Appendix B

Basic Ideas about Sets and Arithmetic

Introduction

The language of sets is used throughout mathematics. Although there is an axiomatic, formal way of talking about sets, our use of sets in this book is primarily intuitive. There are several simple ideas and notations that you should be familiar and comfortable with, which are presented in this appendix. Some of these ideas—especially *set-builder notation, subset,* and the *empty set*—are discussed more fully in Section 2.1. This appendix also lists some basic assumptions about arithmetic which will be used in this text.

How to Describe a Set

A set is simply a collection of objects. The objects in a set are usually called its *elements* or *members*. There are several common ways to describe a set or specify what its elements are.

a) *Listing:* A small set can often best be specified by actually listing the elements that belong to it. Braces—$\{\ \}$—are used to enclose the elements. For example:

$\{1, 2, 3, 4\}$ is the set containing the first four positive integers;
$\{1, 2, 3, 4, 6, 12\}$ is the set of positive integers that divide "evenly" into 12.

The order in which objects are listed does not matter; the set $\{1, 2, 3\}$ is the same as the set $\{2, 3, 1\}$. Also, repeating an element in the listing does not change the set: $\{1, 2, 3\}$ is the same as the set $\{1, 2, 1, 3\}$.

b) *Listing with ellipsis:* "Ellipsis" is the name for the mathematical "etcetera" symbolized by three dots (...). It means that a list should continue in the pattern already indicated. The set goes on "forever" if nothing comes after the ellipsis, or the three dots can be used to represent the "middle" of the set, if some object is given at the end. For example:

$\{2, 4, 6, 8, \ldots\}$ is the set of positive even integers, which goes on indefinitely.

Because it is infinite, this set could not be described by a list without ellipsis.

$\{a, b, c, d, \ldots, m\}$ is the set of letters in the first half of the alphabet. Here ellipsis is simply a convenience, to avoid having to list all the letters individually.

Ellipsis must be used with caution, to be sure it is not ambiguous. $\{2, 4, \ldots\}$ is not a good use of ellipsis, because this could represent the set of powers of 2—$\{2, 4, 8, 16, 32, \ldots\}$—as well as the set of positive even integers—$\{2, 4, 6, 8, 10, \ldots\}$.

c) *Verbal description:* Often this is the simplest way to describe a set. It helps give a sense of the relationship between the elements, i.e., explain why they are being grouped together. For example: "the set of points in the plane with integer coordinates" describes a set that includes such points as $(4, -7)$, $(0, 2)$, and $(-3, 0)$, but excludes $(3.2, 6)$ or $(2, \pi)$.

d) *Set-builder notation:* This is a special type of verbal description, in which a symbol such as x or y is used to represent a possible element, and then a condition about that symbol is given to indicate whether an element is in the set or not. For example:

$$\{x : x \text{ is a real number and } 4 \leq x < 7\}.$$

The colon (:) is read as "such that". Thus, this set is "the set of all x such that x is a real number and 4 is less than or equal to x which is less than 7." More colloquially, we would describe this as "the set of all real numbers which are greater than or equal to 4 and less than 7." The real numbers 4, 5.3, and 6.19 belong to this set; 7, −2.6, and 8.3 do not. Set-builder notation is a powerful tool, and is discussed more fully in the text (see Section 2.1).

No matter what method is used to describe a set, a set is defined by what its elements are. Therefore, two sets are considered *equal* precisely when they have the same elements, even if the descriptions are different.

For example, "the set of odd integers between 2 and 8" and "the set of the first three odd prime numbers" both describe the set whose elements are 3, 5, and 7. Therefore these two sets are equal.

Elements and Subsets

No matter what method is used to indicate which objects belong to a set, it is often convenient to designate the set itself by a symbol; capital letters are usually used. For example, we might use the letter A to represent the set described in d) above:

$$A = \{x : x \text{ is a real number and } 4 \leq x < 7\}$$

The symbol "$\in$" is used to indicate that a particular object is an element of a particular set. For example, the real number 5.3 is an element of the set A just described. We can represent this fact symbolically by the notation "$5.3 \in A$". (The symbol "$\in$" is variously read as "is an element of", "is a member of", "is in", or "belongs to".) The symbol "$\notin$" means "is not a member of".

If every element of one set is also an element of a second set, we say that the first set is a *subset* of the second. For example, let B represent the set $\{4.5, 5, 5.5, 6, 6.5\}$. Since each one of these numbers belongs to A, we say that B is a subset of A; this is written symbolically "$B \subseteq A$" (read "B is a subset of A", or "B is contained in A"). Any set is considered to be a subset of itself. The notation "$A \not\subseteq B$" means "A is not a subset of B".

Caution: Both the "element of" relationship ($\in$) and the "subset of" relationship ($\subseteq$) are sometimes expressed verbally by such phrases as "is in", "is part of", or "is contained in". Keep in mind that the two relationships are quite distinct and are not interchangeable. To avoid confusion, it is sometimes important to insist on the more formal language. Similarly, an individual object, such as the number 6, is different from the set with that object as its only member: $\{6\}$.

A set X is called a *proper subset* of some set Y if X is a subset of Y other than Y itself. This is represented by the notation "$X \subset Y$" (read "X is a proper subset of Y" or "X is properly contained in Y"). (Notice how the "subset" and "proper subset" symbols resemble the symbols for "less than or equal" and "less than".)

Note: some books use the symbol "$\subset$" to mean subset, and write "$\subsetneq$" to mean proper subset.

If X is a subset of Y, we can turn the notation around and write $Y \supseteq X$, and we say that Y is a *superset* of X.

Special Sets

Certain commonly discussed sets have standard symbols associated with them. The following symbols are used throughout this book, and in most mathematics texts, to mean the specific sets described here:

$\varnothing$: this represents the *empty set* (also known as the *null set*), that is, the set with no elements in it. The empty set is a subset of every set.

N: this represents the *natural numbers*, that is, the set of positive integers $\{1, 2, 3, 4, \ldots\}$.

W: this represents the *whole numbers*, that is, the positive integers together with zero: $\{0, 1, 2, 3, \ldots\}$.

Z: this represents the *integers*, that is, positive, negative, and zero: $\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$.

Q: this represents the *rational numbers*, that is, all numbers which can be represented as fractions, whether positive or negative (this includes all the integers as well).

R: this represents the *real numbers*, that is, all the numbers that correspond to points on the real number line.

C: this represents the *complex numbers*, that is, all numbers which are combinations of real and imaginary numbers.

We occasionally use notation such as $\mathbf{Q}^+$ for positive rational numbers or $\mathbf{R}^{\geq 0}$ for nonnegative real numbers.

In addition, we use the following standard notation for intervals on the real number line:

$$[a, b] = \{x : a \leq x \leq b\}$$

$$[a, b) = \{x : a \leq x < b\}$$

$$(a, b] = \{x : a < x \leq b\}$$

$$(a, b) = \{x : a < x < b\}$$

$$[a, \infty) = \{x : a \leq x\}$$

$$(a, \infty) = \{x : a < x\}$$

$$(-\infty, b] = \{x : x \leq b\}$$

$$(-\infty, b) = \{x : x < b\}$$

$$(-\infty, \infty) = \mathbf{R}$$

Making New Sets from Old Ones

Sets can be combined to create other sets, just as numbers are combined arithmetically to give other numbers. A general process by which objects are combined to make a new object is called an *operation*. (Addition, subtraction, etc., are operations on numbers.) If an operation works with two objects to create a third, it is called a *binary operation*. (We define this term formally in Section 7.1.) There are several standard binary operations for combining sets:

The *union* of two sets X and Y is the set which contains precisely those elements which belongs to either X or Y (or both). The combined set is written $X \cup Y$ (read "X union Y"). In set-builder notation, we can write

$$X \cup Y = \{w : w \in X \text{ or } w \in Y\}.$$

For example, if $X = \{1, 4, 6\}$ and $Y = \{2, 4, 9\}$, then

$$X \cup Y = \{1, 2, 4, 6, 9\}.$$

Notice that the element 4 belongs to both X and Y, and it is included (once) in the union.

The *intersection* of two sets X and Y is the set which contains precisely those elements which belong to both X and Y. This set is written $X \cap Y$ (read "X intersect Y"). In set-builder notation, we can write

$$X \cap Y = \{w : w \in X \text{ and } w \in Y\}.$$

If $X \cap Y = \varnothing$, that is, if X and Y have no elements in common, then we say that X and Y are *disjoint*. Notice that $X \cap Y$ is always a subset of X, Y, and $X \cup Y$. For example, if $X = \{1, 4, 6\}$ and $Y = \{2, 4, 9\}$, then

$$X \cap Y = \{4\}.$$

The two binary operations just described — union and intersection — can be extended naturally to work with more than two sets at a time. Thus, the union of several sets consists of those elements which belong to any of the given sets, and the intersection consists of those elements which belong to all of the given sets.

The next operation can only be applied to two sets at a time:

> The *difference* of two sets X and Y is the set which contains precisely those elements which belong to X but not to Y. The combined set is written $X - Y$ (read "X minus Y"). (There may be objects in Y which are not in X; we can't "remove" these from X.) In set-builder notation, we can write
>
> $$X - Y = \{w : w \in X \text{ and } w \notin Y\}.$$
>
> For example, if $X = \{1, 4, 6\}$ and $Y = \{2, 4, 9\}$, then
>
> $$X - Y = \{1, 6\}.$$

Often a mathematical discussion will have a *universe of discourse,* a set $\mathbf{U}$ which represents the overall framework of the given situation. Usually this universe is some standard set such as $\mathbf{R}$ or $\mathbf{Z}$. In this context, we define the *complement* of an individual set A to be the set difference $\mathbf{U} - A$, i.e., the set consisting of those elements of the given universe which do not belong to A. This difference is abbreviated symbolically as A' (read "A prime" or "A complement"). For example, if the universe is $\mathbf{R}$, and A is the interval $[4, 7)$, then A' is the set $(-\infty, 4) \cup [7, \infty)$.

Keep in mind that the idea of the complement of a set is only meaningful if some universe has been specified, and that the meaning of A' will change if the universe is changed.

While union, intersection, and difference are binary operations, complementation is an example of a *unary operation* — a way of defining a new set which starts with just one set.

We define two other important set operations — *Cartesian product* and *power set* — in Section 2.1.

Example 1. Operations on Sets Suppose $\mathbf{U} = \{1, 2, 3, \ldots, 12\}$. Define the sets A, B, C, D, and E as follows:

$$A = \{1, 2, 3, 4, 5\}, \quad B = \{2, 4, 6, 8\}, \quad C = \{9, 10, 11\},$$
$$D = \{2, 6\}, \quad \text{and} \quad E = \varnothing.$$

Find each of the following:

a) $A \cup B$.
b) $A \cup C$.
c) $B \cup E$.
d) $A \cap B$.
e) $B \cap C$.
f) $A \cap E$.

g) $A - B$.
h) $A - C$.
i) A'.

SOLUTIONS

a) $\{1, 2, 3, 4, 5, 6, 8\}$.
b) $(1, 2, 3, 4, 5, 9, 10, 11\}$.
c) $\{2, 4, 6, 8\}$.
d) $\{2, 4\}$.
e) $\emptyset$.
f) $\emptyset$.
g) $\{1, 3, 5\}$.
h) $\{1, 2, 3, 4, 5\}$.
i) $\{6, 7, 8, 9, 10, 11, 12\}$. □

Cardinality

The number of elements in a finite set X is called its *cardinality*. We will use the notation $|X|$ for the cardinality of the set X, although there are other commonly used notations, such as $n(X)$ and $\#(X)$. For example, if T is the set $\{2, 4, 7, 11, 16\}$, then the cardinality of T is 5, and we can write $|T| = 5$.

In Section 8.1, we give a more formal and abstract definition of cardinality, which can be applied to infinite sets as well. However, the definition given here is used everywhere except Chapter 8.

Assumptions about Arithmetic

Throughout this text, we will be assuming certain basic facts about arithmetic. With certain exceptions that will be clearly noted, we will prove all the other properties that we develop. If a proof done in this text seems to leave out a step, or to make an assumption that seems unjustified, the missing reasoning should be found in the list here, or be a simple variation of, or deduction from, these assumptions. You should refer to this list if you are in doubt, and seek justification of any reasoning that seems unclear.

Since the set **C** of complex numbers includes all the other sets of numbers we are concerned with, we will state our most general properties in terms of that set. In other words, all objects in the following conditions are assumed to be elements of **C**. If a particular property applies only to some smaller set, we will indicate that. (The labels for the properties listed here are defined and discussed primarily in Section 7.1. Many may be familiar to you.)

a) General properties of operations:

i) $a + b = b + a$ (commutativity of addition).
ii) $a \cdot b = b \cdot a$ (commutativity of multiplication).

iii) $(a + b) + c = a + (b + c)$ (associativity of addition).
iv) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ (associativity of multiplication).
v) $a(b + c) = ab + ac$ (distributivity of multiplication over addition).
vi) $a(-b) = -(ab)$ (rule of signs).
vii) If $a + b = a + c$, then $b = c$ (cancellation property for addition).
viii) If $ab = ac$ and $a \neq 0$, then $b = c$ (cancellation property for multiplication).

b) Properties of 0, 1, and inverses:
i) $a + 0 = a$ (identity for addition).
ii) $a \cdot 0 = 0$.
iii) $a \cdot 1 = a$ (identity for multiplication).
iv) $a + (-a) = 0$ (inverse for addition).
v) $-(-a) = a$.
vi) $a - b = a + (-b)$ (definition of subtraction).

For properties vii) and viii), $a \neq 0$:

vii) $a \cdot a^{-1} = 1$ (inverse for multiplication).
viii) $(a^{-1})^{-1} = a$.

For property ix), $b \neq 0$:

ix) $a \div b = a \cdot b^{-1}$ (definition of division).

c) Closure properties:

i) If a and b are both real numbers, then $a + b$, $a - b$, and $a \cdot b$ are real numbers; if, further, $b \neq 0$, then $a \div b$ is also a real number.
ii) If a and b are both rational numbers, then $a + b$, $a - b$, and $a \cdot b$ are rational numbers; if, further, $b \neq 0$, then $a \div b$ is also a rational number.
iii) If a and b are both integers, then $a + b$, $a - b$, and $a \cdot b$ are integers.
iv) If a and b are both natural numbers, then $a + b$ and $a \cdot b$ are natural numbers.

d) Properties of real numbers—inequalities (for $a, b, c \in \mathbf{R}$):
i) If $a > 0$, then $1/a > 0$ and $-a < 0$; if $a < 0$, then $1/a < 0$ and $-a > 0$.
ii) If $a > b$, then $a + c > b + c$.
iii) If $a > b$ and $c > 0$, then $ac > bc$ and $a \div c > b \div c$.
iv) If $a > b$ and $c < 0$, then $ac < bc$.
v) For any two real numbers, exactly one of the following is true: $a < b$, $a = b$, $a > b$ (trichotomy property).
vi) If $a > b$ and $b > c$, then $a > c$ (transitivity property of inequality).

e) Properties of real numbers—exponents (for $a, b, c \in \mathbf{R}$, $a > 0$):
i) $a^{b+c} = a^b a^c$.
ii) $(a^b)^c = a^{bc}$.

iii) $(ab)^c = a^c b^c$.

iv) $a^{-b} = 1/a^b$.

f) Properties of natural numbers (for $a, b \in \mathbf{N}$):

i) $0 < 1 \leq a$ (i.e., 1 is positive, and is the smallest natural number).

ii) If $a > b$, then $a \geq b + 1$ (i.e., there are no natural numbers between b and $b + 1$).