Mahler measure of the A-polynomial

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Outline

History

$\text{PSL}(2, \mathbb{C})$ A-polynomial

Mahler measure

Bloch-Wigner dilogarithm

Mahler measure of $\overline{A}_0(\ell, m)$

Examples
In 2000 David Boyd observed (numerically) that the two-variable Mahler measure of A-polynomials were equal to sums of hyperbolic volumes. In many cases it was equal to the volume.

In 2003 Boyd and Rodrigues-Villegas explained this observation and gave a technique to compute the Mahler measures of (tempered) two-variable polynomials.

In this talk I will explain:

- How this technique works for A-polynomials.
- Why A-polynomials are natural examples which work.
An *ideal tetrahedron* is a geodesic tetrahedron in hyperbolic 3-space $\mathbb{H}^3$ with all its four vertices on the sphere at infinity.

Every edge gets a complex number called the *edge parameter*. Isometry classes $\leftrightarrow \{ z \in \mathbb{C} \mid \text{Im}(z) > 0 \}$. An ideal tetrahedron with edge parameter $z$ is denoted by $\triangle(z)$.

An *ideal triangulation* of a cusped hyperbolic 3-manifold $N$ is a decomposition into hyperbolic ideal tetrahedra.
Parameter Space

Let $N$ be one-cusped hyperbolic 3-manifold triangulated with $n$ tetrahedra.

- At every edge the tetrahedra close up and their parameters multiply to 1. This gives **gluing equations**:
  \[
  \prod_{i=1}^{n} z_i^{r_{ij}} (1 - z_i)^{r_{ij}'} = \pm 1, \quad j = 1, \ldots, n.
  \]

- The cusp torus gives **completeness equations**:
  \[
  \ell(z) = \prod_{i=1}^{n} z_i^{l_i} (1 - z_i)^{l_i'} = 1
  \]
  \[
  m(z) = \prod_{i=1}^{n} z_i^{m_i} (1 - z_i)^{m_i'} = 1
  \]

- $P(N) = \{z = (z_1, \ldots, z_n) \in \mathbb{C}^n \mid \text{satisfy gluing equations}\}$ is called the **parameter space** of $N$. $P_0(N)$ is the component containing the complete parameter $z^0$. 
Define $\text{Hol} : P_0(N) \to \mathbb{C}^2$ as $\text{Hol}(z) = (\ell(z), m(z))$. The image is a curve in $\mathbb{C}^2$ and let $\overline{A}_0(\ell, m)$ be its defining equation.

**Thm** (C) $\overline{A}_0(\ell, m)$ is the component of the $\text{PSL}(2, \mathbb{C})$ A-polynomial corresponding to the component containing the complete structure.

For knot complements

$$\overline{A}_0(\ell^2, m^2) = A_0(\ell, m)A_0(-\ell, m)$$

In general all four factors of the $\text{SL}(2, \mathbb{C})$ A-polynomial can appear with signs on $\ell$ and $m$. 
Mahler measure

Let $p(x_1, \ldots, x_n) \in \mathbb{C}[x_1^\pm, \ldots, x_n^\pm]$. The logarithmic Mahler measure of $p$ is defined as

$$m(p) = \frac{1}{(2\pi i)^n} \int_{\mathbb{T}^n} \log |p(x_1, \ldots, x_n)| \frac{dx_1}{x_1} \cdots \frac{dx_n}{x_n}$$

- $m(p_1 \cdot p_2) = m(p_1) + m(p_2)$.
- Jensen’s formula: $$\frac{1}{2\pi i} \int_{S^1} \log |x - \alpha| \frac{dx}{x} = \log^+ |\alpha|$$
- Let $p(x) = a_0 \prod_{i=1}^{n} (x - \alpha_i)$. Then $m(p) = \log |a_0| + \sum_{i=1}^{n} \log^+ |\alpha_i|$, where $\log^+ |\alpha| = \max\{0, \log |\alpha|\}$. 
Volume Form or Regulator

Let \( p(x, y) \in \mathbb{Z}[x, y] \) be irreducible polynomial.

\[ X = \{(x, y) \in \mathbb{C}^2 \mid p(x, y) = 0\} \]

\( \tilde{X} \) = smooth projective completion of \( X \)
\( \mathbb{C}(\tilde{X}) = \) field of meromorphic functions on \( \tilde{X} \)

For \( f, g \in \mathbb{C}(\tilde{X}) \), the **Volume form** is defined as

\[
\eta(f, g) = \log |f| \; d\text{arg} \; g - \log |g| \; d\text{arg} \; f
\]

\( \eta \in H^1(\tilde{X} - S; \mathbb{R}) \) where \( S = \) zeros and poles of \( f \& g \).
Mahler measure of $p(x, y)$

Write $p(x, y) = a_0(y) \prod_{j=1}^{m}(x - x_j(y))$ where $x_j$’s are algebraic functions of $y$ on $\tilde{X}$. By Jensen’s formula

$$\frac{1}{(2\pi i)^2} \int_{\mathbb{T}^2} \log |x - x_j(y)| \frac{dx}{x} \frac{dy}{y} = \frac{1}{2\pi i} \int_{S^1} \log^+ |x_j(y)| \frac{dy}{y}$$

Let $\gamma_j = \{(x, y) \in \tilde{X} | |y| = 1, |x_j| \geq 1\}$ be an oriented path in $\tilde{X}$.

On $\gamma_j$, $\frac{dy}{y} = d \log |y| + id \arg y = id \arg y$. 

\[ i \eta(x_j, y) = i(\log |x_j| \ d \arg y - \log |y| \ d \arg x_j) \]
\[ = i \log |x_j| \ d \arg y \]
\[ = \log |x| \frac{dy}{y} \]

Prop \[ m(p(x, y)) = m(a_0(y)) + \sum_{i=1}^{n} \frac{1}{2\pi} \int_{\gamma_i} \eta(x_j, y) \]
**Bloch-Wigner dilogarithm**

Lobachevsky function: \( L(\theta) = - \int_0^\theta \log |2 \sin u| \, du \)

\[ \text{vol}(\triangle(z)) = L(\alpha) + L(\beta) + L(\gamma) \] where \( \alpha, \beta, \gamma \) are the dihedral angles of \( \triangle(z) \).

Classical dilogarithm: \( \text{Li}_2(z) = \sum_{n=1}^\infty \frac{z^n}{n^2}, \ |z| < 1 \)

It can be analytically extended to \( \mathbb{C} - (1, \infty) \) as

\[ \text{Li}_2(z) = - \int_0^z \frac{\log(1-u)}{u} \, du \]

The **Bloch-Wigner dilogarithm** is defined as

\( D(z) = \text{Im}(\text{Li}_2(z)) + \log |z| \arg(1 - z) \)
Properties of $D(z)$

- $D(z)$ is real analytic on $\mathbb{C} - \{0, 1\}$.

- $D(e^{i\theta}) = L(\theta)$

- **Thm** $\text{vol}(\triangle(z)) = D(z)$.
  This follows from the 5-term relation and other functional equations of $D(z)$.

- **Thm** $\eta(z, 1 - z) = dD(z)$.

If we can express $\eta(x, y)$ in terms of $\eta(z, 1 - z)$'s then we can use Stokes Theorem to evaluate $m(p(x, y))$ in terms of $D(z)$ and get hyperbolic volumes.
Exactness of Volume Form

Let $F = \mathbb{C}(\tilde{X})$, there are maps

\[ \wedge^2_{\mathbb{Z}}(F^*) \xrightarrow{\text{sym}} K_2(F) \xrightarrow{\eta} H^1(\tilde{X}; \mathbb{R}) \]

where $\text{sym}(f \wedge g) = \{f, g\}$ and $\eta(\{f, g\}) = \eta(f, g)$.

For $x, y, z_i \in F^*$, suppose in $\wedge^2_{\mathbb{Z}}(F^*)$ we can show

\[ x \wedge y = \sum_{i=1}^{n} z_i \wedge (1 - z_i) \]

Then $\eta(x, y) = \sum_{i=1}^{n} \eta(z_i, 1 - z_i) = \sum_{i=1}^{n} dD(z)$.
Let $X = P_0(N)$ and let $\ell, m, z_i \in F = \mathbb{C}(\tilde{P}_0(N))$.

**Thm** (C) In $\bigwedge^2_{\mathbb{Z}}(F^*)$, $\ell \wedge m = \sum_{i=1}^n z_i \wedge (1 - z_i)$.

$$\implies \eta(\ell, m) = d\left(\sum_{i=1}^n D(z_i)\right)$$

$$\sum_{i=1}^n D(z_i) = \text{vol}(N(z))$$

Hence $\eta(\ell, m)$ gives variation of volume under deformation and hence is called the volume form.

Exactness of $\eta(\ell, m)$ was directly shown by Hodgson and Neumann-Zagier.
Mahler measure of $\overline{A}_0(\ell, m)$

Let $\gamma_j = \{|m| = 1, |\ell_j| \geq 1\}$.
Let each $\gamma_j$ have $c_j$ components.
Let $\omega_{ijk}^1$ and $\omega_{ijk}^2$ be lifts of the end points of $\gamma_j$ to $P_0(N)$.

$$m(\overline{A}_0(\ell, m)) = \frac{1}{2\pi} \sum_{j=1}^{m} \int_{\gamma_j} \eta(\ell_j, m)$$

$$= \frac{1}{2\pi} \sum_{j=1}^{m} \sum_{k=1}^{c_j} \sum_{i=1}^{n} (D(\omega_{ijk}^2) - D(\omega_{ijk}^1))$$
Remarks

• Since $\overline{A}_0(1, 1) = 0$ and $(1, 1)$ corresponds to the complete structure, $\text{vol}(N)$ always appears as a summand in above.

• Conjugate lifts of $(1, 1)$ to $P_0(N)$ correspond to different complex embeddings of the invariant trace field of $N$.

These give conjugate volumes in the summand.

$$\sum_{i=1}^{n}[\omega_{ijk}^s]$$ are elements of the Bloch group $\mathcal{B}(\mathbb{C})$. 

Examples

• $K = 4_1$, $\pi m(\bar{A}_0(\ell, m)) = \text{vol}(S^3 - K)$.

• $K = 6_2$, $\pi m(\bar{A}_0(\ell, m)) = \text{vol}(S^3 - K) + V_2$, where $V_2$ is the conjugate volume given by the Borel regulator.

• $K = k5_{15} \cong m240$, $\pi m(\bar{A}_0(\ell, m)) = \text{vol}(S^3 - K) + V_2 + V_3$, where $V_2 = \text{vol}(m240(0, 1))$ and $V_3 = \text{vol}(m240(0, 2))$.

Marc Culler has a program which computes A-polynomials. In addition it also computes the necessary information to compute its Mahler measure (numerically).
Neumann-Zagier matrices

Let $J_{2k} = \begin{pmatrix} 0 & \text{Id}_k \\ -\text{Id}_k & 0 \end{pmatrix}$ be the symplectic matrix.

A $(n + 2) \times 2n$ matrix $U$ is called a Neumann-Zagier matrix if it satisfies

$UJ_{2n}U^t = 2 \begin{pmatrix} J_2 & 0 \\ 0 & 0 \end{pmatrix}$

**Thm** (Neumann-Zagier 85) The exponents of the gluing and completeness equation satisfy the above condition.

Starting with any NZ matrix $U$, we can form “gluing” and “completeness” equations to obtain an A-polynomial. We can compute its Mahler measure using this method.
Com On Nhieu Lam
Thank You Very Much