Homework 3

Topology I, Math 70800, Spring 2021

Instructor: Abhijit Champanerkar

Due: Monday Mar 29th

(upload pdf scan on CUNY BlackBoard in Homework Section)

Topic: Mayer-Vietoris sequence, cellular homology

Reading [1]

1. Read Example 2.23 from page 125 which describes the generator of homology of S^n explicitly.

2. Read proof of the Five-Lemma on page 129.

3. Read the proof for infinite dimensional case of (a) equivalence of simplicial and singular homology (page 130), (b) equivalence of cellular and singular homology (pages 138-139).

4. Read details of the proof of Proposition 2.20 (computation of degree in terms of local degrees) on pages 135-136, and the cellular boundary formula on pages 140-141.

5. Read Proposition 2.33, page 137 about degree of suspensions.

Problems

1. Compute the homology of the following spaces using the Mayer-Vietoris sequence.

   (a) Closed surfaces M_g and N_g (orientable and non-orientable surface of genus g).

   (b) The 2-complex obtained by identifying the sides of a n-sided polygon using the word a^n.

   (c) Prove that $\tilde{H}_n(X) \cong \tilde{H}_n(SX)$ where SX is the suspension of X.

2. Compute the cellular homology of the following spaces after choosing an appropriate cell structure.

[1] All section, chapter, page and example numbers refer to the book "Algebraic Topology" by Allen Hatcher freely available at `http://www.math.cornell.edu/~hatcher/AT/ATpage.html`

(a) Closed surface M_g and N_g (orientable and non-orientable surface of genus g).

(b) Let X be a closed surface given by the identification of sides of an octagon using the word $abca^{-1}dc^{-1}db$. Compute Identify the surface.

(c) S^n with the cell structure with two cells in each dimension and the $k-$skeleton is S^k (described in the last paragraph on page 7).

(d) $S^1 \times (S^1 \vee S^1)$.

(e) $\mathbb{R}P^n/\mathbb{R}P^m$ for $m < n$.

(f) $\mathbb{R}P^\infty$ (described in the last paragraph on page 7).

3. Let X be a CW complex. Prove that $H_n(X^n)$ is free. (Hint: Look at cellular boundary map from $H_n(X^n, X^{n-1})$ to $H_{n-1}(X^{n-1}, X^{n-2})$.

4. Read about products of CW complexes from pages 523-525. Prove that for finite CW complexes X and Y, $\chi(X \times Y) = \chi(X)\, \chi(Y)$.

5. (a) Let X be a finite CW complex and $p : Y \to X$ be a $k-$sheeted cover. Show that $\chi(Y) = k\, \chi(X)$.

   (b) Using (a), show that M_g covers M_h if and only if $g = n(h-1)+1$, where n is the number of sheets in the covering (M_g is the orientable surface of genus g).

6. Show that any finitely generated abelian group arises as the first homology group of a connected topological space.

7. Section 2.1, page 131-133, Problems 17, 18, 22, 29.

Hand-in: 1c, 2bd,5ab, 29