

Review for Final Exam

Geometry, MTH 329 Spring 2026

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- Final will be held on **Wednesday May 20th** at same time and room as our class and will be 100 minutes long.
 - **Syllabus:** Concurrence (Ceva's Theorem), Euler's Polyhedral Theorem, Chapter 11, Chapters 12 and Chapter 15 (topics covered in the class on Hyperbolic Geometry and Euclidean, Spherical and Hyperbolic isometries)
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Definitions & Examples:

\mathbb{E}^2 : Euclidean distance formula, angles, triangles, quadrilaterals, circles, perpendicular bisector, equidistant points, Isometries, Translations, Reflections, Rotations, Composition of isometries, Three Reflection Theorem (statement), Fixed points of isometries.

\mathbb{H}^2 : Hyperbolic plane, Hyperbolic lines, Hyperbolic isometries, translations, dilations, reflections, inversions, hyperbolic distance, hyperbolic angles, hyperbolic triangles, area, Fixed point of isometries.

Straight edge and compass constructions:

1. Construct a regular pentagon, hexagon, octagon, decagon.
2. Construct the image of a point under inversion. There are 3 cases.
(See https://youtu.be/_0bk_bL07cg?si=C2qov61Lh-1E3ExC)
3. Construct hyperbolic geodesic between two points in $\overline{\mathcal{H}}$.
4. Construct hyperbolic geodesic between two points in $\overline{\mathcal{D}}$.
5. Construct hyperbolic geodesic perpendicular to a vertical geodesic at a given point in \mathcal{H} .
6. Find common perpendicular to two ultra parallel lines in \mathcal{H} one of which is a vertical geodesic.

Problems

1. State Ceva's theorem and using Ceva's theorem prove the following:
 - (a) The 3 medians are concurrent
 - (b) The 3 angle bisectors are concurrent
 - (c) The 3 altitudes are concurrent
2. State Euler's Polyhedral Theorem with correct hypothesis and illustrate it for a given polyhedra e.g. cube.
3. Can there be a polyhedron consisting of 12 hexagonal faces and every vertex of degree 4?

Problems: Euclidean Isometries

4. Let f be any isometry of \mathbb{R}^2 . Prove that
 - (a) f preserves angles.
 - (b) f takes lines to lines.
 - (c) that f takes circles to circles.
5. Let f be an isometry of \mathbb{R}^2 given by three reflections, $f = \rho_c \circ \rho_b \circ \rho_a$.
 - (a) Suppose the three lines intersect at a single point. Describe the isometry. f .
 - (b) Suppose that a is parallel to b , and c is perpendicular to a . Describe the isometry.
 - (c) Suppose the 3 lines are parallel. Describe the isometry.
 - (d) Suppose the 3 lines mutually intersect each other i.e. form a triangle. Describe the isometry.
6. Let f be reflection in the line $x = 1$, and let g be an anti-clockwise rotation of $\pi/4$ about $(0, 0)$. Describe $f \circ g$.
7. Give example of two different type of isometries which preserve the $x - axis$.
8. Find translation of \mathbb{R}^2 which takes the line $x = y$ to the lines $x = y + 2$.

9. Show that the composition of a rotation by π about $(0, 0)$ followed by a rotation by π about $(2, 0)$ is a translation. Draw pictures.
10. Describe all the isometries of \mathbb{R}^2 which fix the origin $(0, 0)$. Use your answer to describe all isometries of \mathbb{R}^2 which take $(0, 0)$ to $(1, 1)$ as a product of a translation and one other isometry.
11. Classify the type of the isometry f as reflection, translation, rotation or glide reflection:
 - (a) f is product of 10 isometries and fixes the point $(1, 1)$.
 - (b) f is composition of two translations.
 - (c) f fixes a line and is a product of 7 reflections.
 - (d) f has no fixed points and is a product of 5 reflections.

Problems: Hyperbolic Geometry

12. Prove that there is a unique hyperbolic line passing through two points in the hyperbolic plane (there are two cases).
13. Given a hyperbolic line L and a point P outside it, prove that there are infinitely many hyperbolic lines parallel to L in the following cases:
 - (a) L is vertical half-line and P is a point outside it.
 - (b) L is half circle and P is a point outside the circle.
 - (c) L is half circle and P is a point inside the circle.
14. For the given hyperbolic line L and a point P outside it, find 3 hyperbolic lines through P and parallel to L .
 - (a) $L : x = 0, P = 2 + 2i$.
 - (b) $L : x = 2, P = -4 + i$.
 - (c) $L : x^2 + y^2 = 1, P = (3, 1)$.
 - (d) $L : x^2 + y^2 = 4, P = (1, 4)$.

15. The area of a hyperbolic triangle with angles α, β, γ is $\pi - (\alpha, \beta, \gamma)$. The angle between geodesics which are asymptotically parallel is zero. Compute area of the following hyperbolic polygons:

- (a) Triangle with sides with hyperbolic lines $x = 0, x = 1$ and $x^2 + y^2 = 1$.
- (b) Triangle with sides with hyperbolic lines $x = 5, x = 6$ and $(x - 5)^2 + y^2 = 1$ (hint: translations along x-axis are hyperbolic isometries).
- (c) Quadrilateral with sides $x = 0, x = 2, x^2 + y^2 = 1, (x - 2)^2 + y^2 = 1$ (hint: divide into triangles).

16. Classify the type of the isometry g as inversion, hyperbolic, parabolic or rotation and describe it:

- (a) g is composition of inversions in the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.
- (b) g is composition of reflections in lines $x = 1$ and $x = 3$.
- (c) g is composition of reflection in line $x = 3$ and inversion in circle $x^2 + y^2 = 9$.
- (d) g fixes the circle $x^2 + y^2 = 1$ and is a product of 5 reflections.

17. For the three statements below, fill in the chart with **Y**(Yes) or **N**(No) in each space.

X_1 : Any two distinct lines intersect in a unique point.

X_2 : For a line L and point Q outside L , a unique line is parallel to L through Q .

X_3 : The sum of angles of a triangle are greater than π .

	X_1	X_2	X_3
\mathbb{H}^2			
\mathbb{S}^2			
\mathbb{E}^2			