of CW complexes from pages 523-525. Prove that for finite CW complexes X and Y, $\chi(X \times Y) = \chi(X)\,\chi(Y)$.

5. Let X be a finite CW complex and $p : Y \to X$ be a $k-$sheeted cover. Show that $\chi(Y) = k\,\chi(X)$.

6. Using (a), show that M_g covers M_h if and only if $g = n(h-1)+1$, where n is the number of sheets in the covering (M_g is the orientable surface of genus g).

7. Show that any finitely generated abelian group arises as the first homology group of a connected topological space.

8. (Homology of knot complements)

   (a) Let S^1 be the unit circle in $\mathbb{R}^3 \subset S^3 = \mathbb{R}^3 \cup \infty$. Let V be a open solid torus neighbourhood of S^1 and $U = S^3 - S^1$. Using the Mayer-Vietoris sequence for reduced homology of $S^3 = U \cup V$ compute $\widetilde{H}_*(S^3 - S^1)$.

   (b) Let K be any smooth embedding of S^1 in S^3 (called a knot in S^3), and let V be a open solid torus neighbourhood of K (smoothness ensures this tubular neighbourhood) and $U = S^3 - K$. Compute $\widetilde{H}_*(S^3 - K)$ using the Mayer-Vietoris sequence.

   (c) Does the homology detect different embeddings of S^1 in S^3 i.e. does homology detect knottedness ?

Hand-in: 1c, 2bd,5, 8b