Quotient Spaces

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- Quotient topology/spaces/maps are also known as Identification topology/spaces/maps.
- $X \cong Y$ means X is homemorphic to Y.

Let X be a set. A **partition** of X is a family of disjoint subsets of X whose union is X. If $\sim$ is an equivalence relation (a relation which is reflexive, symmetric and transitive) on X, then $\sim$ induces a partition of X which consists of equivalence classes.

Let X be a topological space and $\mathcal{P}$ be a partition of X (possibly induced by some equivalence relation). Let Y be the set whose points are elements of $\mathcal{P}$. Y is called a **quotient space** of X. In case $\mathcal{P}$ is induced by a $\sim$ then Y is denoted as $Y = X/\sim$.

Let $\pi : X \to Y$ be the map which sends x to the set in $\mathcal{P}$ which contains it. π is called the projection map. The **quotient topology** on Y is the topology in which $V \subset Y$ is open if and only if $\pi^{-1}(V) \subset X$ is open. It is easy to check that this gives a topology on Y. π is continuous and surjective by definition. If X is compact or connected then so is Y.

A map $f : X \to Y$ is called a **quotient map** if $V \subset Y$ is open if and only if $f^{-1}(V) \subset X$ is open. The projection map is a quotient map. A surjective, continuous, open or closed map is a quotient map. If X is compact and Y is Hausdorff, then any surjective, continuous map is a quotient map.

Note that in Example 1 below, $S^1 \subset \mathbb{R}^2$ and has the subspace topology. Example 1 says that S^1 with the subspace topology is homeomorphic to the quotient space $[0,1]/\sim$. This is not obvious and is proved using Theorem 1. See Section 4.2, *Basic Topology* by Armstrong for more details.

Theorem 1. *Let X be compact and Y be Hausdorff. Let $f : X \to Y$ be a continuous and onto map. Let $X^* = \{f^{-1}(y) \mid y \in Y\}$ and give X^* the quotient topology. Then X^* is homeomorphic to Y.*

Examples:

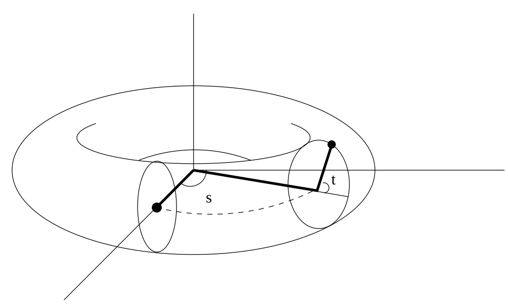
1. $S^1 \cong [0,1]/\sim$ where $0 \sim 1$ and $x \sim x$ for all $x \neq 0, 1$.

2. (a) The torus $T^2 \cong [0,1] \times [0,1]/\sim$ where $(x, 0) \sim (x, 1)$, $(0, y) \sim (1, y)$ for all $x, y \in [0,1]$ and $(x, y) \sim (x, y)$ otherwise.

   (b) (Rigourous) Let T^2 be the torus defined as a quotient space of the square. Let $b > a > 0$. Consider the map $F : [0,1] \times [0,1] \to \mathbb{R}^3$ defined by $F(s,t) = ((b + a\cos(2\pi t))\cos(2\pi s), (b + a\cos(2\pi t))\sin(2\pi s), a\sin(2\pi t))$. Show that:

      i. F is a quotient map onto its image.
      ii. F factors to a map f from T^2 to $\mathbb{R}^3$.
      iii. f is a homeomorphism onto its image.

   Note: The image of F is an imbedding of the torus in $\mathbb{R}^3$ thought of as a boundary of a doughnut. This shows that the torus as a quotient space of $[0,1] \times [0,1]$ is homeomorphic to the torus we draw in $\mathbb{R}^3$.



3. Let X be a topological space and $A \subset X$. Let $\mathcal{P}$ be a partition of X which consists of the sets A and $\{x\}$ for $x \in X - A$. Let X/A denote the quotient space with respect to this partition. In X/A, the set A is identified to a point.

   For example, let $S^n = \{\overline{x} \in \mathbb{R}^{n+1} \mid |\overline{x}| = 1\}$ be the n-sphere and let $D^n = \{\overline{x} \in \mathbb{R}^n \mid |\overline{x}| \leq 1\}$ be the closed unit ball in $\mathbb{R}^n$. Note that the boundary of D^n is S^{n-1}. Then $D^n/S^{n-1} \cong S^n$. This is rigourously proved as follows:

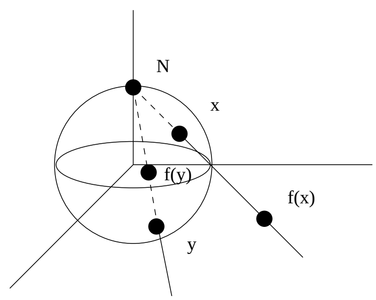
Let $S^n = \{\overline{x} \in \mathbb{R}^{n+1} \mid |\overline{x}| = 1\ \}$. The n-sphere S^n is the set of all unit vectors in $\mathbb{R}^{n+1}$. Let $D^n = \{\overline{x} \in \mathbb{R}^n \mid |\overline{x}| \leq 1\ \}$. D^n is the closed unit ball in $\mathbb{R}^n$. Note that the boundary of D^n is S^{n-1}. This exercise will show that S^n is obtained by identifying the boundary of D^n to one point. Let $N = (0, \ldots, 0, 1) \in S^n$ be the north pole. Define $f : S^n - N \longrightarrow \mathbb{R}^n$ by

$$f(\overline{x}) = (\frac{x_1}{1 - x_{n+1}}, \ldots, \frac{x_n}{1 - x_{n+1}})$$

This map is called the *s*tereographic projection. Define $g : \mathbb{R}^n \rightarrow S^n - N$ by

$$g(\overline{x}) = (\frac{2x_1}{|\overline{x}|^2 + 1}, \ldots, \frac{2x_n}{|\overline{x}|^2 + 1}, \frac{|\overline{x}|^2 - 1}{|\overline{x}|^2 + 1})$$

We can use any any point of S^n instead of the north pole. The picture below illustrates the stereographic projection for S^2.



(a) Show that f and g are continuous and inverses of each other. This shows that $S^n - N$ and $\mathbb{R}^n$ are homeomorphic.

(b) Let B^n be the open unit ball in $\mathbb{R}^n$. Define $h : B^n \longrightarrow \mathbb{R}^n$ and $k : \mathbb{R}^n \longrightarrow B^n$ by $h(\overline{x}) = \dfrac{\overline{x}}{1 - |\overline{x}|}$, $k(\overline{x}) = \dfrac{\overline{x}}{1 + |\overline{x}|}$. Show that h and k are continuous and inverses of each other. This shows that B^n is homeomorphic to $\mathbb{R}^n$.

(c) Define $F : D^n \longrightarrow S^n$ by

$$F(\overline{x}) = \begin{cases} g(h(\overline{x})) & \text{if } |\overline{x}| < 1 \\ N & \text{if } |\overline{x}| = 1 \end{cases}$$

Show that F is a quotient map.

(d) Show that D^n/S^{n-1} is homeomorphic to S^n.

4. (Attaching maps) We can use continuous functions to glue two spaces to each other. Let X and Y be topological spaces, $A \subset Y$ and $f : A \to X$ be a continuous function. Define a relation on $X \sqcup Y$ (disjoint union of X and Y) as follows:

   - $a \sim f(a)$ for $a \in A$
   - $x \sim x$ for $x \in X - f(A)$
   - $y \sim y$ for $x \in Y - A$

   The quotient space $(X \sqcup Y)/\sim$ is denoted by $X \cup_f Y$ and f is called the attaching map. For example:

   (a) $X = Y = D^2$, $A = S^1$ and $f : S^1 \to D^2$ is the inclusion. Then $(X \cup_f Y)/ \cong S^2$

   (b) Let S_1 and S_2 be closed surfaces. For $i = 1, 2$, let B_i be an open neighbourhood of some point in S_i homeomorphic to the open disk in $\mathbb{R}^2$. Then $\partial(S_i - B_i) \simeq S^1$ for $i = 1, 2$. Take any homeomorphism $f : \partial(S_1 - B_1) \to \partial(S_2 - B_2)$. Then $S_1 \cup_f S_2$ is called the **connect sum** of S_1 and S_2 and is independent of the choices of the neighbourhoods and the map f. It is denoted as $S_1 \# S_2$.

   For example S^2 with g handles is homeomorphic to the connect sum of g tori and S^2 with g cross caps is homeomorphic to the connect sum of g cross surfaces (projective planes).

5. (Projective spaces) Define the following quotient spaces:

   - For $\overline{x} \in S^n$, let $\overline{x} \sim -\overline{x}$. Let $X = S^n/\sim$.
   - For $\overline{x} \in D^n$, let $\overline{x} \sim -\overline{x}$ if $|\overline{x}| = 1$ otherwise $\overline{x} \sim \overline{x}$. Let $Y = D^n/\sim$.
   - For $\overline{x}, \overline{y} \in \mathbb{R}^{n+1} - \{\overline{0}\}$, let $\overline{x} \sim \overline{y}$ if $\overline{y} = \alpha\,\overline{x}$ for some non-zero real α. Let $Z = \mathbb{R}^{n+1}/\sim$.

   Show that

(a) The spaces X, Y and Z are homeomorphic to each other. This space is known as the n-dimensional real projective space and denoted by $\mathbb{RP}^n$.

(b) $\mathbb{RP}^2$ is homeomorphic to the projective plane P^2.