Homework 6

Topology I, Math 70800, Spring 2021

Instructor: Abhijit Champanerkar

Due: Friday May 28th

Topic: Cup products, Duality and Products

Reading [1]

1. Read the examples 3.7 (orientable surfaces), 3.8 (non-orientable surfaces), 3.9 on pages 207-209, 3.15 on page 217.
2. Read about relative cup products discussed on page 209.
3. Read about comology rings on page 211-212.
4. Read statements of Theorems 3.12 about cohomology rings of $\mathbb{R}\mathrm{P}^n$ and $\mathbb{C}\mathrm{P}^n$, and use them in problems below.
5. Read the details of the proof of Theorem 3.14.
6. Read Page 230- 233, Introduction to Poincare duality.

Problems

1. Show that the homology and cohomology groups of the following pairs of spaces are the same. Are the spaces homeomorphic to each other ? Are the spaces homotopic to each other ?

   (a) $X = \mathbb{C}\mathrm{P}^2$, $Y = S^2 \vee S^4$

   (b) $X = \mathbb{R}\mathrm{P}^3$, $Y = \mathbb{R}\mathrm{P}^2 \vee S^3$

   (c) $X = \mathbb{R}^3 - \big(\{(0,0,z) | z \in \mathbb{R}\} \cup \{(x,y,0) | x^2 + y^2 = 1\}\big)$ (complement of the Hopf link in S^3), $Y = \mathbb{R}^3 - \big(\{(0,0,z) | z \in \mathbb{R}\} \cup \{(x,y,0) | (x-2)^2 + y^2 = 1\}\big)$ (complement of the unlink on two components in S^3).

[1] All section, chapter, page and example numbers refer to the book "Algebraic Topology" by Allen Hatcher freely available at `http://www.math.cornell.edu/~hatcher/AT/ATpage.html`

2. Use the ideas in Examples 3.7 and 3.8 to construct co-cycles and cup products for the following CW complexes with given coefficients obtained by identifying a polygon using the following words.

   (a) a^5 with $\mathbb{Z}_5$ coefficients.

   (b) a^3b^3 with $\mathbb{Z}_3$ coefficients.

3. Let $X = S^2$, $Y = T^2$, the torus and Z denote the Klein bottle.

   (a) Prove that $f^* : H^2(Y;\mathbb{Z}) \to H^2(X;\mathbb{Z})$ is trivial for any map $f : X \to Y$.

   (b) Prove that $f_* : H_2(X;\mathbb{Z}) \to H_2(Y;\mathbb{Z})$ is trival.

   (c) Can you say the same about g^* for any map $g : Y \to X$.

   (d) Prove that $f^* : H^2(Z;\mathbb{Z}_2) \to H^2(X;\mathbb{Z}_2)$ is trivial for any map $f : X \to Z$.

   (e) Prove that $f^* : H^2(Y;\mathbb{Z}_2) \to H^2(Z;\mathbb{Z}_2)$ is trivial for any map $f : Z \to Y$.

   (f) Prove that $f^* : H^2(Z;\mathbb{Z}_2) \to H^2(Y;\mathbb{Z}_2)$ is trivial for any map $f : Y \to Z$.

4. Let M_g denote the closed orientable surface of genus $g \geq 1$.

   (a) Prove that for each nonzero $\alpha \in H^1(M_g;\mathbb{Z})$, there exists $\beta \in H^1(M_g;\mathbb{Z})$ with $\alpha \cup \beta \neq 0$.

   (b) Prove that for any map $f : M_m \to M_n$ with $n > m$, the map $f^* : H^2(M_n) \to H^2(M_m)$ is trivial.

   (c) Prove that M_g is not homotopy equivalent to a wedge sum $X \vee Y$ of CW complexes, each with nontrivial reduced homology.

5. Prove that a degree 1 map between manifolds induces a surjection on the fundamental groups.

6. Let M be an odd dimensional manifold with boundary. Show that $2\chi(M) = \chi(\partial M)$ (Hint: Double the manifold along its boundary).

7. Page 228-230: 1, 2, 3, 8

8. Page 257 - 258: 2, 3, 4, 5, 6, 7, 9, 11

9. Page 280: 1, 4

Hand-in:1b, 2b, 3a, 4a, 6ab(page 258).