5

Euclidean Versus Hyperbolic Geometry

Euclidean geometry and hyperbolic geometry are compared and contrasted.

5.1 Euclid's Postulates Revisited

We now reconsider Euclid's five postulates one at a time and show that the first four remain valid in the context of hyperbolic geometry, whereas the parallel postulate does not.

Postulate 1. *To draw a straight line from any point to any point.*

That this postulate also holds in the hyperbolic plane was already noted following the proof of Theorem 4.2.1.

Postulate 2. *To produce a finite straight line continuously in a straight line.*

Let γ be any geodesic containing the two points P and Q. We will show that if P, while moving along the geodesic γ, approaches the x-axis, then $h(P, Q)$ becomes indefinitely large. We assume here that γ is a bowed geodesic centered at $C(c, 0)$, leaving the case of the straight geodesics as Exercise 9. Let the coordinates of P and Q relative to a standard polar coordinate system placed at C be (r_P, α) and (r_Q, β). Then, by Proposition 4.1.1,

$$\lim_{\alpha\to 0} h(P, Q) = \lim_{\alpha\to 0} \ln \frac{\csc\beta - \cot\beta}{\csc\alpha - \cot\alpha} = \lim_{\alpha\to 0} \ln[(\csc\beta - \cot\beta)(\csc\alpha + \cot\alpha)],$$

which is clearly infinite since $\csc 0 = \cot 0 = \infty$.

Postulate 3. *To describe a circle with any center and radius.*

In a sense the existence of circles in the upper half-plane is quite obvious. Given any point C, a positive real number r, and any ray (half-geodesic) γ emanating from C, it follows from Postulate 2 that there is a point P_γ on γ that is at a hyperbolic distance of r from C. The locus of all such points P_γ is the hyperbolic circle with center C and hyperbolic radius r. The following theorem, however, may come as a great surprise to the reader.

Theorem 5.1.1 *Every Euclidean circle in the upper half-plane is also a hyperbolic circle.*

PROOF: Let q be a Euclidean circle with center O and diameter BC perpendicular to the x-axis (Fig. 5.1). If q is to be proven to be also a hyperbolic circle, then common sense dictates that

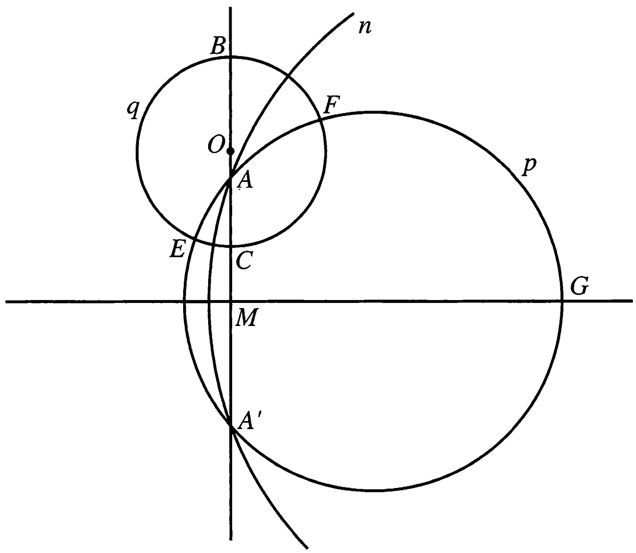

Figure 5.1

its hyperbolic center A should be the hyperbolic midpoint of the segment BC. It follows from Example 4.2.3 that

$$AM = \sqrt{BM \cdot CM}. \tag{1}$$

We will now show that every geodesic p through A divides q into two hyperbolically congruent parts. (When this is accomplished, we will of course be very close to our goal.) This hyperbolic congruence will be the hyperbolic reflection I_p, namely the inversion that leaves every point of p fixed. Proposition 3.1.7 states that this can only happen provided that the circles p and q are orthogonal. However, and this is what makes this proof work, p and q are orthogonal if and only if the hyperbolic reflection in the original circle q fixes the circle p. Thus, it is now necessary to prove that the inversion $I_{O,k}$, where k is the Euclidean radius of q, fixes the circle p.

The inversion $I_{O,k}$ of course fixes the intersections E and F of p and q. Let A' be the reflection of A in the x-axis. It is clear that A' is on p. Note that if $O = (\cdot, b)$, then $B = (\cdot, b+k)$, $C = (\cdot, b-k)$, and, by (1) above, $A = (\cdot, \sqrt{b^2 - k^2})$. Moreover, since

$$OA \cdot OA' = (b - \sqrt{b^2 - k^2})(b + \sqrt{b^2 - k^2}) = k^2,$$

it follows that A' is also on $I_{O,k}(p)$. Thus, A' is on both p and $I_{O,k}(p)$ which implies that $p = I_{O,k}(p)$. This means that p and q are orthogonal, so $q = I_p(q)$. In other words, every hyperbolic straight line through A divides q into two hyperbolically congruent parts.

We conclude the proof by showing that the hyperbolic length of the arc AE is a constant in the sense that it is independent of the position of p, as long as p passes through A. This is accomplished by producing a hyperbolic reflection that transforms AE onto AC. Let G be either of the intersections of p with the x-axis, and let n be the bowed geodesic centered at G and passing through A. Since $I_n(A) = A$

and $I_n(A') = A'$, it follows that $I_n(p) = BM$. However, by the first part of the proof, $I_n(q) = q$, and hence I_n transforms E, which is the intersection of q and p, onto C, which is the intersection of q and BM. Note that were we to choose the other intersection to play the role of G, then E would actually be transformed to B. In either case, the same conclusion is reached: the hyperbolic length of the arc AE is constant, so q is also a hyperbolic circle centered at A.

Q.E.D.

Proposition 5.1.2 *If a Euclidean circle has Euclidean center (h, k) and a Euclidean radius r < k, then it has the hyperbolic center (H, K), and the hyperbolic radius R, where*

$$H = h, \qquad K = \sqrt{k^2 - r^2}, \qquad R = \frac{1}{2}\ln\frac{k+r}{k-r}.$$

PROOF: Let B and C be, respectively, the points of the circle that lie directly above and below (h, k). It is clear that their coordinates are $(h, k+r)$ and $(h, k-r)$. It then follows from Proposition 4.1.3 that the hyperbolic diameter of this circle equals

$$\ln\frac{k+r}{k-r}$$

which gives us the desired expression for R. Similar considerations yield the value of K, and the fact that $H = h$ is clear. Conversely, when the expression for R is inverted, it yields

$$\frac{k}{r} = \frac{e^{2R}+1}{e^{2R}-1} = \coth R.$$

If this expression is solved for k and r simultaneously with $K^2 = k^2 - r^2$, we obtain

$$r^2 = \frac{K^2}{\coth^2 R - 1} = K^2 \sinh^2 R$$

and

$$k^2 = K^2 + r^2 = K^2[1 + \sinh^2 R] = K^2 \cosh^2 R$$

Q.E.D.

Corollary 5.1.3 *Every hyperbolic circle is also a Euclidean circle.*

□

Because of the crucial role that the number π plays in Euclidean geometry, the hyperbolic length of a circle is also of interest. Suppose, therefore, that q is a circle with Euclidean center $C(h, k)$ and Euclidean radius r. The circle q is then the graph of the equation

$$(x-h)^2 + (y-k)^2 = r^2.$$

Let $P(x, y)$ denote an arbitrary point on the circle, and let t denote the angle from the positive x-axis to the radius CP. Then, the circle has the parametric equations:

$$x = h + r\cos t, \qquad y = k + r\sin t.$$

Hence, $dx = -r\sin t\,dt$, $dy = r\cos t\,dt$, and the hyperbolic length of q is

$$\int_q \frac{\sqrt{dx^2 + dy^2}}{y} = 2\int_{-\pi/2}^{\pi/2} \frac{rdt}{k + r\sin t} = \left[2r\frac{2}{\sqrt{k^2 - r^2}} \tan^{-1} \frac{k\tan(t/2) + r}{\sqrt{k^2 - r^2}}\right]_{-\pi/2}^{\pi/2}$$

$$= \frac{4r}{\sqrt{k^2 - r^2}} \left[\tan^{-1} \frac{k + r}{\sqrt{k^2 - r^2}} - \tan^{-1} \frac{-k + r}{\sqrt{k^2 - r^2}}\right]$$

$$= \frac{4r}{\sqrt{k^2 - r^2}} \left[\tan^{-1} \sqrt{\frac{k + r}{k - r}} + \tan^{-1} \sqrt{\frac{k - r}{k + r}}\right] = \frac{2\pi r}{\sqrt{k^2 - r^2}}$$

because of the trigonometric identity $\tan^{-1} x + \tan^{-1} \frac{1}{x} = \frac{\pi}{2}$.

Consequently, the ratio of the hyperbolic length of a circle to the hyperbolic length of its diameter is

$$\frac{\dfrac{2\pi r}{\sqrt{k^2 - r^2}}}{\ln \dfrac{k + r}{k - r}},$$

a quantity that clearly depends on both k and r. In Exercise 8, the readers are requested to evaluate some limiting values of this ratio.

Postulate 4. *That all right angles are equal to one another.*

We will show that every two hyperbolic right angles are in fact congruent. More specifically, given any right angle α, we will show that there is a sequence of hyperbolic rigid motions that carry α to the fixed right angle δ of Fig. 5.2, whose vertex is assumed to have ordinate 1. It is clear that the reflection in the y-axis carries the angle δ_1 onto δ and that the inversion $I_{O,1}$ carries the angles δ_2 and δ_3 onto δ and δ_1, respectively. Moreover, if k is the ordinate of the common vertex of angles γ, γ_1, γ_2, and γ_3, then the inversion $I_{O,\sqrt{k}}$ carries these angles onto the angles δ, δ_1, δ_2, and δ_3 respectively, and it is

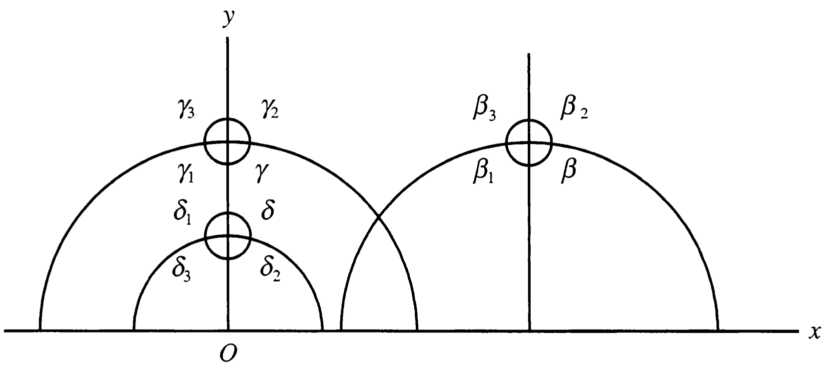

Figure 5.2

now also clear that there is a hyperbolic translation that carries the angles β, β_1, β_2, and β_3 onto the respective γs. These considerations demonstrate that all right angles one of whose sides is a straight geodesic are indeed congruent. Suppose now that α is a right angle both of whose sides p and q are bowed geodesics (Fig. 5.3). If d is the Euclidean length of the line segment AB, then, by Theorem 3.1.3 the inversion $I_{A,d}$ maps the geodesic p into a straight geodesic, thus showing that α is congruent to a right angle one of whose sides is a straight geodesic. Since those angles were already shown to be hyperbolically congruent to δ, we are done.

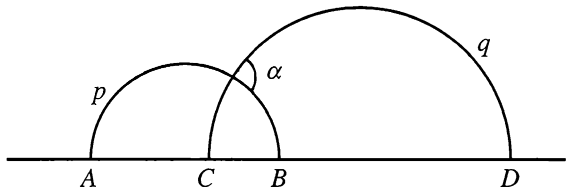

Figure 5.3

We digress here to point out that the above argument can be easily modified to demonstrate a more general result. A hyperbolic angle is said to be in *standard position* if one of its sides is the ray $\{(0, y) \mid 1 \le y < \infty\}$ and the other is in the half-plane $\{(x, 0) \mid x > 0\}$. For example, the angle δ of Fig. 5.2 is in standard position.

Proposition 5.1.4 *In hyperbolic geometry, every angle is congruent to an angle in standard position.*

□

Playfair's Postulate (Euclid's Postulate 5). *Given a straight line m and a point P not on m, there is a unique straight line that is parallel to m and contains P.*

If we define two nonintersecting geodesics as parallel, then it is easy to see that Playfair's Postulate no longer holds in the upper half-plane. In fact, the geodesics p, q, and r of Fig. 5.4 all pass through the point P and are all parallel to the geodesic s. The reader can no doubt construct many other such parallels.

There is of course another way to look at parallel lines. In Euclidean geometry, given a straight line m, the locus of all those points P whose distance from m is a constant d is a pair of straight lines

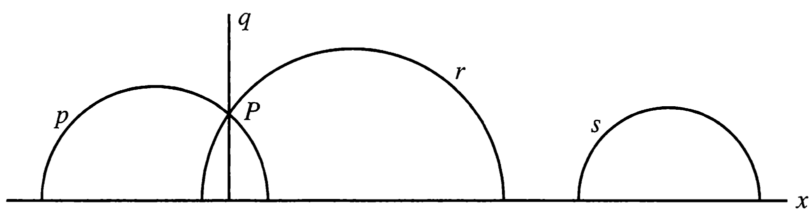

Figure 5.4

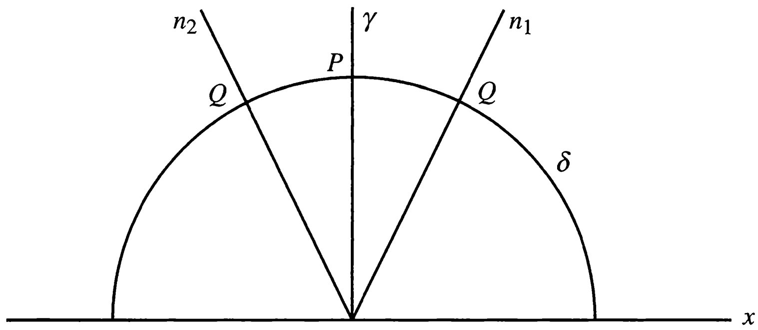

Figure 5.5

p and p' which are both parallel to m and which lie at a distance d on either of its sides. While the hyperbolic analog of this is not quite so simple, it is interesting enough to merit some discussion. The resulting configuration has an elegance all its own.

First, we need to discuss the hyperbolic distance from a point to a geodesic. Given a geodesic γ and a point P not on γ, it follows from Proposition 12 of Euclid that there is a geodesic δ containing P that is perpendicular to γ. (The reader who is uncomfortable with this line of reasoning can prove this directly as a statement about orthogonal circles.) It also follows from Euclid's Propositions 17 and 27 that the geodesic δ is in fact unique. In complete analogy with Euclidean geometry, the hyperbolic distance of the point P from the geodesic γ is the hyperbolic length of the unique geodesic segment that joins P to γ and is perpendicular to γ.

Suppose first that γ is a straight geodesic. Referring to Fig. 5.5, it is clear that the bowed geodesics δ that are positioned symmetrically about γ are all orthogonal to it. Moreover, it follows from Corollary 4.1.2 that regardless of the position of the point P on the geodesic γ, the hyperbolic length of the geodesic segment PQ of δ has a constant value, say d. Thus, the locus of points Q that are at a constant hyperbolic distance from γ consists of the two Euclidean straight lines n_1 and n_2, which, while they are not hyperbolic straight lines, may still be said to be parallel to it. Note that it is very tempting to say here that n_1 and n_2 intersect γ at infinity.

If γ is a bowed geodesic, let A and B be its Euclidean endpoints, and let r be its Euclidean radius (Fig. 5.6). By Theorem 3.1.3, the inversion $I_{A,2r}$ transforms γ into a Euclidean ray that emanates from B. Since this ray must also be a geodesic, it follows that γ is transformed into the straight geodesic γ' that lies directly above B. Let m_1 and m_2 be the above described locus of all the points that lie at a constant hyperbolic distance d from γ'. Since the inversion $I_{A,2r}$ is a hyperbolic rigid motion, it follows that the curves

$$n_1 = I_{A,2r}(m_1) \quad \text{and} \quad n_2 = I_{A,2r}(m_2)$$

constitute the locus of all the points that are at a constant hyperbolic distance d from the geodesic γ. However, in view of Theorem 3.1.3, we may now conclude that this locus consists of the two circular arcs joining A and B that are depicted in Fig. 5.6. Since the Euclidean straight lines m_1 and m_2 make the same angle with the geodesic γ', the same holds for the angles that n_1 and n_2 make with γ at the point B. Consequently, if n_1' is the reflection of n_1 in the x-axis, then n_2 and n_1' together form a complete Euclidean circle.

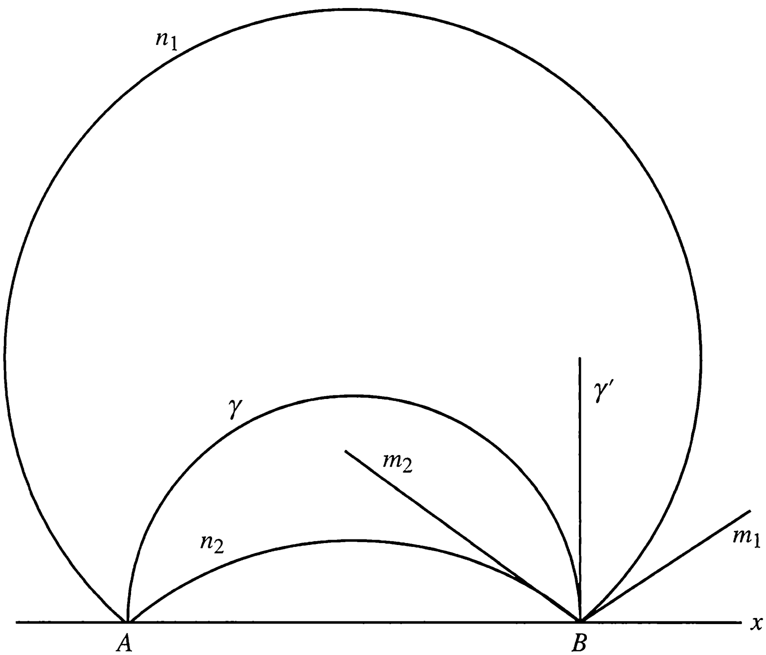

Figure 5.6

Example 5.1.5
Let us find the curves that lie at the constant hyperbolic distance 1 from the y-axis. It follows from the above considerations that these are two Euclidean rays from the origin. If one ray has inclination $\alpha < \pi/2$ from the positive x-axis, then, by Proposition 4.1.1,

$$\ln \frac{\csc(\pi/2) - \cot(\pi/2)}{\csc \alpha - \cot \alpha} = 1.$$

This yields the equations

$$\csc \alpha - \cot \alpha = e^{-1} \quad \text{and} \quad \csc \alpha + \cot \alpha = e$$

from which we conclude that

$$\alpha = \sin^{-1}\left(\frac{2}{e + e^{-1}}\right) \approx 40^\circ.$$

Thus, the locus in question consists of the two rays from the origin whose inclinations from the positive and negative directions along the x-axis are 40°.

Example 5.1.6
Describe the locus of those points that lie at the constant distance of 1 from the bowed geodesic with Euclidean center (2, 0) and Euclidean radius 3 (Fig. 5.7). By the discussion preceding Example 5.1.5,

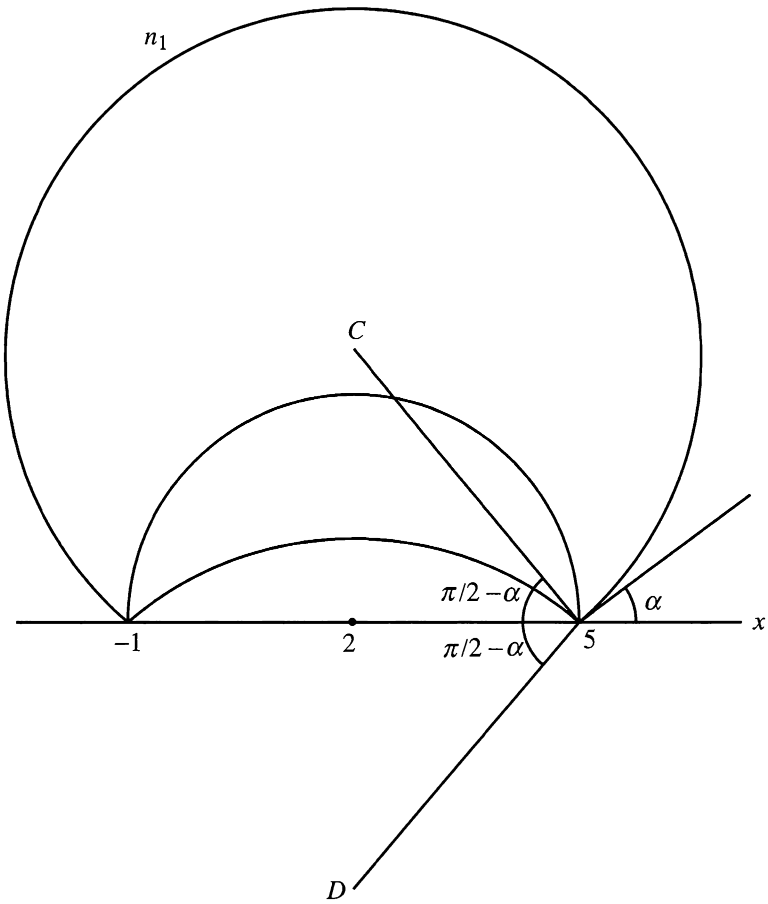

Figure 5.7

this locus consists of two arcs of Euclidean circles that pass through the points $(-1, 0)$ and $(5, 0)$ whose tangents, at those points, make an angle of

$$\alpha = \sin^{-1}\left(\frac{2}{e + e^{-1}}\right) \approx 40^\circ$$

with the x-axis. Since the radius through the point of contact is perpendicular to the tangent, it now follows that the centers C and D of these arcs have coordinates $\left(2, \pm 3\tan\left(\frac{\pi}{2} - \alpha\right)\right) \approx (2, \pm 3.52)$.

Exercises 5.1

1. Find the hyperbolic center and radius of the Euclidean circle with center (5, 4) and radius 3. Find both its hyperbolic and Euclidean circumferences.
2. Find the Euclidean center and radius of the circle with hyperbolic center (5, 4) and radius 3.