Sample problems for Exam 2 Math 301 Introduction to Proof Fall 2023

- Syllabus for Exam 2: Chapters 3, 4 from the book.
- Best way to prepare for the exam is to review definitions and study problems on homework, quizzes and the sample problems below.
- Please review the Function Practice Problems Fun Pack.
- Exam 2 will be held on Wednesday Nov 29th.
- Review for Exam 2 will be held Monday Nov 27th.
- (1) Define the following precisely using quantifiers: function, injective function, surjective function, bijective function, subset, powerset, divisor, GCD, prime number, composite number, even number, odd number.
- (2) Find 2 examples of a function $f: \mathbb{N} \to \mathbb{Z}$ which is
 - (a) injective but not surjective
 - (b) neither injective nor surjective
 - (c) bijective
- (3) Find an example of a function $f \colon \mathbb{R} \to \mathbb{R}$ which is
 - (a) surjective but not injective
 - (b) neither injective nor surjective
 - (c) bijective
- (4) Find explicit bijections between the following subintervals of \mathbb{R} .
 - (a) (0,1) and $(1,\infty)$
 - (b) $(1,\infty)$ and $(0,\infty)$
 - (c) $(0,\infty)$ and $(-\infty,0)$
 - (d) (0, 1) and $(-\infty, 0)$
- (5) What do the following statements mean in ordinary language? Write out their negations using quantifiers.
 - (a) $(\forall n \in N) (x \neq 3^n)$
 - (b) $(\exists a, b \in \mathbb{N})(\sqrt{2} = a/b)$
 - (c) $(\forall b \in B)(\exists a \in A)(f(a) = b)$, assume $f: A \to B$
 - (d) $(\forall n \in \mathbb{N})(\exists m \in \mathbb{N})(n^2 n = 2m)$
 - (e) $(\exists A \subset \mathbb{R})(\forall x \in \mathbb{R})(\exists a \in A)(x > a)$
 - (f) $(\forall n \in \mathbb{N})(\exists p \in \mathbb{N})((p > n) \text{ and } [(\forall q \in \mathbb{N})(q | p \Rightarrow (q = 1) \text{ or } q = p)])$

- (6) State the negation of the following statements, using appropriate quantifiers. (a) π^2 is irrational.
 - (b) The function f is injective but not surjective.
 - (c) The integer n is divisible by two distinct primes.
 - (d) There is an injective function $f: A \to B$.
 - (e) The function $f \colon \mathbb{R} \to \mathbb{R}$ is bounded above but not below.
- (7) Write out precise statements and careful proofs, or give counterexamples, of the following statements.
 - (a) Composition of injective functions is injective.
 - (b) Composition of surjective functions is surjective.
 - (c) Sum of the squares of any three consecutive integers, plus one, is divisible by 3.
 - (d) Product of two consecutive integers is divisible by 3.
 - (e) Product of two consecutive integers is divisible by 2. (Hint: COnsider 2 case: n is even or n is odd).
 - (f) If f is injective, then $f \circ g$ is injective.
 - (g) If f and $f \circ g$ are surjective, then g is surjective.
 - (h) If $f \circ g \colon \mathbb{R} \to \mathbb{R}$ is decreasing, and $g \colon \mathbb{R} \to \mathbb{R}$ is decreasing, then f is decreasing.
 - (i) If $x \in \mathbb{R}$ and $x^2 \leq x$, then $x \leq 1$.
- (8) Let $f: X \to Y$ be a function. If $A \subseteq Y$, let $f^{-1}(A)$ be the pre-image of A in X. Show that this defines a function from $\mathcal{P}(Y)$ to $\mathcal{P}(X)$. Can you say when it is injective or surjective?
- (9) Explore the definition of an inverse function. Prove that a function $f: A \to B$ has an inverse if and only if it is both surjective and injective.
- (10) Suppose that $f: X \to Y$ and let $A \subseteq X$ and $B \subseteq Y$.
 - (a) Prove or give a counterexample: $f^{-1}(f(A)) \subseteq A$
 - (b) Prove or give a counterexample: $B \subseteq f(f^{-1}(B))$
 - (c) Prove or give a counterexample: $f(\overline{A} \cup f^{-1}(B)) = f(A) \cup B$.
 - (d) Prove or give a counterexample: $f^{-1}(f(A) \cap B) = A \cap f^{-1}(B)$

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