Spherical Triangles and Girard’s Theorem

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Let $S^2$ denote the unit sphere in $\mathbb{R}^3$ i.e. the set of all unit vectors i.e. the set $\{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 + z^2 = 1 \}$. 

A great circle in $S^2$ is a circle which divides the sphere in half. In other words, a great circle is the intersection of $S^2$ with a plane passing through the origin.
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Great circles are straight lines

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Given two distinct points on $S^2$, there is a great circle passing through them obtained by the intersection of $S^2$ with the plane passing through the origin and the two given points.
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You can similarly verify the other three Euclid’s postulates for geometry.
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The angle at both the vertices are equal. Both diangles bounded by two great circles are congruent to each other.
Area of a diangle

**Proposition**

Let $\theta$ be the angle of a diangle. Then the area of the diangle is $2\theta$. 

Proof:

The area of the diangle is proportional to its angle. Since the area of the sphere, which is a diangle of angle $2\pi$, is $4\pi$, the area of the diangle is $2\theta$.

Alternatively, one can compute this area directly as the area of a surface of revolution of the curve $z = \sqrt{1 - y^2}$ by an angle $\theta$. This area is given by the integral

$$\int_{1-\theta}^{1+\theta} z \sqrt{1 + (z')^2} \, dy.$$ 

This is very similar to the formula for the length of an arc of the unit circle which subtends an angle $\theta$. 

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More Examples. Take ballon, ball and draw on it.

![Spherical Triangle](image-url)
Girard’s Theorem: Area of a spherical triangle

Girard’s Theorem

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\[ \triangle ABC \] as shown above is formed by the intersection of three great circles.

Vertices \( A \) and \( D \) are antipodal to each other and hence have the same angle. Similarly for vertices \( B, E \) and \( C, F \). Hence the triangles \( \triangle ABC \) and \( \triangle DEF \) are antipodal (opposite) triangles and have the same area.

Assume angles at vertices \( A, B \) and \( C \) to be \( \alpha, \beta \) and \( \gamma \) respectively.
Let $R_{AD}$, $R_{BE}$ and $R_{CF}$ denote pairs of diangles as shown. Then $\triangle ABC$ and $\triangle DEF$ each gets counted in every diangle.
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$$\text{Area}(S^2) = \text{Area}(R_{AD}) + \text{Area}(R_{BE}) + \text{Area}(R_{CF}) - 4X$$

$$4\pi = 4\alpha + 4\beta + 4\gamma - 4X$$

$$X = \alpha + \beta + \gamma - \pi$$
Corollary

Let $R$ be a spherical polygon with $n$ vertices and $n$ sides with interior angles $\alpha_1, \ldots, \alpha_n$. Then $\text{Area}(R) = \alpha_1 + \ldots + \alpha_n - (n - 2)\pi$. 
Corollary

Let $R$ be a spherical polygon with $n$ vertices and $n$ sides with interior angles $\alpha_1, \ldots, \alpha_n$. Then $\text{Area}(R) = \alpha_1 + \ldots + \alpha_n - (n - 2)\pi$.

**Proof:** Any polygon with $n$ sides for $n \geq 4$ can be divided into $n - 2$ triangles.

The result follows as the angles of these triangles add up to the interior angles of the polygon.