Homework 2

Topology I, Math 70800, Spring 2017
Instructor: Abhijit Champanerkar
Due: Tuesday March 14th before class

Reading [1]

1. Read about the idea and motivation for homology from pages 97-101 of the textbook.
2. Proof of Proposition 2.9 on page 111, Section 2.1.
3. Read about finitely generated abelian groups and the computatbility of homology groups from the given handout.
4. Read proof of the Five-Lemma on page 129.
5. Read about *Barycentric Subdivision of Simplices* from pages 119 - 120, and *Iterated Barycentric Subdivision* from page 123.
6. Read about the *Naturality of Exact sequences* on page 127.

Problems

1. (a) For an exact sequence $A \xrightarrow{f} B \to C \to D \xrightarrow{g} E$ show that $C = 0$ iff f is surjective and g is injective.
   (b) Using this prove that the inclusion $A \xrightarrow{i} X$ induces isomorphisms on all homology groups iff $H_n(X, A) = 0$ for all n.
2. Let $r : X \to A$ be a retraction and let $i : A \to X$ be the inclusion map. Show that $i_* : H_*(A) \to H_*(X)$ is a monomorphism onto a direct summand.
3. Show that chain homotopy of chain maps is an equivalence relation.
4. If $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$ is exact, then f is surjective if and only if h is surjective.

[1] All section, chapter, page and example numbers refer to the book "Algebraic Topology" by Allen Hatcher freely available at `http://www.math.cornell.edu/~hatcher/AT/ATpage.html`

5. (a) If $0 \to A \to B \to C \to 0$ is a short exact sequence of of vector spaces and linear maps, then show that $\mathrm{dim\ B} = \mathrm{dim\ A} + \mathrm{dim\ C}$.

   (b) If $0 \to A \to B \to C \to 0$ is a short exact sequence of finitely generated abelian groups, then show that $\mathrm{rank\ B} = \mathrm{rank\ A} + \mathrm{rank\ C}$. (Hint: Extend a maximally independent subset of A to a maximally independent subet of B).

   (c) If $0 \to A_n \to A_{n-1} \to \ldots \to A_1 \to A_0 \to 0$ is an exact squence of finitely generated abelian groups, then $\sum_{i=0}^{n}(-1)^i \mathrm{rank\ A_i} = 0$.

6. For each of the following exact sequences say as much as possible about the abelian group G and/or the unknown homomorphism α.

   (a) $0 \to \mathbb{Z} \to G \to \mathbb{Z} \to 0$

   (b) $0 \to \mathbb{Z} \to G \to \mathbb{Z}_2 \to 0$

   (c) $0 \to \mathbb{Z} \to G \to \mathbb{Z}_n \to 0$

   (d) $0 \to \mathbb{Z}_4 \to G \to \mathbb{Z}_2 \to 0$

   (e) $0 \to \mathbb{Z}_{p^m} \to G \to \mathbb{Z}_{p^n} \to 0$

   (f) $0 \to \mathbb{Z}_3 \to G \to \mathbb{Z}_2 \to \mathbb{Z} \xrightarrow{\alpha} \mathbb{Z} \to 0$

   (g) $0 \to \mathbb{Z} \xrightarrow{\alpha} \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}_2 \to 0$

   (h) $0 \to G \xrightarrow{\alpha} \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z}_2 \to 0$

7. Show that $H_0(X, A) = 0$ iff A meets eah path-component of X.

8. Compute the relative homology groups for the following pairs (X, A).

   (a) $X = S^2$ and $A = \{a_1, \ldots, a_n\}$

   (b) $X = T^2$ and $A =$meridian.

   (c) $X = \mathbb{R}$ and $A = \mathbb{Q}$.

   (d) $X = T^2$ and $A =$ meridian and longitude.

9. Compute the homology of space X gives below, by finding a homotopy equivalent space (usually a deformation retract) Y whose homology you have computed. Please justify the homotopy equivalence (or deformation retract).

   (a) X is orientable surface of genus g with b boundary components.

   (b) X is non-orientable surface of genus g with b boundary components.

   (c) $X = \mathbb{R}^3 - \{(0, 0, z) | z \in \mathbb{R}\}$

   (d) $X = \mathbb{R}^3 - \big(\bigcup_{i=0}^{n} \{(i, 0, z) | z \in \mathbb{R}\}\big)$

(e) $X = \mathbb{R}^3 - \{(x, y, 0) | x^2 + y^2 = 1\}$

(f) X is a torus with n meridional disks attached (a meridional disk a disk which bounds a meridian inside the torus).

(g) $X = \mathbb{R}^3 - \big(\{(0, 0, z) | z \in \mathbb{R}\} \cup \{(x, y, 0) | x^2 + y^2 = 1\}\big)$

Hand-in: 2, 5c, 8ad, 9fg