## Sample problems for Exam 2 Math 301 Introduction to Proof Fall 2023

- Syllabus for Exam 2: Chapters 3, 4 from the book.
- Best way to prepare for the exam is to review definitions and study problems on homework, quizzes and the sample problems below.
- Please review the Function Practice Problems Fun Pack.
- Exam 2 will be held on Wednesday Nov 29th.
- Review for Exam 2 will be held Monday Nov 27th.
(1) Define the following precisely using quantifiers: function, injective function, surjective function, bijective function, subset, powerset, divisor, GCD, prime number, composite number, even number, odd number.
(2) Find 2 examples of a function $f: \mathbb{N} \rightarrow \mathbb{Z}$ which is
(a) injective but not surjective
(b) neither injective nor surjective
(c) bijective
(3) Find an example of a function $f: \mathbb{R} \rightarrow \mathbb{R}$ which is
(a) surjective but not injective
(b) neither injective nor surjective
(c) bijective
(4) Find explicit bijections between the following subintervals of $\mathbb{R}$.
(a) $(0,1)$ and $(1, \infty)$
(b) $(1, \infty)$ and $(0, \infty)$
(c) $(0, \infty)$ and $(-\infty, 0)$
(d) $(0,1)$ and $(-\infty, 0)$
(5) What do the following statements mean in ordinary language? Write out their negations using quantifiers.
(a) $(\forall n \in N)\left(x \neq 3^{n}\right)$
(b) $(\exists a, b \in \mathbb{N})(\sqrt{2}=a / b)$
(c) $(\forall b \in B)(\exists a \in A)(f(a)=b)$, assume $f: A \rightarrow B$
(d) $(\forall n \in \mathbb{N})(\exists m \in \mathbb{N})\left(n^{2}-n=2 m\right)$
(e) $(\exists A \subset \mathbb{R})(\forall x \in \mathbb{R})(\exists a \in A)(x>a)$
(f) $(\forall n \in \mathbb{N})(\exists p \in \mathbb{N})((p>n)$ and $[(\forall q \in \mathbb{N})(q \mid p \Rightarrow(q=1)$ or $\mathrm{q}=\mathrm{p})])$
(6) State the negation of the following statements, using appropriate quantifiers.
(a) $\pi^{2}$ is irrational.
(b) The function $f$ is injective but not surjective.
(c) The integer $n$ is divisible by two distinct primes.
(d) There is an injective function $f: A \rightarrow B$.
(e) The function $f: \mathbb{R} \rightarrow \mathbb{R}$ is bounded above but not below.
(7) Write out precise statements and careful proofs, or give counterexamples, of the following statements.
(a) Composition of injective functons is injective.
(b) Composition of surjective functons is surjective.
(c) Sum of the squares of any three consecutive integers, plus one, is divisible by 3 .
(d) Product of two consecutive integers is divisible by 3 .
(e) Product of two consecutive integers is divisible by 2. (Hint: COnsider 2 case: $n$ is even or $n$ is odd).
(f) If $f$ is injective, then $f \circ g$ is injective.
(g) If $f$ and $f \circ g$ are surjective, then $g$ is surjective.
(h) If $f \circ g: \mathbb{R} \rightarrow \mathbb{R}$ is decreasing, and $g: \mathbb{R} \rightarrow \mathbb{R}$ is decreasing, then $f$ is decreasing.
(i) If $x \in \mathbb{R}$ and $x^{2} \leqslant x$, then $x \leqslant 1$.
(8) Let $f: X \rightarrow Y$ be a function. If $A \subseteq Y$, let $f^{-1}(A)$ be the pre-image of $A$ in $X$. Show that this defines a function from $\mathcal{P}(Y)$ to $\mathcal{P}(X)$. Can you say when it is injective or surjective?
(9) Explore the definition of an inverse function. Prove that a function $f: A \rightarrow B$ has an inverse if and only if it is both surjective and injective.
(10) Suppose that $f: X \rightarrow Y$ and let $A \subseteq X$ and $B \subseteq Y$.
(a) Prove or give a counterexample: $f^{-1}(f(A)) \subseteq A$
(b) Prove or give a counterexample: $B \subseteq f\left(f^{-1}(B)\right)$
(c) Prove or give a counterexample: $f\left(A \cup f^{-1}(B)\right)=f(A) \cup B$.
(d) Prove or give a counterexample: $f^{-1}(f(A) \cap B)=A \cap f^{-1}(B)$

