Homework 2

Topology I, Math 70800, Spring 2021
Instructor: Abhijit Champanerkar
Due: Monday Mar 8th
(upload pdf scan on CUNY BlackBoard in Homework Section)
Topic: Relative homology, exact sequences

Reading [1]

1. Read about the idea and motivation for homology from pages 97-101 of the textbook.

2. Proof of Proposition 2.9 on page 111, Section 2.1.

3. Read about finitely generated abelian groups and the computatbility of homology groups attached to this homework.

4. Read proof of the Five-Lemma on page 129.

5. Read about *Barycentric Subdivision of Simplices* from pages 119 - 120, and *Iterated Barycentric Subdivision* from page 123.

6. Read about the *Naturality of Exact sequences* on page 127.

Problems

1. Show that $H_0(X, A) = 0$ iff A meets eah path-component of X.

2. Compute the relative homology groups for the following pairs (X, A).

   (a) $X = S^2$ and $A = \{a_1, \ldots, a_n\}$

   (b) $X = T^2$ and $A =$meridian.

   (c) $X = \mathbb{R}$ and $A = \mathbb{Q}$.

   (d) $X = T^2$ and $A =$ meridian and longitude.

[1] All section, chapter, page and example numbers refer to the book "Algebraic Topology" by Allen Hatcher freely available at `http://www.math.cornell.edu/~hatcher/AT/ATpage.html`

3. Compute the homology of space X gives below, by finding a homotopy equivalent space (usually a deformation retract) Y whose homology you have computed. Please justify the homotopy equivalence (or deformation retract).

   (a) X is orientable surface of genus g with b boundary components.

   (b) X is non-orientable surface of genus g with b boundary components.

   (c) $X = \mathbb{R}^3 - \{(0,0,z)|z \in \mathbb{R}\}$

   (d) $X = \mathbb{R}^3 - \big(\bigcup_{i=0}^{n}\{(i,0,z)|z \in \mathbb{R}\}\big)$

   (e) $X = \mathbb{R}^3 - \{(x,y,0)|x^2+y^2=1\}$

   (f) X is a torus with n meridional disks attached (a meridional disk a disk which bounds a meridian inside the torus).

   (g) $X = \mathbb{R}^3 - \big(\{(0,0,z)|z \in \mathbb{R}\} \cup \{(x,y,0)|x^2+y^2=1\}\big)$

4. (a) For an exact sequence $A \xrightarrow{f} B \to C \to D \xrightarrow{g} E$ show that $C = 0$ iff f is surjective and g is injective.

   (b) Using this prove that the inclusion $A \xrightarrow{i} X$ induces isomorphisms on all homology groups iff $H_n(X,A) = 0$ for all n.

5. Let $r : X \to A$ be a retraction and let $i : A \to X$ be the inclusion map. Show that $i_* : H_*(A) \to H_*(X)$ is a monomorphism onto a direct summand.

6. Show that chain homotopy of chain maps is an equivalence relation.

Problems below are practice problems on exact sequences.

7. If $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$ is exact, then f is surjective if and only if h is surjective.

8. For each of the following exact sequences say as much as possible about the abelian group G and/or the unknown homomorphism α.

   (a) $0 \to \mathbb{Z} \to G \to \mathbb{Z} \to 0$

   (b) $0 \to \mathbb{Z} \to G \to \mathbb{Z}_2 \to 0$

   (c) $0 \to \mathbb{Z} \to G \to \mathbb{Z}_n \to 0$

   (d) $0 \to \mathbb{Z}_4 \to G \to \mathbb{Z}_2 \to 0$

   (e) $0 \to \mathbb{Z}_{p^m} \to G \to \mathbb{Z}_{p^n} \to 0$

   (f) $0 \to \mathbb{Z}_3 \to G \to \mathbb{Z}_2 \to \mathbb{Z} \xrightarrow{\alpha} \mathbb{Z} \to 0$

   (g) $0 \to \mathbb{Z} \xrightarrow{\alpha} \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}_2 \to 0$

   (h) $0 \to G \xrightarrow{\alpha} \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z}_2 \to 0$

9. (a) If $0 \to A \to B \to C \to 0$ is a short exact sequence of of vector spaces and linear maps, then show that $\mathrm{dim\ B = dim\ A + dim\ C}$.

   (b) If $0 \to A \to B \to C \to 0$ is a short exact sequence of finitely generated abelian groups, then show that $\mathrm{rank\ B = rank\ A + rank\ C}$. (Hint: Extend a maximally independent subset of A to a maximally independent subet of B).

   (c) If $0 \to A_n \to A_{n-1} \to \ldots \to A_1 \to A_0 \to 0$ is an exact squence of finitely generated abelian groups, then $\sum_{i=0}^{n}(-1)^i \mathrm{rank\ A_i} = 0$.

Hand-in: 2ad, 3fg, 9c

In particular, if $G = G_1 \oplus G_2$, then $G/G_1 \simeq G_2$.

Of course, one can have $G/G_1 \simeq G_2$ without its following that $G = G_1 \oplus G_2$; that is, G_1 may be a subgroup of G without being a direct summand in G. For instance, the subgroup $n\mathbf{Z}$ of the integers is not a direct summand in $\mathbf{Z}$, for that would mean that

$$\mathbf{Z} \simeq n\mathbf{Z} \oplus G_2$$

for some subgroup G_2 of $\mathbf{Z}$. But then G_2 is isomorphic to $\mathbf{Z}/n\mathbf{Z}$, which is a group of finite order, while no subgroup of $\mathbf{Z}$ has finite order.

Incidentally, we shall denote the group $\mathbf{Z}/n\mathbf{Z}$ of integers modulo n simply by $\mathbf{Z}/n$, in accordance with current usage.

The fundamental theorem of finitely generated abelian groups

There are actually two theorems that are important to us. The first is a theorem about subgroups of free abelian groups. We state it here, and give a proof in §11:

Theorem 4.2. *Let F be a free abelian group. If R is a subgroup of F, then R is also a free abelian group. If F has rank n, then R has rank $r \leq n$; furthermore, there is a basis $e_1, \ldots, e_n$ for F and integers $t_1, \ldots, t_k$ with $t_i > 1$ such that*

(1) *$t_1e_1, \ldots, t_ke_k, e_{k+1}, \ldots, e_r$ is a basis for R.*

(2) *$t_1 \mid t_2 \mid \cdots \mid t_k$, that is, t_i divides t_{i+1} for all i.*

The integers $t_1, \ldots, t_k$ are uniquely determined by F and R, although the basis $e_1, \ldots, e_n$ is not.

An immediate corollary of this theorem is the following:

Theorem 4.3 (The fundamental theorem of finitely generated abelian groups). *Let G be a finitely generated abelian group. Let T be its torsion subgroup.*

(a) *There is a free abelian subgroup H of G having finite rank β such that $G = H \oplus T$.*

(b) *There are finite cyclic groups $T_1, \ldots, T_k$, where T_i has order $t_i > 1$, such that $t_1 \mid t_2 \mid \cdots \mid t_k$ and*

$$T = T_1 \oplus \cdots \oplus T_k.$$

(c) *The numbers β and $t_1, \ldots, t_k$ are uniquely determined by G.*

The number β is called the **betti number** of G; the numbers $t_1, \ldots, t_k$ are called the **torsion coefficients** of G. Note that β is the rank of the free abelian group $G/T \simeq H$. The rank of the subgroup H and the orders of the subgroups T_i are uniquely determined, but the subgroups themselves are not.

Proof. Let S be a finite set of generators $\{g_i\}$ for G; let F be the free abelian group on the set S. The map carrying each g_i to itself extends to a homomorphism carrying F *onto* G. Let R be the kernel of this homomorphism. Then $F/R \simeq G$. Choose bases for F and R as in Theorem 4.2. Then

$$F = F_1 \oplus \cdots \oplus F_n$$

where F_i is infinite cyclic with generator e_i; and

$$R = t_1 F_1 \oplus \cdots \oplus t_k F_k \oplus F_{k+1} \oplus \cdots \oplus F_r.$$

We compute the quotient group as follows:

$$F/R \simeq (F_1/t_1F_1 \oplus \cdots \oplus F_k/t_kF_k) \oplus (F_{r+1} \oplus \cdots \oplus F_n).$$

Thus there is an isomorphism

$$f : G \to (\mathbf{Z}/t_1 \oplus \cdots \oplus \mathbf{Z}/t_k) \oplus (\mathbf{Z} \oplus \cdots \oplus \mathbf{Z}).$$

The torsion subgroup T of G must be mapped to the subgroup $\mathbf{Z}/t_1 \oplus \cdots \oplus \mathbf{Z}/t_k$ by f, since any isomorphism preserves torsion subgroups. Parts (a) and (b) of the theorem follow. Part (c) is left to the exercises. □

This theorem shows that any finitely generated abelian group G can be written as a finite direct sum of cyclic groups; that is,

$$G \simeq (\mathbf{Z} \oplus \cdots \oplus \mathbf{Z}) \oplus \mathbf{Z}/t_1 \oplus \cdots \oplus \mathbf{Z}/t_k.$$

with $t_i > 1$ and $t_1 \mid t_2 \mid \cdots \mid t_k$. This representation is in some sense a "canonical form" for G. There is another such canonical form, derived as follows:

Recall first the fact that if m and n are relatively prime positive integers, then

$$\mathbf{Z}/m \oplus \mathbf{Z}/n \simeq \mathbf{Z}/mn.$$

It follows that any finite cyclic group can be written as a direct sum of cyclic groups whose orders are powers of primes. Theorem 4.3 then implies that for any finitely generated group G,

$$G \simeq (\mathbf{Z} \oplus \cdots \oplus \mathbf{Z}) \oplus (\mathbf{Z}/a_1 \oplus \cdots \oplus \mathbf{Z}/a_s)$$

where each a_i is a power of a prime. This is another canonical form for G, since the numbers a_i are uniquely determined by G (up to a rearrangement), as we shall see. The numbers a_i are called the **invariant factors** of G.

EXERCISES

1. Show that if G is a finitely generated abelian group, every subgroup of G is finitely generated. (This result does not hold for non-abelian groups.)
2. (a) Show that if G is free, then G is torsion-free.
   (b) Show that if G is finitely generated and torsion-free, then G is free.

*§11. THE COMPUTABILITY OF HOMOLOGY GROUPS

We have computed the homology groups of some familiar spaces, such as the sphere and the torus and the Klein bottle. Now we ask the question whether one can in fact compute homology groups in general. For finite complexes, the answer is affirmative. In this section, we present an explicit algorithm for carrying out the computation.

First, we prove a basic theorem giving a "normal form" for homomorphisms of finitely generated free abelian groups. The proof is constructive in nature. One corollary is the theorem about subgroups of free abelian groups that we stated earlier as Theorem 4.2. A second corollary is a theorem concerning standard bases for free chain complexes. And a third corollary gives our desired algorithm for computing the homology groups of a finite complex.

First, we need two lemmas with which you might already be familiar.

Lemma 11.1. *Let A be a free abelian group of rank n. If B is a subgroup of A, then B is free abelian of rank $r \leq n$.*

Proof. We may without loss of generality assume that B is a subgroup of the n-fold direct product $\mathbf{Z}^n = \mathbf{Z} \times \cdots \times \mathbf{Z}$. We construct a basis for B as follows:

Let $\pi_i : \mathbf{Z}^n \to \mathbf{Z}$ denote projection on the ith coordinate. For each $m \leq n$, let B_m be the subgroup of B defined by the equation

$$B_m = B \cap (\mathbf{Z}^m \times \mathbf{0}).$$

That is, B_m consists of all $\mathbf{x} \in B$ such that $\pi_i(\mathbf{x}) = 0$ for $i > m$. In particular, $B_n = B$. Now the homomorphism

$$\pi_m : B_m \to \mathbf{Z}$$

carries B_m onto a subgroup of $\mathbf{Z}$. If this subgroup is trivial, let $\mathbf{x}_m = \mathbf{0}$; otherwise, choose $\mathbf{x}_m \in B_m$ so that its image $\pi_m(\mathbf{x}_m)$ generates this subgroup. We assert that the non-zero elements of the set $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ form a basis for B.

First, we show that for each m, the elements $\mathbf{x}_1, \ldots, \mathbf{x}_m$ generate B_m. (Then, in particular, the elements $\mathbf{x}_1, \ldots, \mathbf{x}_n$ generate B.) It is trivial that $\mathbf{x}_1$ generates B_1; indeed if d is the integer $\pi_1(\mathbf{x}_1)$, then

$$\mathbf{x}_1 = (d, 0, \ldots, 0)$$

and B_1 consists of all multiples of this element.

Assume that $\mathbf{x}_1, \ldots, \mathbf{x}_{m-1}$ generate B_{m-1}; let $\mathbf{x} \in B_m$. Now $\pi_m(\mathbf{x}) = k\pi_m(\mathbf{x}_m)$ for some integer k. It follows that

$$\pi_m(\mathbf{x} - k\mathbf{x}_m) = 0,$$

so that $\mathbf{x} - k\mathbf{x}_m$ belongs to B_{m-1}. Then

$$\mathbf{x} - k\mathbf{x}_m = k_1\mathbf{x}_1 + \cdots + k_{m-1}\mathbf{x}_{m-1}$$

by the induction hypothesis. Hence $\mathbf{x}_1, \ldots, \mathbf{x}_m$ generate B_m.

Second, we show that for each m, the non-zero elements in the set $\{\mathbf{x}_1, \ldots, \mathbf{x}_m\}$ are independent. The result is trivial when $m = 1$. Suppose it true for $m - 1$. Then we show that if

$$\lambda_1 \mathbf{x}_1 + \cdots + \lambda_m \mathbf{x}_m = \mathbf{0},$$

then it follows that for each i, $\lambda_i = 0$ whenever $\mathbf{x}_i \neq \mathbf{0}$; independence follows.

Applying the map π_m, we derive the equation

$$\lambda_m \pi_m(\mathbf{x}_m) = 0.$$

From this equation, it follows that either $\lambda_m = 0$ or $\mathbf{x}_m = \mathbf{0}$. For if $\lambda_m \neq 0$, then $\pi_m(\mathbf{x}_m) = 0$, whence the subgroup $\pi_m(B_m)$ is trivial and $\mathbf{x}_m = \mathbf{0}$ by definition. We conclude two things:

$$\lambda_m = 0 \quad \text{if} \quad \mathbf{x}_m \neq \mathbf{0},$$

$$\lambda_1 \mathbf{x}_1 + \cdots + \lambda_{m-1} \mathbf{x}_{m-1} = \mathbf{0}.$$

The induction hypothesis now applies to show that for $i < m$,

$$\lambda_i = 0 \quad \text{whenever} \quad \mathbf{x}_i \neq \mathbf{0}. \quad \square$$

For later use, we generalize this result to arbitrary free abelian groups:

Lemma 11.2. *If A is a free abelian group, any subgroup B of A is free.*

Proof. The proof given for the finite case generalizes, provided we assume that the basis for A is indexed by a well-ordered set J having a largest element. (And the well-ordering theorem, which is equivalent to the axiom of choice, tells us this assumption is justified.)

We begin by assuming A equals a direct sum of copies of $\mathbf{Z}$; that is, A equals the subgroup of the cartesian product $\mathbf{Z}^J$ consisting of all tuples $(n_\alpha)_{\alpha \in J}$ such that $n_\alpha = 0$ for all but finitely many α. Then we proceed as before.

Let B be a subgroup of A. Let B_β consist of those elements $\mathbf{x}$ of B such that $\pi_\alpha(\mathbf{x}) = 0$ for $\alpha > \beta$. Consider the subgroup $\pi_\beta(B_\beta)$ of $\mathbf{Z}$; if it is trivial define $\mathbf{x}_\beta = \mathbf{0}$, otherwise choose $\mathbf{x}_\beta \in B_\beta$ so $\pi_\beta(\mathbf{x}_\beta)$ generates the subgroup.

We show first that the set $\{\mathbf{x}_\alpha \mid \alpha \leq \beta\}$ generates B_β. This fact is trivial if β is the smallest element of J. We prove it in general by transfinite induction. Given $\mathbf{x} \in B_\beta$, we have

$$\pi_\beta(\mathbf{x}) = k\pi_\beta(\mathbf{x}_\beta)$$

for some integer k. Hence $\pi_\beta(\mathbf{x} - k\mathbf{x}_\beta) = 0$. Consider the set of those indices α for which $\pi_\alpha(\mathbf{x} - k\mathbf{x}_\beta) \neq 0$. (If there are none, $\mathbf{x} = k\mathbf{x}_\beta$ and we are through.) All of these indices are less than β, because $\mathbf{x}$ and $\mathbf{x}_\beta$ belong to B_β. Furthermore, this set of indices is *finite*, so it has a largest element γ, which is less than β. But this means that $\mathbf{x} - k\mathbf{x}_\beta$ belongs to B_γ, whence by the induction hypothesis, $\mathbf{x} - k\mathbf{x}_\beta$ can be written as a linear combination of elements $\mathbf{x}_\alpha$ with each $\alpha \leq \gamma$.

Second, we show that the non-zero elements in the set $\{\mathbf{x}_\alpha \mid \alpha \leq \beta\}$ are independent. Again, this fact is trivial if β is the smallest element of J. In general, suppose

$$\lambda_{\alpha_1}\mathbf{x}_{\alpha_1} + \cdots + \lambda_{\alpha_k}\mathbf{x}_{\alpha_k} + \lambda_\beta \mathbf{x}_\beta = \mathbf{0},$$

where $\alpha_i < \beta$. Applying π_β, we see that

$$\lambda_\beta \pi_\beta(\mathbf{x}_\beta) = 0.$$

As before, it follows that either $\lambda_\beta = 0$ or $\mathbf{x}_\beta = \mathbf{0}$. We conclude that

$$\lambda_\beta = 0 \qquad \text{if} \qquad \mathbf{x}_\beta \neq \mathbf{0},$$

and

$$\lambda_{\alpha_1}\mathbf{x}_{\alpha_1} + \cdots + \lambda_{\alpha_k}\mathbf{x}_{\alpha_k} = \mathbf{0}.$$

The induction hypothesis now implies that $\lambda_{\alpha_i} = 0$ whenever $\mathbf{x}_{\alpha_i} \neq \mathbf{0}$. $\square$

We now prove our basic theorem. First we need a definition.

Definition. Let G and G' be free abelian groups with bases $a_1, \ldots, a_n$ and $a_1', \ldots, a_m'$, respectively. If $f : G \to G'$ is a homomorphism, then

$$f(a_j) = \sum_{i=1}^{m} \lambda_{ij} a_i'$$

for unique integers λ_{ij}. The matrix (λ_{ij}) is called the **matrix of** f relative to the given bases for G and G'.

Theorem 11.3. *Let G and G' be free abelian groups of ranks n and m, respectively; let $f : G \to G'$ be a homomorphism. Then there are bases for G and G' such that, relative to these bases, the matrix of f has the form*

$$B = \left[\begin{array}{ccc|c} b_1 & & 0 & \\ & \ddots & & 0 \\ 0 & & b_l & \\ \hline & 0 & & 0 \end{array}\right]$$

where $b_i \geq 1$ and $b_1 \mid b_2 \mid \cdots \mid b_l$.

This matrix is in fact uniquely determined by f (although the bases involved are not). It is called a **normal form** for the matrix of f.

Proof. We begin by choosing bases in G and G' arbitrarily. Let A be the matrix of f relative to these bases. We shall give shortly a procedure for modifying these bases so as to bring the matrix into the normal form described. It is called "the reduction algorithm." The theorem follows. □

basis of G'

row reduction

Consider the following "elementary row operations" on an integer matrix A:

(1) Exchange row i and row k.

(2) Multiply row i by -1.

(3) Replace row i by (row i) + q(row k), where q is an integer and $k \neq i$.

Each of these operations corresponds to a change of basis in G'. The first corresponds to an exchange of a_i' and a_k'. The second corresponds to replacing a_i' by $-a_i'$. And the third corresponds to replacing a_k' by $a_k' - qa_i'$, as you can readily check.

G basis of

column operation

There are three similar "column operations" on A that correspond to changes of basis in G.

We now show how to apply these six operations to an arbitrary matrix A so as to reduce it to our desired normal form. We assume A is not the zero matrix, since in that case the result is trivial.

Before we begin, we note the following fact: If c is an integer that divides each entry of the matrix A, and if B is obtained from A by applying any one of these elementary operations, then c also divides each entry of B.

The reduction algorithm

Given a matrix $A = (a_{ij})$ of integers, not all zero, let $\alpha(A)$ denote the smallest non-zero element of the set of numbers $|a_{ij}|$. We call a_{ij} a **minimal entry** of A if $|a_{ij}| = \alpha(A)$.

The reduction procedure consists of two steps. The first brings the matrix to a form where $\alpha(A)$ is as small as possible. The second reduces the dimensions of the matrix involved.

Step 1. We seek to modify the matrix by elementary operations so as to *decrease* the value of the function α. We prove the following:

If the number $\alpha(A)$ fails to divide some entry of A, then it is possible to decrease the value of α by applying elementary operations to A; and conversely.

The converse is easy. If the number $\alpha(A)$ divides each entry of A, then it will divide each entry of any matrix B obtained by applying elementary operations to A. In this situation, it is not possible to reduce the value of α by applying elementary operations.

To prove the result itself, we suppose a_{ij} is a minimal entry of A that fails to divide some entry of A. If the entry a_{ij} fails to divide some entry a_{kj} in its *column*, then we perform a division, writing

$$\frac{a_{kj}}{a_{ij}} = q + \frac{r}{a_{ij}},$$

where $0 < |r| < |a_{ij}|$. Signs do not matter here; q and r may be either positive or negative. We then replace (row k) of A by (row k) $-$ q(row i). The result is to replace the entry a_{kj} in the kth row and jth column of A by $a_{kj} - qa_{ij} = r$. The value of α for this new matrix is at most $|r|$, which is less than $\alpha(A)$.

A similar argument applies if a_{ij} fails to divide some entry in its *row.*

Finally, suppose a_{ij} divides each entry in its row and each entry in its column, but fails to divide the entry a_{st}, where $s \neq i$ and $t \neq j$. Consider the following four entries of A:

$$\begin{matrix} a_{ij} & \cdots & a_{it} \\ \vdots & & \vdots \\ a_{sj} & \cdots & a_{st} \end{matrix}$$

Because a_{ij} divides a_{sj}, we can by elementary operations bring the matrix to the form where the entries in these four places are as follows:

$$\begin{matrix} a_{ij} & \cdots & a_{it} \\ \vdots & & \vdots \\ 0 & \cdots & a_{st} + la_{it} \end{matrix}$$

If we then replace (row i) of this matrix by (row i) + (row s), we are back in the previous situation, where a_{ij} fails to divide some entry in its row.

Step 2. At the beginning of this step, we have a matrix A whose minimal entry divides every entry of A.

Apply elementary operations to bring a minimal entry of A to the upper left corner of the matrix and to make it positive. Because it divides all entries in its row and column, we can apply elementary operations to make all the other entries in its row and column into zeros. Note that at the end of this process, the entry in the upper left corner divides all entries of the matrix.

One now begins Step 1 again, applying it to the smaller matrix obtained by ignoring the first row and first column of our matrix.

Step 3. The algorithm terminates either when the smaller matrix is the zero matrix or when it disappears. At this point our matrix is in normal form. The only question is whether the diagonal entries $b_1, \ldots, b_l$ successively divide one another. But this is immediate. We just noted that at the end of the first application of Step 2, the entry b_1 in the upper left corner divides all entries of the matrix. This fact remains true as we continue to apply elementary operations. In particular, when the algorithm terminates, b_1 must divide each of $b_2, \ldots, b_l$.

A similar argument shows b_2 divides each of $b_3, \ldots, b_l$. And so on.

It now follows immediately from Exercise 4 of §4 that the numbers $b_1, \ldots, b_l$ are uniquely determined by the homomorphism f. For the number l of non-zero entries in the matrix is just the rank of the free abelian group $f(G) \subset G'$. And those numbers b_i that are greater than 1 are just the torsion coefficients $t_1, \ldots, t_k$ of the quotient group $G'/f(G)$.

Applications of the reduction algorithm

Now we prove the basic theorem concerning subgroups of free abelian groups, which we stated in §4.

Proof of Theorem 4.2. Given a free abelian group F of rank n, we know from Lemma 11.1 that any subgroup R is free of rank $r \leq n$. Consider the *inclusion* homomorphism $j : R \to F$, and choose bases $a_1, \ldots, a_r$ for R and $e_1, \ldots, e_n$ for F relative to which the matrix of j is in the normal form of the preceding theorem. Because j is a monomorphism, this normal form has no zero columns. Thus $j(a_i) = b_i e_i$ for $i = 1, \ldots, r$, where $b_i \geq 1$ and $b_1 \mid b_2 \mid \cdots \mid b_r$. Since $j(a_i) = a_i$, it follows that $b_1 e_1, \ldots, b_r e_r$ is a basis for R. □

Now we prove the "standard basis theorem" for free chain complexes.

Definition. A **chain complex** $\mathcal{C}$ is a sequence

$$\cdots \longrightarrow C_{p+1} \xrightarrow{\partial_{p+1}} C_p \xrightarrow{\partial_p} C_{p-1} \longrightarrow \cdots$$

of abelian groups C_i and homomorphisms ∂_i, indexed with the integers, such that $\partial_p \circ \partial_{p+1} = 0$ for all p. The pth **homology group** of $\mathcal{C}$ is defined by the equation

$$H_p(\mathcal{C}) = \ker \partial_p / \operatorname{im} \partial_{p+1}.$$

If $H_p(\mathcal{C})$ is finitely generated, its betti number and torsion coefficients are called the betti number and torsion coefficients of $\mathcal{C}$ in dimension p.

Theorem 11.4 (Standard bases for free chain complexes). *Let $\{C_p, \partial_p\}$ be a chain complex; suppose each group C_p is free of finite rank. Then for each p there are subgroups U_p, V_p, W_p of C_p such that*

$$C_p = U_p \oplus \boxed{V_p \oplus W_p,} \quad \ker \partial_p = Z_p$$

where $\partial_p(U_p) \subset W_{p-1}$ and $\partial_p(V_p) = 0$ and $\partial_p(W_p) = 0$. Furthermore, there are bases for U_p and W_{p-1} relative to which $\partial_p : U_p \to W_{p-1}$ has a matrix of the form

$$B = \begin{bmatrix} b_1 & & 0 \\ & \ddots & \\ 0 & & b_l \end{bmatrix},$$

where $b_i \geq 1$ and $b_1 \mid b_2 \mid \cdots \mid b_l$.

Proof. Step 1. Let

$$Z_p = \ker \partial_p \qquad \text{and} \qquad B_p = \operatorname{im} \partial_{p+1}.$$

Let W_p consist of all elements c_p of C_p such that some non-zero multiple of c_p

belongs to B_p. It is a subgroup of C_p, and is called the group of **weak boundaries.** Clearly

$$B_p \subset W_p \subset Z_p \subset C_p.$$

(The second inclusion uses the fact that C_p is torsion-free, so that the equation $mc_p = \partial_{p+1} d_{p+1}$ implies that $\partial_p c_p = 0$.) We show that W_p is a direct summand in Z_p.

Consider the natural projection

$$Z_p \to H_p(\mathcal{C}) \to H_p(\mathcal{C})/T_p(\mathcal{C}),$$

where $T_p(\mathcal{C})$ is the torsion subgroup of $H_p(\mathcal{C})$. The kernel of this projection is W_p; therefore, $Z_p/W_p \simeq H_p/T_p$. The latter group is finitely generated and torsion-free, so it is free. If $c_1 + W_p, \ldots, c_k + W_p$ is a basis for Z_p/W_p, and $d_1, \ldots, d_l$ is a basis for W_p, then it is straightforward to check that $c_1, \ldots, c_k, d_1, \ldots, d_l$ is a basis for Z_p. Then $Z_p = V_p \oplus W_p$, where V_p is the group with basis $c_1, \ldots, c_k$.

Step 2. Suppose we choose bases $e_1, \ldots, e_n$ for C_p, and $e_1', \ldots, e_m'$ for C_{p-1}, relative to which the matrix of $\partial_p : C_p \to C_{p-1}$ has the normal form

$$\begin{array}{c} \\ e_1' \\ \vdots \\ e_l' \\ e_{l+1}' \\ \vdots \\ e_m' \end{array}
\begin{array}{c} \begin{array}{cccccc} e_1 & \cdots & e_l & e_{l+1} & \cdots & e_n \end{array} \\
\left[\begin{array}{ccc|c} b_1 & & 0 & \\ & \ddots & & 0 \\ 0 & & b_l & \\ \hline & & & \\ & 0 & & 0 \\ & & & \end{array}\right] \end{array}$$

where $b_i \geq 1$ and $b_1 \mid b_2 \mid \cdots \mid b_l$. Then the following hold:

(1) $e_{l+1}, \ldots, e_n$ is a basis for Z_p.

(2) $e_1', \ldots, e_l'$ is a basis for W_{p-1}.

(3) $b_1 e_1', \ldots, b_l e_l'$ is a basis for B_{p-1}.

We prove these results as follows: Let c_p be the general p-chain. We compute its boundary; if

$$c_p = \sum_{i=1}^{n} a_i e_i, \qquad \text{then} \qquad \partial_p c_p = \sum_{i=1}^{l} a_i b_i e_i'.$$

To prove (1), we note that since $b_i \neq 0$, the p-chain c_p is a cycle if and only if $a_i = 0$ for $i = 1, \ldots, l$. To prove (3), we note that any $p-1$ boundary $\partial_p c_p$ lies in the group generated by $b_1 e_1', \ldots, b_l e_l'$; since $b_i \neq 0$, these elements are independent. Finally, we prove (2). Note first that each of $e'_1, \dots, e'_l$ belongs to W_{p-1}, since $b_i e'_i = \partial e_i$. Conversely, let

$$c_{p-1} = \sum_{i=1}^{m} d_i e'_i$$

be a $p-1$ chain and suppose $c_{p-1} \in W_{p-1}$. Then c_{p-1} satisfies an equation of the form

$$\lambda c_{p-1} = \partial_p c_p = \sum_{i=1}^{l} a_i b_i e'_i$$

for some $\lambda \neq 0$. Equating coefficients, we see that $\lambda d_i = 0$ for $i > l$, whence $d_i = 0$ for $i > l$. Thus $e'_1, \dots, e'_l$ is a basis for W_{p-1}.

Step 3. We prove the theorem. Choose bases for C_p and C_{p-1} as in Step 2. Define U_p to be the group spanned by $e_1, \dots, e_l$; then

$$C_p = U_p \oplus Z_p.$$

Using Step 1, choose V_p so that $Z_p = V_p \oplus W_p$. Then we have a decomposition of C_p such that $\partial_p(V_p) = 0$ and $\partial_p(W_p) = 0$. The existence of the desired bases for U_p and W_{p-1} follows from Step 2. □

Note that W_p and $Z_p = V_p \oplus W_p$ are uniquely determined subgroups of C_p. The subgroups U_p and V_p are not uniquely determined, however.

Theorem 11.5. *The homology groups of a finite complex K are effectively computable.*

Proof. By the preceding theorem, there is a decomposition

$$C_p(K) = U_p \oplus V_p \oplus W_p$$

where $Z_p = V_p \oplus W_p$ is the group of p-cycles and W_p is the group of weak p-boundaries. Now

$$H_p(K) = Z_p/B_p \simeq V_p \oplus (W_p/B_p) \simeq (Z_p/W_p) \oplus (W_p/B_p).$$

The group Z_p/W_p is free and the group W_p/B_p is a torsion group; computing $H_p(K)$ thus reduces to computing these two groups.

Let us choose bases for the chain groups $C_p(K)$ by orienting the simplices of K, once and for all. Then consider the matrix of the boundary homomorphism $\partial_p : C_p(K) \to C_{p-1}(K)$ relative to this choice of bases; the entries of this matrix will in fact have values in the set $\{0,1,-1\}$. Using the reduction algorithm described earlier, we reduce this matrix to normal form. Examining Step 2 of the preceding proof, we conclude from the results proved there the following facts about this normal form:

(1) The rank of Z_p equals the number of zero columns.

(2) The rank of W_{p-1} equals the number of non-zero rows.

(3) There is an isomorphism

$$W_{p-1}/B_{p-1} \simeq \mathbf{Z}/b_1 \oplus \mathbf{Z}/b_2 \oplus \cdots \oplus \mathbf{Z}/b_l.$$

Thus the normal form for the matrix of $\partial_p : C_p \to C_{p-1}$ gives us the torsion coefficients of K in dimension $p - 1$; they are the entries of the matrix that are greater than 1. This normal form also gives us the rank of Z_p. On the other hand, the normal form for $\partial_{p+1} : C_{p+1} \to C_p$ gives us the rank of W_p. The difference of these numbers is the rank of Z_p/W_p—that is, the betti number of K in dimension p. $\square$

EXERCISES

1. Show that the reduction algorithm is not needed if one wishes merely to compute the betti numbers of a finite complex K; instead all that is needed is an algorithm for determining the rank of a matrix. Specifically, show that if A_p is the matrix of $\partial_p : C_p(K) \to C_{p-1}(K)$ relative to some choice of basis, then

$$\beta_p(K) = \text{rank } C_p(K) - \text{rank } A_p - \text{rank } A_{p+1}.$$

2. Compute the homology groups of the quotient space indicated in Figure 11.1. [*Hint:* First check whether all the vertices are identified.]

3. Reduce to normal form the matrix

$$\begin{bmatrix} 2 & 6 & 4 \\ 4 & -7 & 4 \\ 4 & 8 & 4 \end{bmatrix}.$$

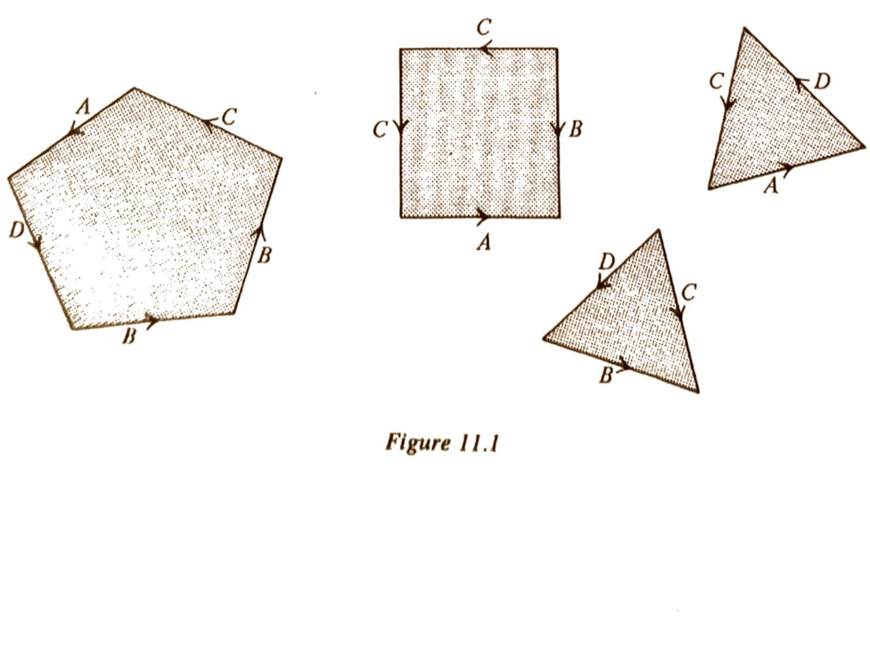

Figure 11.1