Symmetry of hypersurfaces and the Hopf Lemma

Abstract. In 1945, S.S. Chern provided the following characterization of spheres in three-dimensional Euclidean space: Let M be a closed convex surface satisfying

$$F(\kappa_1, \kappa_2) = 1,$$

where κ_1 and κ_2 denote the principal curvatures, and F is elliptic in the sense that $\partial_{\kappa_i} F > 0$. Then M must be a sphere.

Important special cases include

$$F(\kappa_1, \kappa_2) = \kappa_1 + \kappa_2 \quad \text{and} \quad F(\kappa_1, \kappa_2) = \kappa_1 \kappa_2,$$

corresponding to prescribed mean curvature and prescribed Gaussian curvature, respectively.

Nirenberg and I explored extensions of this problem and proposed the following conjecture: Let M be a closed convex surface in three-dimensional Euclidean space, and let F be elliptic. Suppose that for any two points (X_1, X_2, X_3) and $(X_1, X_2, \hat{X}_3)$ on M with $X_3 \geq \hat{X}_3$, the inequality

$$F(\kappa_1, \kappa_2)(X_1, X_2, X_3) \leq F(\kappa_1, \kappa_2)(X_1, X_2, \hat{X}_3)$$

holds. Then M must be symmetric about some hyperplane $X_3 = \textit{constant}$.

In this talk, I will survey developments in this area and present open problems, both related to resolving this conjecture and to broader conjectures concerning extensions of the Hopf Lemma.