Knots, graphs and Khovanov homology II

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Outline

Turaev surface

Ribbon graphs

Quasi-trees

Homological width

Conclusion
Let $D$ be a link diagram and let $s_A$ and $s_B$ be the all–$A$ and all–$B$ states of $D$.

Turaev constructed a cobordism between $s_A$ and $s_B$:

Let $\Gamma \subset S^2$ be the 4–valent projection of $D$ at height 0. Put $s_A$ at height 1, and $s_B$ at height $-1$, joined by saddles:
Turaev surface $F(D)$: Attach $|s_A| + |s_B|$ discs to all boundary circles above.

Turaev genus of $D$, $g_T(D) := g(F) = (c(D) + 2 - |s_A| - |s_B|)/2$.

Turaev genus of non-split link $L := g_T(L) = \min_D g_T(D)$. 
Properties

- Non-split link $L$ is alternating iff $g_T(L) = 0$.

- $D$ is alternating on the Turaev surface.

- $g_T(L) \leq dalt(L) = \min$ number of crossing changes to make $L$ alternating.

- Measures “distance” from alternating.

- Turaev surface can be constructed for any two complementary states of $D$. 
Turaev’s Proof of Tait’s conjecture

**Conjecture** (Tait) A reduced alternating diagram $D$ has minimal crossing number among all diagrams for the alternating link $L$.

The proof follows from three claims:

- Although defined for diagrams, the Jones polynomial $V_L(t)$ is a link invariant.

- $s_A$ and $s_B$ contribute the extreme terms $\pm t^\alpha$ and $\pm t^\beta$ of $V_L(t)$.

- span $V_\ell(t) = \alpha - \beta \leq c(\ell) - g_T(\ell)$, with equality if $\ell$ is alternating (generally, adequate).
Since $D$ is alternating on the Turaev surface, we can generalize the Tait graph construction to get graphs on surfaces.

The Turaev surface $F(D)$ can be checkerboard colored with $|s_A|$ white regions (height $> 0$), and $|s_B|$ black regions (height $< 0$).

Let $G_A, G_B \subset F(D)$ be the adjacency graphs for respective regions. $G_A$ and $G_B$ are embedded and

$$v(G_A) = |s_A|, \quad e(G_A) = c(D), \quad f(G_A) = |s_B|$$

$G_A$ (and $G_B$) give a cell decomposition of $F(D)$.

If $D$ is alternating, $G_A$ and $G_B$ are dual Tait graphs on $F(D) = S^2$. 
Example: Pretzel links

Let $p_i, q_j \in \mathbb{N}$. The pretzel link $P(p_1, \ldots, p_n, -q_1, \ldots, -q_m)$, is a link with diagram of the form

- If $m = 0$, then the pretzel link is alternating and $g_T = 0$.

- If $m > 0$ then the pretzel link is non-alternating and can be embedded on the torus. Hence $g_T = 1$. 
Ribbon graphs

An (oriented) ribbon graph $G$ is a multi-graph (loops and multiple edges allowed) that is embedded in an oriented surface $F$, such that its complement is a union of 2-cells. The genus $g(G) := g(F)$.

Example
Algebraic definition

$G$ can also be described by a triple of permutations $(\sigma_0, \sigma_1, \sigma_2)$ of the set $\{1, 2, \ldots, 2n\}$ such that

- $\sigma_1$ is a fixed-point-free involution.
- $\sigma_0 \circ \sigma_1 \circ \sigma_2 = \text{Identity}$

This triple gives a cell complex structure for the surface of $G$ such that

- Orbits of $\sigma_0$ are vertices.
- Orbits of $\sigma_1$ are edges.
- Orbits of $\sigma_2$ are faces.

The genus $g(G) = (2 - v(G) + e(G) - f(G))/2$. 
Ribbon graph example

\[ \sigma_0 = (1234)(56) \]
\[ \sigma_1 = (14)(25)(36) \]
\[ \sigma_2 = (1)(246)(35) \]

\[ \sigma_0 = (1234)(56) \]
\[ \sigma_1 = (13)(26)(45) \]
\[ \sigma_2 = (152364) \]
Ribbon graph from any state of a link diagram

$G_A, G_B$ defined earlier as checkerboard graphs on Turaev surface $F(D)$ were ribbon graphs. We can construct the ribbon graph $G_s$ directly from any state $s$ of $D$:

1. For each crossing of $D$, attach an edge between state circle(s).
2. Collapse each state circle of $s$ to a vertex of $G_s$. 

2. (2006) Dasbach, Futer, Kalfagianni, Lin, and Stoltzfus showed that $V_L(t)$ can be recovered from BRT polynomial of $G_A$. 
Example: From diagram to Tait graph and ribbon graph
For a planar graph, a spanning tree is a spanning subgraph whose regular neighbourhood has one boundary component.

Example
A quasi-tree of a ribbon graph is a spanning ribbon subgraph with one face. The genus of a quasi-tree is its genus as a ribbon graph.

**Example:** Genus one quasi-tree of a genus two ribbon graph.
Every quasi-tree corresponds to an ordered chord diagram,
Let $D$ be a connected link diagram, $G$ its Tait graph, $G_A$ its all-$A$ ribbon graph.

**Theorem.** (C-Kofman-Stoltzfus) Quasi-trees of $G_A$ are in one-one correspondence with spanning trees of $G$:

$$Q_j \leftrightarrow T_v \quad \text{where} \quad v + j = \left( V(G) + E_+(G) - V(G_A) \right) / 2$$

$Q_j$ is quasi-tree of genus $j$, and $T_v$ is spanning tree with $v$ positive edges.
Graphs and Ribbon graphs

Graphs $\leftrightarrow$ Ribbon graphs

Spanning trees $\leftrightarrow$ Quasi-trees

Tutte polynomial $\leftrightarrow$ BRT polynomial

Activity w.r.t. spanning trees $\leftrightarrow$ Activity w.r.t. quasi-trees

Spanning tree expansion of the Tutte polynomial $\leftrightarrow$ Quasi-tree expansion of the BRT polynomial
Quasi-trees and Khovanov homology

**Theorem** (C-Kofman-Stoltzfus) For a knot diagram $D$, there exists a quasi-tree complex $\mathbb{C}(G_A) = \{\mathbb{C}^u_v(G_A), \partial\}$ that is a deformation retract of the reduced Khovanov complex, where

$$\mathbb{C}^u_v(G_A) = \mathbb{Z}\langle Q \subset G_A \mid u(Q) = u, -g(Q) = v \rangle.$$

From above, if $Q_j$ is quasi-tree of genus $j$, and $T_v$ is spanning tree with $v$ positive edges,

$$v + j = (V(G) + E_+(G) - V(G_A))/2.$$
Corollary (C-Kofman-Stoltzfus) For any knot $K$, the width of its reduced Khovanov homology $w_{KH}(K) \leq 1 + g_T(K)$.

Proof. For any ribbon graph $G$, $g(G) = \max_{Q \subseteq G} g(Q)$. Therefore, the quasi-tree complex $\mathbb{C}(G_A)$ has at most $1 + g(G_A)$ rows.
Turaev genus and homological width

- Using $w_{KH}(K)$, we get lower bounds for $g_T(K)$. In particular, $g_T(T(3, q)) \xrightarrow{q \to \infty} \infty$.

- For an adequate knot $K$ with an adequate diagram $D$, T. Abe showed $g_T(K) = g_T(D) = w_{KH}(K) - 1 = c(K) - \text{span} V_K(t)$.

- Dasbach and Lowrance also proved bounds in terms of $g_T(K)$ for the Ozsváth-Szabó $\tau$ invariant and the Rasmussen’s $s$ invariant.

- Similar bounds for homological width of knot Floer homology in terms of $g_T(K)$ were obtained by Adam Lowrance.
Related open problems

1. Find families of homologically thin knots with $g_T(K) > 1$? Generally, are there any lower bounds independent of knot homology?

2. Which operations on knots preserve or increase Turaev genus? For e.g. for adequate knots $g_T(K \# K') = g_T(K) + g_T(K')$ and $g_T$ is preserved under mutation. How about non-adequate knots?

3. How is the Turaev genus related to the topology and hyperbolic geometry of knot complements?
Computer programs to study Knots & Links

- SnapPy (study hyperbolic knots, links and 3-manifolds) by Weeks, Culler and Dunfield.
- Knotscape (old program to study knots) by Thistlethwaite.
- knot by Kodama.
- LinKnot by Jablan and Sazdanovic.
- KnotTheory by Dror Bar-Natan.
- KhoHo by Shumakovitch to compute Khovanov Homology.
- KnotAtlas (database) by Dror Bar-Natan.
- Table of Knot Invariants (database) by Livingston.


7. V. G. Turaev. *A simple proof of the Murasugi and Kauffman...*
Questions

Thank You

Slides available from:
http://www.math.csi.cuny.edu/abhijit/