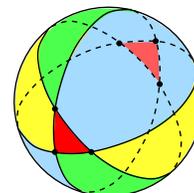


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# Notes on Spherical Geometry



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## 1. VECTORS AND PLANES IN $\mathbb{R}^3$

To review vector, dot and cross products, lines and planes in  $\mathbb{R}^3$ . see Chapter 12 from the book *Calculus: Early Transcendentals*, by Rogawski and Adams, 3rd Edition (ISBN-13: # 978-1-4641-1488-5). In addition to computations of dot and cross products, and equations of lines and planes, we will also use the following propositions:

**Proposition 1.** *There are infinitely many planes containing a fixed line.*

**Proposition 2.** *Two planes with a point in common intersect in a straight line. The direction vector of this line is given by the cross product of their normal vectors.*

### Problems.

- (1) Find all values of  $b$  for which the vectors  $\langle 4, -2, 7 \rangle$  and  $\langle b^2, b, 0 \rangle$  are orthogonal.
- (2) Let  $\vec{v} = \langle 2, 0, 0 \rangle$  and  $\vec{w} = \langle -1, 0, 1 \rangle$ . Find a vector orthogonal to  $\vec{v}$  and  $\vec{w}$ .
- (3) Show that if  $\vec{v}$  and  $\vec{w}$  lie in the  $yz$ -plane then  $\vec{v} \times \vec{w}$  is a multiple of  $\vec{i}$ .
- (4) Let  $\vec{i} = \langle 1, 0, 0 \rangle$ ,  $\vec{j} = \langle 0, 1, 0 \rangle$ ,  $\vec{k} = \langle 0, 0, 1 \rangle$  be the standard basis vectors of  $\mathbb{R}^3$ . Using the geometry of the cross product compute the following:
  - (a)  $\vec{i} \times \vec{j}$
  - (b)  $3\vec{j} \times \vec{k}$
  - (c)  $\vec{i} \times 2\vec{k}$
  - (d)  $4\vec{j} \times 3\vec{j}$
- (5) Find the equation of the plane parallel to the plane  $3x - 7y + 2z = 13$  and passing through the point  $(0, -2, 3)$ .
- (6) Find the equation of line of intersection of the  $xy$ -plane and  $yz$ -plane. Also find the angle between the planes.
- (7) Find the area of the triangle with vertices  $(1, 2, 3)$ ,  $(3, 1, 5)$  and  $(4, 5, 6)$ .
- (8) Determine whether the points  $(1, 1, 2)$ ,  $(0, 1, 0)$  and  $(1, 2, 3)$  are collinear.
- (9) Find the equation of line of intersection of the planes  $y - z = 5$  and  $x - z = 7$ . Also find the angle between the planes.
- (10) Find the equation of the plane passing through the points  $(1, 3, 2)$ ,  $(0, 3, 0)$  and  $(2, 4, 3)$ .
- (11) Find the equation of a plane passing through point  $(2, 1, 0)$  and parallel to the plane  $x - 2y + 5z = 3$ . Does this plane pass through the origin?
- (12) Find equation of line of intersection between the planes  $x - y + 2z = 0$  and  $x + 2y - z = 0$ . Also find the angle between the planes.

## 2. POINTS AND LINES ON $\mathbb{S}^2$

**Important conventions:**

- (1) For a point  $P = (a, b, c) \in \mathbb{R}^3$  its *position vector*  $\vec{OP} = \langle a, b, c \rangle$ .
- (2) Normal vectors to planes will be assumed to be of length 1.

**Points:** The points in spherical geometry are the points on the unit sphere i.e.

$$\mathbb{S}^2 = \{(x, y, z) \mid x^2 + y^2 + z^2 = 1\} = \{P \in \mathbb{R}^3 \mid \|\vec{OP}\| = 1\}.$$

**Definition 3. (Antipodes)** The *antipodal map* is a transformation of  $\mathbb{S}^2$  which takes a point to its diametrically opposite point. Its has a simple formula given by

$$A : \mathbb{S}^2 \rightarrow \mathbb{S}^2, \quad A(x, y, z) = (-x, -y, -z).$$

The points  $P = (a, b, c)$  and  $-P := A(P) = (-a, -b, -c)$  are called *antipodal points* or *antipodes*.

**Proposition 4.** (1) The line joining  $P$  and  $-P$  passes through the origin. (2) Conversely any line in through the origin intersects the unit sphere in a pair of antipodal points.

*Proof.* (1) The line given by the parametric equation  $\gamma(t) = t \vec{OP}$  contains the origin when  $t = 0$ , the point  $P$  when  $t = 1$  and the point  $-P$  when  $t = -1$  collinear in  $\mathbb{R}^3$ .

(2) A line  $L$  through the origin has parametric equation  $\gamma(t) = t \vec{u}$ , where  $\vec{u} \neq \vec{0}$  is the direction vector of the  $L$ .

$$L \cap \mathbb{S}^2 = \{Q \in \mathbb{S}^2 \mid OQ = t \vec{u}\} \implies \|t \vec{u}\| = 1 \implies |t| \|\vec{u}\| = 1 \implies t = \pm / \|\vec{u}\|.$$

Hence the position vectors of the intersection points are negatives of each other, hence the intersection points are antipodes. □

**Definition 5. (Lines)** A line on  $\mathbb{S}^2$  is a *great circle* i.e. the largest circle that can be drawn on  $\mathbb{S}^2$ . Equivalently, its the interesection of a plane through origin and  $\mathbb{S}^2$ . See Figure 1(a).

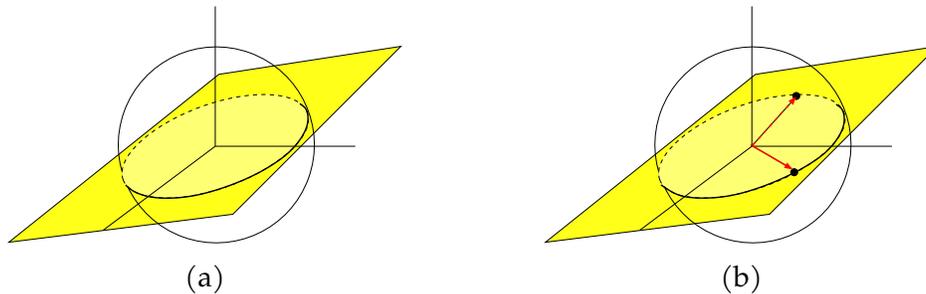


FIGURE 1. (a) Great circle obtained as intersection of plane through origin and  $\mathbb{S}^2$ . (b) Great circle passing through two non-antipodal points.

**Equations and poles:** Every plane through origin  $O$  has the equation  $ax + by + cz = 0$  where  $\vec{n} = \langle a, b, c \rangle$  is the normal vector of the plane. We will denote this plane by  $\mathcal{P}_{\vec{n}}$ . The great circle  $\mathcal{P}_{\vec{n}} \cap \mathbb{S}^2$  is denoted by  $L_{\vec{n}}$ . We will refer to  $ax + by + cz = 0$  as the *equation* of  $L_{\vec{n}}$ .

By convention  $\vec{n}$  has length one i.e.  $a^2 + b^2 + c^2 = 1$ . Hence the endpoint of  $\vec{n}$  can be thought of as a point on  $\mathbb{S}^2$ , called the *pole* of  $L_{\vec{n}}$ . For any point  $P \in \mathbb{S}^2$ ,  $L_{\vec{OP}}$  is the line with pole  $P$  or the *polar line* of  $P$ .

**Proposition 6.** *Antipodal points determine the same polar line i.e. if  $Q = -P$  then  $L_{\vec{OP}} = L_{\vec{OQ}}$ .*

*Proof.* Let  $P = (a, b, c)$  then since  $Q = -P = (-a, -b, -c)$ . Equation of  $L_{\vec{OP}}$  is

$$ax + by + cz = 0 \implies -(ax + by + cz) = 0 \implies (-a)x + (-b)y + (-c)z = 0,$$

which is the equation for the great circle  $L_{\vec{OQ}}$ . □

**Proposition 7.** *If a point lies on a great circle, then its antipode also lies on it i.e if  $P \in L_{\vec{n}}$  then  $-P \in L_{\vec{n}}$ .*

*Proof.* Let  $\vec{n} = \langle a, b, c \rangle$  and let  $P = (x_0, y_0, z_0) \in L_{\vec{n}}$ . Then

$$ax_0 + by_0 + cz_0 = 0 \implies -(ax_0 + by_0 + cz_0) = 0 \implies a(-x_0) + b(-y_0) + c(-z_0) = 0,$$

which implies the point  $-P = (-x_0, -y_0, -z_0) \in L_{\vec{n}}$ . □

**Proposition 8.** *Any two distinct great circles intersect in a pair of antipodal points i.e. if  $\vec{n}_2 \neq \pm \vec{n}_1$  then  $L_{\vec{n}_1} \cap L_{\vec{n}_2} = \{P, -P\}$ .*

*Proof.* Since great circles are intersection of  $\mathbb{S}^2$  with a plane passing through origin, the intersection of distinct great circles is given by the intersection of the corresponding two planes with  $\mathbb{S}^2$ . The intersection of distinct planes with a common point (in this case the origin) is a line through the origin. It follows from Proposition 4 that a line through the origin intersects  $\mathbb{S}^2$  in antipodal points. These points are the end points of the unit vectors in the direction of  $\pm \vec{n}_1 \times \vec{n}_2$ . □

Proposition 8 implies that there are no parallel lines on  $\mathbb{S}^2$ . **Hence Euclid's 5th postulate holds trivially in Spherical Geometry. But Playfair's Axiom does not hold in Spherical Geometry !**

**Theorem 9.** *Any two distinct points on  $\mathbb{S}^2$  lie on a great circle. If the points are antipodes there are infinitely many great circles which contain them, if not then it is unique.*

*Proof.* Let  $P$  and  $Q$  be two distinct points on  $\mathbb{S}^2$ . We have two cases depending on whether they are antipodes or not.

*Case 1:  $Q = -P$*  If the points are antipode, then the line joining  $P$  and  $Q$  passes through the origin (Proposition 4). There are infinitely many planes containing this line (Proposition ??), all of which contain the origin. Hence the intersection of any of these planes with  $\mathbb{S}^2$  give a great circle containing  $P$  and  $Q$ .

*Case 2:  $Q \neq -P$*  If the points are not antipode, then the points  $O, P$  and  $Q$  are non-collinear. Hence there is a unique plane which contains them whose normal vector is the unit vector in

the direction of  $\vec{OP} \times \vec{OQ}$ . Hence the intersection of this plane with  $\mathbb{S}^2$  give a great circle containing  $P$  and  $Q$ . See Figure 1(b)  $\square$

Great circles can be traversed indefinitely. Hence Theorem 9 implies that Euclid's 1st and 2nd postulate hold for Spherical Geometry.

### Problems.

- (1) Write down 15 points on  $\mathbb{S}^2$  explicitly .
- (2) Draw and describe the great circles  $L_{\langle 1,0,0 \rangle}$ ,  $L_{\langle 0,1,0 \rangle}$  and  $L_{\langle 0,0,1 \rangle}$ .
- (3) Find the points of intersection of the following pairs of great circles.
  - (a)  $L_{\langle 1,0,0 \rangle}$  and  $L_{\langle 0,-1,0 \rangle}$ .
  - (b)  $L_{\langle 1/3,1/3,\sqrt{7}/3 \rangle}$  and  $L_{\langle 1/\sqrt{2},1/\sqrt{2},0 \rangle}$ .
- (4) Find the great circle containing the following pairs of points.
  - (a)  $P = (0, 0, -1)$  and  $Q = (0, 1, 0)$ .
  - (b)  $P = (1/2, -1/2, 1/\sqrt{2})$  and  $Q = (2/3, 1/3, -2/3)$ .
- (5) Find infinitely many great circles containing the following pairs of antipodal points.
  - (a)  $P = (1, 0, 0)$  and  $-P = (-1, 0, 0)$ .
  - (b)  $P = (0, 1/2, \sqrt{3}/2)$  and  $-P = (0, -1/2, -\sqrt{3}/2)$ .
- (6) Are the following 3 points collinear.
  - (a)  $P = (0, 0, -1)$ ,  $Q = (0, 1, 0)$  and  $R = (0, 0, 1)$ .
  - (b)  $P = (1/2, -1/2, 1/\sqrt{2})$ ,  $Q = (2/3, 1/3, -2/3)$ , and  $R = (1, 0, 0)$ .

### 3. SPHERICAL DISTANCE FORMULA

**Definition 10.** The **spherical distance** between the points  $P, Q \in \mathbb{S}^2$  is the angle between the respective position vectors (see Figure ??). The spherical distance formula is

$$|PQ|_{\mathbb{S}^2} = \cos^{-1}(\vec{OP} \cdot \vec{OQ}).$$

**Theorem 11.** Let  $p, Q$  and  $R$  be points on  $\mathbb{S}^2$ .

- (1)  $|PQ|_{\mathbb{S}^2} \geq 0$ , &  $|PQ|_{\mathbb{S}^2} = 0$  if and only if  $P = Q$ .
- (2)  $|PQ|_{\mathbb{S}^2} = |QP|_{\mathbb{S}^2}$ .
- (3) (Triangle inequality)  $|PQ|_{\mathbb{S}^2} + |QR|_{\mathbb{S}^2} \geq |PR|_{\mathbb{S}^2}$ .
- (4)  $|\vec{OP} \times \vec{OQ}|^2 = \sin^2 a$ , where  $a = |PQ|_{\mathbb{S}^2}$ .

*Proof.* (1) Follows from the fact that  $\cos^{-1}(x) \in [0, \pi]$ .  $|PQ|_{\mathbb{S}^2} = 0$  implies that the angle between  $\vec{OP}$  and  $\vec{OQ}$  is zero i.e. the vectors are in the same direction. Since both are unit vectors this implies that  $\vec{OP} = \vec{OQ}$  (same direction and magnitude). Hence  $P = Q$ . Conversely if  $P = Q$ ,  $\vec{OP} = \vec{OQ}$  and hence they have same directions. Hence the angle between the vectors is 0.

- (2)  $|PQ|_{\mathbb{S}^2} = \cos^{-1}(\vec{OP} \cdot \vec{OQ}) = \cos^{-1}(\vec{OQ} \cdot \vec{OP}) = |QP|_{\mathbb{S}^2}$ .

(3) The triangle inequality follows from the following vector identities. We will skip details of the proof.

(a) (Cauchy-Schwarz Inequality)  $\vec{u} \cdot \vec{v} \leq |\vec{u}| |\vec{v}|$

(b)  $|\vec{u} \times \vec{v}| = |\vec{u}| |\vec{v}| \sin \theta$ , where  $\theta$  is the angle between  $\vec{u}$  and  $\vec{v}$ .

(c)  $(\vec{u} \times \vec{v}) \cdot (\vec{w} \cdot \vec{z}) = (\vec{u} \cdot \vec{w})(\vec{v} \cdot \vec{z}) - (\vec{u} \cdot \vec{z})(\vec{v} \cdot \vec{w})$

(4)  $|\vec{OP} \times \vec{OQ}|^2 = \sin^2 \theta$ , where  $\theta$  is the angle between  $\vec{OP}$  and  $\vec{OQ}$ . Hence  $\theta = a = |PQ|_{\mathbb{S}^2}$ . □

**Corollary 12.**  $P, Q$  and  $R$  are collinear in that order if and only if  $|PQ|_{\mathbb{S}^2} + |QR|_{\mathbb{S}^2} = |PR|_{\mathbb{S}^2}$ .

*Proof.* SKIP □

**Corollary 13.** (Test for collinearity)  $P, Q$  and  $R$  are collinear if and only if  $\vec{OP} \cdot (\vec{OQ} \times \vec{OR}) = 0$ .

*Proof.*  $P, Q$  and  $R$  are collinear if and only if  $P, Q$  and  $R$  lie in the same plane i.e. they are coplanar. The test for coplanarity is that the vector triple product  $\vec{OP} \cdot (\vec{OQ} \times \vec{OR}) = 0$ . □

### Spherical circles

**Theorem 14.** Spherical circles are intersections of planes with the unit sphere i.e. a circle on  $\mathbb{S}^2$  with center  $P$  with radius  $r$  is given by the intersection of  $\mathbb{S}^2$  with the plane  $\vec{OP} \cdot \langle x, y, z \rangle = \cos r$ .

*Proof.* Let  $C = \{Q \in \mathbb{S}^2 \mid |PQ|_{\mathbb{S}^2} = r\}$ . Then

$$Q = (a, b, c) \in C \iff |PQ|_{\mathbb{S}^2} = r \iff \vec{OP} \cdot \vec{OQ} = \cos r \iff \vec{OP} \cdot \langle a, b, c \rangle = \cos r.$$

Hence  $C = \mathbb{S}^2 \cap \{(x, y, z) \mid \vec{OP} \cdot \langle x, y, z \rangle = \cos r\}$ . See Figure ?? □

**Corollary 15.** The great circle  $L_{\vec{OP}}$  is a spherical circle with center  $P$  and radius  $\pi/2$ .

*Proof.*

$$|PQ|_{\mathbb{S}^2} = \pi/2 \iff \vec{OP} \cdot \vec{OQ} = \cos \pi/2 = 0 \iff \vec{OP} \cdot \vec{OQ} = 0 \iff Q \in L_{\vec{OP}}.$$

□

**Hence Theorem 14 implies that Euclid's 3rd postulate holds for Spherical Geometry.**

### Problems.

(1) Find the spherical distance between the following pairs of points.

(a)  $P = (1, 0, 0), -P = (-1, 0, 0)$ .

(b)  $P = (0, 0, -1), Q = (0, 1, 0)$ .

(c)  $P = (1/2, -1/2, 1/\sqrt{2}), Q = (2/3, 1/3, -2/3)$ .

(d)  $P = (0, 1/2, \sqrt{3}/2)$  and  $-P = (0, -1/2, -\sqrt{3}/2)$ .

(2) Are the following points collinear ?

(a)  $P = (0, 0, -1), Q = (0, 1, 0)$  and  $R = (0, 0, 1)$ .

- (b)  $P = (1/2, -1/2, 1/\sqrt{2})$ ,  $Q = (2/3, 1/3, -2/3)$ , and  $R = (1, 0, 0)$ .
- (3) Use area of surfaces of revolution from Calculus 2 to compute the area of a spherical circle.

#### 4. ANGLES AND PERPENDICULARS

**Definition 16.** The angle between two great circles is the angle between their normals i.e. The angle between the great circles  $L_{\vec{n}_1}$  and  $L_{\vec{n}_2}$  is given by  $\cos^{-1}(\vec{n}_1 \cdot \vec{n}_2)$ .

Two great circles are *perpendicular* if they intersect at right angles. Since the angle in computed between their normals, it is computed in the Euclidean plane spanned by  $\vec{n}_1$  and  $\vec{n}_2$ . Hence all right angles are equal, **verifying Euclides 4th postulate for spherical geometry**. We now prove three useful constructions of perpendiculars.

**Theorem 17.** (1) Let  $L_{\vec{n}_1}$  and  $L_{\vec{n}_2}$  be distinct great circles. Then there exists a unique great circle perpendicular to both.

- (2) Let  $L_{\vec{n}}$  be a great circle and  $P$  be a point outside the great circle. If  $P$  is not a pole of  $L_{\vec{n}}$  then there exists a unique great circle passing through  $P$  and perpendicular to  $L_{\vec{n}}$ .

*Proof.* (1) Since  $L_{\vec{n}_1}$  and  $L_{\vec{n}_2}$  are distinct great circles,  $\vec{n}_1$  is not a multiple of  $\vec{n}_2$  i.e. they have different directions. Hence  $\vec{n}_1 \times \vec{n}_2$  is non-zero vector. Let  $\vec{m}$  be the unit vector in this direction i.e.  $\vec{m} = (\vec{n}_1 \times \vec{n}_2) / \|\vec{n}_1 \times \vec{n}_2\|$ . Since  $\vec{m}$  is perpendicular to both  $\vec{n}_1$  and  $\vec{n}_2$ ,  $L_{\vec{m}}$  intersects  $L_{\vec{n}_1}$  and  $L_{\vec{n}_2}$  at right angles. Uniqueness follows because  $\pm\vec{m}$  define the same great circle.

(2) Since  $P$  is not a pole of  $L_{\vec{n}}$ ,  $\vec{OP}$  is not a multiple of  $\vec{n}$  i.e. they have different directions. Hence  $\vec{n} \times \vec{OP}$  is non-zero vector. Let  $\vec{m}$  be the unit vector in this direction i.e.  $\vec{m} = (\vec{n} \times \vec{OP}) / \|\vec{n} \times \vec{OP}\|$ . Since  $\vec{m}$  is perpendicular to  $\vec{OP}$ ,  $P \in L_{\vec{m}}$ . Since  $\vec{m}$  is perpendicular to  $\vec{n}$ ,  $L_{\vec{m}}$  and  $L_{\vec{n}}$  intersect at right angles. Uniqueness follows because  $\pm\vec{m}$  define the same great circle.  $\square$

**Theorem 18.** Let  $P$  and  $Q$  be distinct points on  $\mathbb{S}^2$ .

- (1) The set of points which are equidistant from  $P$  and  $Q$  is a great circle.  
(2) The equidistant great circle is the perpendicular bisector of the segment  $\overline{PQ}$ .

*Proof.* (1) Let  $P = (a_1, b_1, c_1)$  and  $Q = (a_2, b_2, c_2)$  and  $P \neq Q$ . Let  $A = (x, y, z)$  be a point equidistant from  $P$  and  $Q$ .

$$\begin{aligned}
|AP|_{\mathbb{S}^2} = |BP|_{\mathbb{S}^2} &\iff \cos^{-1}(\vec{OA} \cdot \vec{OP}) = \cos^{-1}(\vec{OA} \cdot \vec{OQ}) \\
&\iff \vec{OP} \cdot \vec{OA} = \vec{OQ} \cdot \vec{OA} \\
&\iff (\vec{OP} - \vec{OQ}) \cdot \vec{OA} = 0 \\
&\iff a_1x + b_1y + c_1z = a_2x + b_2y + c_2z \\
&\iff (a_1 - a_2)x + (b_1 - b_2)y + (c_1 - c_2)z = 0
\end{aligned}$$

Since  $P \neq Q$ ,  $\vec{OP} - \vec{OQ}$  is non-zero. Let  $\vec{m}$  be the unit vector in direction of  $\vec{OP} - \vec{OQ}$ . Hence the set of equidistant points equals  $L_{\vec{m}}$ . See Figure ??.

(2) The segment  $\overline{PQ}$  is contained in the great circle  $L_{\vec{n}}$ , where  $\vec{n}$  is the unit vector in the direction of  $\vec{OP} \times \vec{OQ}$ . For a constant  $\alpha$

$$\begin{aligned}\vec{n} \cdot \vec{m} &= \alpha(\vec{OP} - \vec{OQ}) \cdot (\vec{OP} \times \vec{OQ}) \\ &= \alpha \left( \vec{OP} \cdot (\vec{OP} \times \vec{OQ}) - \vec{OQ} \cdot (\vec{OP} \times \vec{OQ}) \right) \\ &= \alpha(0 - 0) = 0\end{aligned}$$

Hence angle between  $L_{\vec{n}}$  and  $L_{\vec{m}}$  equals  $\cos^{-1}(\vec{n} \cdot \vec{m}) = \cos^{-1}(0) = \pi/2$ . Since  $L_{\vec{m}}$  is the great circle containing the points equidistant from  $P$  and  $Q$ , hence it intersects  $L_{\vec{n}}$  at the midpoint of  $\overline{PQ}$ .  $\square$

### Problems.

- (1) In Theorem 17(2), what happens if  $P$  is a pole of  $L_{\vec{n}}$  ?
- (2) Find angle between the following great circles:
  - (a)  $L_{\vec{i}}$  and  $L_{\vec{k}}$
  - (b)  $L_{\vec{j}}$  and  $L_{\vec{k}}$
  - (c)  $L_{\langle 1/3, 2/3, 2/3 \rangle}$  and  $L_{\langle -3/5, 4/5, 0 \rangle}$
- (3) Find the perpendicular bisector for the segments  $\overline{PQ}$  where  $P$  and  $Q$  are as follows:
  - (a)  $P = (1/2, -1/2, 1/\sqrt{2})$ ,  $Q = (2/3, 1/3, -2/3)$ .
  - (b)  $P = (1/2, -1/2, 1/\sqrt{2})$ ,  $Q = (2/3, 1/3, -2/3)$ .
  - (c)  $P = (1/\sqrt{2}, 0, 1/\sqrt{2})$ ,  $Q = (0, -1/\sqrt{2}, 1/\sqrt{2})$ .

## 5. AREA OF A SPHERICAL TRIANGLE

**Definition 19.** Let  $P, Q \in \mathbb{S}^2$ , the minor segment between  $P$  and  $Q$  is the shorter line segment between  $P$  and  $Q$  i.e.  $|PQ|_{\mathbb{S}^2} \leq \pi$ . Let  $P, Q$  and  $R$  be three non-collinear points on  $\mathbb{S}^2$ . The spherical triangle  $\triangle PQR$  is defined to be the union of the 3 minor segments between the 3 points.

**Theorem 20.** (Girard's Theorem) Let the angles of the spherical triangle  $\triangle PQR$  be  $\alpha, \beta$  and  $\gamma$ . Then

$$\text{area}(\triangle PQR) = \alpha + \beta + \gamma - \pi.$$

For proof of Girard's theorem see the slides posted at:

<http://www.math.csi.cuny.edu/abhijit/329/spherical-triangle.pdf>

**Example 21.** Let  $P = (1, 0, 0)$ ,  $Q = (0, 1, 0)$  and  $R = (0, 0, 1)$ . The triangle  $\triangle PQR$  is such that all of its angles and sides are  $\pi/2$ . See Figure ??.

**Problems.**

(1) Find the area of the triangle with the following vertices:

(a)  $P = (1, 0, 0)$ ,  $Q = (0, 1, 0)$  and  $R = (0, 0, 1)$ .

(b)  $P = (1, 0, 0)$ ,  $Q = (0, -1, 0)$  and  $R = (0, 0, -1)$ .

(c)  $P = (1/2, -1/2, 1/\sqrt{2})$ ,  $Q = (2/3, 1/3, -2/3)$ , and  $R = (1, 0, 0)$ .

## 6. EULER'S POLYHEDRAL FORMULA AND PLATONIC SOLIDS

**Theorem 22.** (Euler's polyhedral formula) Let  $\mathcal{S}$  be a convex polyhedron in  $\mathbb{R}^3$  and let  $v$ ,  $e$  and  $f$  be the number of its vertices, edges and faces respectively. Then

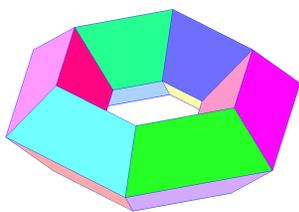
$$v - e + f = 2.$$

For proof of Euler's polyhedral formula and applications see the slides posted at:

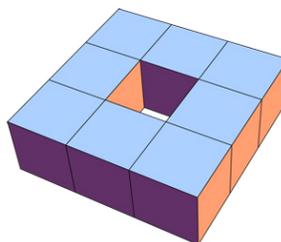
[http://www.math.csi.cuny.edu/abhijit/329/euler\\_slides.pdf](http://www.math.csi.cuny.edu/abhijit/329/euler_slides.pdf)

**Problems.**

- (1) Is a convex polyhedron has four faces and four vertices, how many edges must it have? Is there such a convex polyhedron?
- (2) Can we have a polyhedron consisting of 12 hexagonal faces and every vertex of degree 4?
- (3) Verify Euler's formula for the Platonic solids (figures are from the slides).
- (4) Does Euler's formula hold for the polyhedra in Figure 2(a) and (b)?



(a)



(b)

FIGURE 2. Polyhedral tori

## 7. SPHERICAL COSINE AND SINE LAWS

**Theorem 23.** Let  $\triangle ABC$  be a spherical triangle with angles  $A$ ,  $B$  and  $C$  at the vertices and sides  $a$ ,  $b$  and  $c$  opposite the vertices  $A$ ,  $B$  and  $C$  respectively. Then

(1)  $\cos C = \frac{\cos c - \cos a \cos b}{\sin a \sin b}$ . Similar equations hold for  $\cos A$  and  $\cos B$ .

(2)  $\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}$ .

*Proof.* SKIP

□

**Corollary 24.** (*Spherical Pythagorean Theorem*) Let  $\triangle ABC$  be a spherical triangle as above with angle  $C = \pi/2$ . Then

$$\cos c = \cos a \cos b.$$

**Problems.** To be assigned.

## 8. SPHERICAL ISOMETRIES

**Definition 25.** A spherical isometry is a transformation of  $\mathbb{S}^2$  which preserves spherical distances i.e. a function  $F : \mathbb{S}^2 \rightarrow \mathbb{S}^2$  such that  $|PQ|_{\mathbb{S}^2} = |F(P)F(Q)|_{\mathbb{S}^2}$  for all points  $P, Q \in \mathbb{S}^2$ .

**Remark 26.** How does such an isometry  $F$  look?  $F : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  is a vector field i.e.  $F(x, y, z) = (f(x, y, z), g(x, y, z), h(x, y, z))$  which satisfies the following properties:

- (1)  $F$  preserves the origin i.e.  $F(\vec{O}) = \vec{O}$ .
- (2)  $F$  takes unit vectors to unit vectors (and hence preserves  $\mathbb{S}^2$ ).
- (3)  $F$  preserves the dot product i.e.

$$\vec{u} \cdot \vec{v} = F(\vec{u}) \cdot F(\vec{v}).$$

In fact the last property implies the first two because (1)  $\|\vec{u}\|^2 = \vec{u} \cdot \vec{u}$ , (2)  $\|\vec{u}\| = 0 \iff \vec{u} = 0$ .

Note that composition of two isometries is also an isometry. Here are some examples.

**Example 27.** We will see some examples of reflection and composition of reflections.

- (1)  $F_1(x, y, z) = (-x, y, z)$  is a reflection along the great circle  $L_{\vec{i}}$  i.e. a reflection in the  $yz$ -plane.
- (2)  $F_2(x, y, z) = (x, -y, z)$  is a reflection along the great circle  $L_{\vec{j}}$  i.e. a reflection in the  $xz$ -plane.
- (3)  $F_3(x, y, z) = (x, y, -z)$  is a reflection along the great circle  $L_{\vec{k}}$  i.e. a reflection in the  $xy$ -plane.
- (4) The isometry  $F_1 \circ F_2(x, y, z) = (-x, -y, z)$  is a rotation in the  $z$ -axis by  $\pi$  radians. Similarly  $F_2 \circ F_3(x, y, z) = (x, -y, -z)$  is a rotation in the  $x$ -axis by  $\pi$  radians, and  $F_3 \circ F_1(x, y, z) = (-x, y, -z)$  is a rotation along the  $y$ -axis by  $\pi$  radians.
- (5) The isometry  $F_1 \circ F_2 \circ F_3(x, y, z) = (-x, -y, -z)$  is the antipodal map.

**Definition 28.** (Reflections and Rotations) (1) A reflection of  $\mathbb{S}^2$  in the great circle  $L_{\vec{n}}$  is obtained by the reflecting  $\mathbb{S}^2$  in the plane  $\mathfrak{P}_{\vec{n}}$ . The reflection fixes the line  $L_{\vec{n}}$  pointwise.

(2) A rotation of  $\mathbb{S}^2$  is a rotation along an axis of rotation passing through a pair of antipodal points by some angle  $\theta$ . A rotation fixes a pair of antipodal points.

**Theorem 29.** A reflection in a great circle  $L_{\vec{n}}$  is an isometry of  $\mathbb{S}^2$ .

*Proof.* SKIP

□

**Theorem 30.** Let  $F_1$  and  $F_2$  be reflections in the great circles  $L_{\vec{n}_1}$  and  $L_{\vec{n}_2}$  respectively. Then the composition  $F_1 \circ F_2$  is a rotation of  $\mathbb{S}^2$  with axis the line passing through the points  $L_{\vec{n}_1} \cap L_{\vec{n}_2}$ , and the angle of rotation  $2\theta$ , where  $\theta$  is the angle between  $L_{\vec{n}_1}$  and  $L_{\vec{n}_2}$ .

*Proof.* SKIP □

**Theorem 31.** (Three Reflection Theorem) Every isometry of  $\mathbb{S}^2$  is a composition of at most 3 reflections.

*Proof.* Follows the same proof as for the analogous theorem for  $\mathbb{E}^2$ . Uses the perpendicular bisector as the set of equidistant points. SKIP □

**Classification of spherical isometries** An isometry of  $\mathbb{S}^2$  is either a reflection, a rotation or an “antipodal” maps. Note there are no translations or glide reflections !

**Problems.** To be assigned.