Homework 5

Geometry, MTH 329 Spring 2026

Instructor: Abhijit Champanerkar

Due: Monday May 11th [1] [2]

Reading

1. Sections 15.1, 15.2, 15.3

Problems

Problems # 1 to # 9 from the attached Isometries handout.

Hand-in: Problems 2, 4, 6, 8.1, 8.2

[1] Please upload pdf scan on Brightspace

[2] All section, chapter, page and example numbers refer to the book "Geometry: The Line and the circle" by Carroll and Rykken.

Isometries of Euclidean and Spherical Geometry

MTH 329, Fall 2026

Professor Abhijit Champanerkar

1 Euclidean Isometries

1.1 Classification of Euclidean Isometries

# Reflections	Isometry Type	Orientation	Specified By	Fixed Points	Invariant Set
0	Identity (**Id**)	Preserves	$I(x) = x$	All points	Every point
1	Reflection (**F**)	Reverses	Line L	Line L	Axis L
2	Translation (**T**)	Preserves	Vector $\vec{v}$	None	Lines $\parallel \vec{v}$
2	Rotation (**R**)	Preserves	P and θ	Point P	Circles centered at P
3	Glide Reflection (**G**)	Reverses	Axis $L, \vec{v} \parallel L$	None	Axis L (setwise)

1.2 Orientation Parity

# Reflections	Orientation	Isometry Types
Even	Preserves	Identity, Translation, Rotation
Odd	Reverses	Reflection, Glide Reflection

1.3 Composition (Product) Table

○	**T**	**R**	**F**	**G**
T	T, Id	R	F, G	F, G
R	R	R, T, Id	F, G	F, G
F	F, G	F, G	R, T, Id	R, T, Id
G	F, G	F, G	R, T, Id	R, T, Id

1.4 Problems

Problem 1 *Find a translation which takes the point* $(3, 4)$ *to* $(7, -2)$. *Recall that a translation is a transformation of the form* $t_{a,b}(x, y) = (x + a, y + b)$.

Problem 2 *Find a translation which take line* $x + y = 5$ *to the line* $x + y = 0$.

Problem 3 *Find a rotation which takes the line* $x + y = 0$ *to the* x*-axis. Recall that a rotation about the origin in counterclockwise directions is a transformation of the form* $r_{c,s}(x, y) = (cx - sy, sx + cy)$ *where* $c = \cos\theta, s = \sin\theta$, θ *is angle of rotation.*

Problem 4 *Let* f *be an isometry of the plane given by reflection in three lines as* $f = r_c \circ r_b \circ r_a$. *Suppose the lines* a, b *and* c *are concurrent. Classify the isometry* f.

Problem 5 *An isometry* f *is a composition of 28 reflections and it fixes the origin. Which type of isometry is* f *? Why ?*

Problem 6 (Geometric Identification) *Let* f *and* g *be reflections across two lines* L_1 *and* L_2.

1. *If* $L_1 \parallel L_2$ *and the distance between them is* 3 *cm, describe the isometry* $f \circ g$ *in detail (type and magnitude).*

2. *If* L_1 *and* L_2 *intersect at an angle of* $40°$ *at a point* P, *describe the isometry* $f \circ g$. *Does the order of composition (*$f \circ g$ *vs.* $g \circ f$*) affect the resulting transformation type?*

Problem 7 (The Three-Reflection Theorem) *A glide reflection* G *consists of a reflection across a line* L *and a translation by a vector* $\vec{v}$, *where* $\vec{v} \parallel L$.

1. *Prove that* G *can be expressed as the composition of exactly three reflections* $f_1 \circ f_2 \circ f_3$.

2. *Explain why G cannot be expressed as the composition of only two reflections.*

Problem 8 (Composition Parity) *Using orientation arguments and the classification theorem, determine the resulting isometry type for the following compositions:*

1. $R_1 \circ R_2$*, where* R_1 *and* R_2 *are rotations about distinct centers* P *and* Q*.*

2. $F \circ T$*, where* F *is a reflection and* T *is a non-trivial translation.*

3. $G \circ G$*, where* G *is a glide reflection.*

Problem 9 (Fixed Point Analysis) *Suppose an isometry* ϕ *of the Euclidean plane fixes two distinct points* A *and* B*.*

1. *Prove that* ϕ *must fix every point on the line* L *passing through* A *and* B*.*

2. *Based on the classification of isometries, which specific types could* ϕ *be?*

1.5 Matrix/vector Notations

1 The **translation** $T_{\mathbf{v}}$ by the vector $\mathbf{v} = \begin{pmatrix} a \\ b \end{pmatrix}$ is defined as the mapping:

$$T_{\mathbf{v}} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} x + a \\ y + b \end{pmatrix}$$

2 A **rotation** by angle θ about a point $\mathbf{p} = (a, b)$ can be expressed using 2×2 matrices and vector addition. Let:

$$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}, \quad \mathbf{p} = \begin{pmatrix} a \\ b \end{pmatrix}, \quad R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

The transformation is given by the composition:

$$\mathbf{x}' = R_{\mathbf{p},\theta}(\mathbf{x}) = T_{\mathbf{p}} \circ R_{O,\theta} \circ T_{-\mathbf{p}}(\mathbf{x})$$

In terms of matrix-vector operations, this is written as:

$$\mathbf{x}' = R(\mathbf{x} - \mathbf{p}) + \mathbf{p}$$

Expanding the terms, we obtain:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x-a \\ y-b \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix}$$

3 Let the line L be defined by $y = mx + b$. Let $\theta = \arctan(m)$ be the angle of inclination. The **reflection** of a point $\mathbf{x}$ across L is given by:

$$F_L(\mathbf{x}) = R_\theta F_x R_{-\theta}(\mathbf{x} - \mathbf{p}) + \mathbf{p}$$

Where:

$$\mathbf{p} = \begin{pmatrix} 0 \\ b \end{pmatrix}, \quad R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \quad F_x = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Expanding the linear part $M = R_\theta F_x R_{-\theta}$ yields the Householder reflection matrix:

$$M = \begin{pmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{pmatrix}$$

The full transformation in matrix-vector form is:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{pmatrix} \begin{pmatrix} x \\ y-b \end{pmatrix} + \begin{pmatrix} 0 \\ b \end{pmatrix}$$

Explanation

The reflection F_L across the line $L : y = mx + c$ is achieved through the following sequence:

1. **Translate** the line so it passes through the origin: $T_{-\mathbf{p}}$ where $\mathbf{p} = \begin{pmatrix} 0 \\ c \end{pmatrix}$.
2. **Rotate** the line by $-\theta$ so it aligns with the x-axis: $R_{-\theta}$ where $\theta = \arctan(m)$.
3. **Reflect** across the x-axis: $F_x = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.
4. **Invert Rotation**: R_θ.
5. **Invert Translation**: $T_{\mathbf{p}}$.

The composition is written as:

$$F_L = T_{\mathbf{p}} \circ R_\theta \circ F_x \circ R_{-\theta} \circ T_{-\mathbf{p}}$$

2 Spherical Isometries

2.1 The Three Reflection Theorem

Every isometry of the sphere S^2 can be expressed as the composition of at most three reflections across great circles.

Unlike Euclidean space, S^2 lacks translations and glide reflections because all great circles intersect. Therefore, all orientation-preserving isometries (except the identity) are rotations.

This leads to a natural classification based on the number of reflections required:

Reflections	**Resulting Isometry**	**Orientation**	**Fixed-Point Set**
0	Identity	Preserving	Entire S^2
1	Reflection	Reversing	Great Circle
2	Rotation	Preserving	Two Antipodal Points
3	Rotary Reflection	Reversing	None (typically)

Table 1: Classification of isometries in S^2

2.2 Orientation-Preserving Isometries

Identity Fixes every point on the sphere (zero reflections).

Rotation Every orientation-preserving isometry of S^2 (other than the identity) is a rotation about some axis. According to Euler's Rotation Theorem, any composition of rotations is itself a single rotation. It fixes exactly two antipodal points (the poles).

2.3 Orientation-Reversing (Indirect) Isometries

These correspond to elements of $O(3)$ with determinant -1.

Reflection Fixes a great circle (the "equator" of the reflection).

Rotary Reflection (Antipodal Map + Rotation) A composition of a rotation about an axis and a reflection across a plane perpendicular to that axis.

Note: The Antipodal Map ($x \mapsto -x$) is a specific type of rotary reflection. In S^2, the antipodal map is orientation-reversing because the dimension is even ($n = 2$).

Fixed Points A simple reflection fixes a great circle, while a generic rotary reflection (that is not a pure reflection) fixes no points on the sphere.

Since there are no parallel lines in S^2, every isometry must either fix points or be a "rotary" transformation.

2.4 Classification and the Antipodal Map

Orientation-preserving isometries belong to the subgroup $SO(3)$, while orientation-reversing isometries have a determinant of -1. A fundamental result is that the composition of three reflections is equivalent to a rotation followed by the **antipodal map**.

$$M = R_1 R_2 R_3 \tag{1}$$

Since the product of two reflections $R_1 R_2$ is a rotation $Rot(\theta)$, we examine the third reflection. Any reflection R across a plane with normal v can be factored as:

$$R = (-I) \cdot Rot_v(\pi) \tag{2}$$

where $-I$ is the antipodal map. Consequently, the total isometry becomes:

$$M = Rot(\theta) \cdot (-I) \cdot Rot_v(\pi) = (-I) \cdot Rot(\phi) \tag{3}$$

In 3D, the antipodal map $x \mapsto -x$ is orientation-reversing because $\det(-I) = (-1)^3 = -1$. In the specific context of S^2, this inversion ensures that any orientation-reversing isometry that is not a simple reflection is a rotary reflection.