Homework 5

Topology I, Math 70800, Spring 2021

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Due: Monday May 3rd

(upload pdf scan on CUNY BlackBoard in Homework Section)

Topic: Universal Coefficient Theorem

Reading [1]

1. (Split sequences) Read the proof of the Splitting Lemma, page 147.
2. (Homology of Groups) Read in detail about Homology of groups and Proposition 2.45, pages 148 - 149.
3. Read the discussion and example which will help understand the last part of Theorem 2A.1 ($H_1(X)$ and $\pi_1(X)$), page 168.
4. (Homological Algebra) Read proof of Lemmas 3.1 (page 194) about uniqueness of free resolutions.
5. (Right exactness of $-\otimes G$, Tor) Read proofs of Lemmas 3A.1, 3A.2 and Proposition 3A.5 (Properties of Tor), pages 262 - 265.

Problems

1. (Left exactness of Hom) For an abelian group A, denote by $A^* = \mathrm{Hom}(\mathrm{A}, \mathrm{G})$. If $A \to B \to C \to 0$, then show that $A^* \leftarrow B^* \leftarrow C^* \leftarrow 0$ is exact.
2. Show that $\mathrm{Tor}(\mathrm{A}, \mathrm{B})$ is always a torsion group.
3. Use the Universal Coefficient Theorem to show that if $H_*(X;\mathbb{Z})$ is finitely generated, then for any coefficient field F, we have $\chi(X) = \sum_n (-1)^n \dim\, H_n(X;F)$.
4. Using the Universal Coefficient Theorems (or using the definition), compute the homology and cohomology groups with the given coefficient groups (assume homology with $\mathbb{Z}$ coefficients):

[1] All section, chapter, page and example numbers refer to the book "Algebraic Topology" by Allen Hatcher freely available at `http://www.math.cornell.edu/~hatcher/AT/ATpage.html`

(a) M_g, the closed orientable surface of genus g with coefficients $\mathbb{Q}$, $\mathbb{Z}$, $\mathbb{Z}_2$, $\mathbb{Z}_4$.

(b) N_g, the closed non-orientable surface of genus g with coefficients $\mathbb{Q}$, $\mathbb{Z}$, $\mathbb{Z}_2$, $\mathbb{Z}_4$.

(c) $\mathbb{R}P^n$ with coefficients $\mathbb{Q}$, $\mathbb{Z}$, $\mathbb{Z}_3$ and $\mathbb{Z}_4$.

5. Compute the cohomology groups of $X = \{p\}$ using the definition.

6. Show that a degree d map of S^n induces an automorphism of $H^n(S^n; G)$ given by multiplication by d.

7. Show that for any map $f : T \to K$, the induced map $f_* : H_2(T; \mathbb{Z}_2) \to H_2(K; \mathbb{Z}_2)$ is trivial.

8. Let $\mathcal{C}$ be a free chain complex. Show that if $H_i(\mathcal{C})$ is finitely generated for all i, then $H_p(\mathcal{C}; \mathbb{Z}_n) \cong H^p(\mathcal{C}; \mathbb{Z}_n)$.

9. Page 204 - 206: Problems 4, 5, 6, 8

Hand-in: 2, 4b, 5, 7, 8