Nonlinearity **17** (2004) 953–974 PII: S0951-7715(04)68637-3

Density of global trajectories for filtered Navier–Stokes equations*

Jesenko Vukadinovic

Department of Mathematics, Van Vleck Hall, 480 Lincoln Dr., Madison, WI 53706, USA

E-mail: vukadino@math.wisc.edu

Received 8 September 2003, in final form 23 January 2004
Published 27 February 2004
Online at stacks.iop.org/Non/17/953 (DOI: 10.1088/0951-7715/17/3/011)

Recommended by E S Titi

Abstract

For two-dimensional periodic Kelvin-filtered Navier–Stokes systems, both positively and negatively invariant sets $\mathcal{M}_n$, consisting of initial data for which solutions exist for all negative times and exhibiting a certain asymptotic behaviour backwards in time, are investigated. They are proven to be rich in the sense that they project orthogonally onto the sets of lower modes corresponding to the first n distinct eigenvalues of the Stokes operator. In general, this yields the density in the phase space of trajectories of global solutions, but with respect to a weaker norm. This result applies equally to the two-dimensional periodic Navier–Stokes equations (NSEs) and the two-dimensional periodic Navier–Stokes-α model. We designate a subclass of filters for which the density follows in the strong topology induced by the (energy) norm of the phase space, as originally conjectured for the NSEs by Bardos and Tartar (1973 *Arch. Ration. Mech. Anal.* **50** 10–25).

Mathematics Subject Classification: 35Q30, 76D05, 35B40, 35B41

1. Introduction

In the Eulerian representation of fluids, Navier–Stokes equations (NSEs) express the relationship between velocity $v(x, t)$, density $\rho(x, t)$, and pressure $p(x, t)$ at point $x \in \mathbb{R}^d$, $d = 2, 3$, and at time $t \in \mathbb{R}$. In the case of incompressible homogenous fluids, the density is constant in space and time. Assuming for simplicity that $\rho \equiv 1$, the balance of momentum yields

$$\partial_t v - \nu\Delta v + (v \cdot \nabla)v + \nabla p = f,$$

* This work was partially supported by the NSF grant DMS-0074460 while the author was a graduate student at Indiana University.

where $\nu > 0$ represents the kinematic viscosity and f represents a given body forcing. The constraint of incompressibility can be expressed through the divergence-free condition $\nabla \cdot v = 0$. The system is nonlinear and nonlocal. Due to viscosity, the total kinetic energy is dissipated forward in time, providing the strongest source of information currently known about NSEs. The existence and uniqueness of a strong (regular) solution to the initial value problem $v(0) = v_0$, subject to adequate boundary conditions, has been established definitively for all positive times only for space dimension $d = 2$. In this case, the equations define a forward regularizing flow for which there exists a compact absorbing set. This, in turn, gives rise to the existence of the global attractor—the essential object for understanding the long time behaviour of the solutions. In the three-dimensional case, the energy dissipation is used to construct Leray weak solutions for all positive times. In general, however, regularity and uniqueness of such solutions is guaranteed for a finite interval of time only.

In both theoretical and numerical studies of NSEs, certain approximations are often useful. Usually, they are solutions of partial differential equations that have the NSE as a limiting case. According to the properties of the NSE that they preserve, we distinguish two important classes of approximations. One class, which includes Galerkin approximations and Leray regularization (mollified equations), preserves the energy balance equation for the NSEs, but alters the vorticity equation. In the other, which includes Kelvin-filtered NSEs (KFNSEs), the opposite is the case: the vorticity equation has the same structure as the one for the NSE, whereas the energy balance equation is approximated. The KFNSEs have the following form:

$$\partial_t v - \nu \Delta v + (u \cdot \nabla) v + (\nabla u)^T \cdot v + \nabla Q = f, \qquad u = \mathcal{F} v, \quad \nabla \cdot v = 0.$$

The functions v and u denote the Eulerian fluid velocity and the filtered Eulerian fluid velocity, respectively. The inviscid one-dimensional version of those equations was originally derived using the Euler–Poincaré variational framework for the special choice of filter $\mathcal{F} = (I - \alpha^2 \Delta)^{-1}$ (see [20]). The so-called Navier–Stokes-α model was obtained in [2, 11], by generalizing the equation to higher dimensions, and adding the viscous term in an *ad hoc* fashion. Due to the special geometric and physical properties, this system is used to model turbulent flows (see [2, 15]). The equations can also be derived in a more general setting by filtering the velocity in the NSE in a manner that preserves a form of the Kelvin circulation theorem, or the structure of the vorticity equation (see [15]). In this paper, the filtering operator $\mathcal{F}$ will be given using Fourier multipliers $\mathcal{F} = \phi(-\Delta)^{-1} = \phi^{-1}(-\Delta)$, where $\phi : [0, \infty) \to [1, \infty)$ is a nondecreasing convex function with $\phi(0) = 1$. If $\phi \equiv 1$, the KFNSEs are nothing else but the NSEs.

Many dissipative differential equations of fluid dynamics, including the NSE and the KFNSE, can be written in the following form:

$$\dot{v} + \nu A v + B(v, v) = f, \tag{1.1}$$

where A is a closed positive self-adjoint linear operator and B is a bilinear form. Other examples include complex Ginzburg–Landau equations, Lorenz equations, Kuramoto–Sivashinsky equations and the original Burger's equation. Under certain conditions, these systems are solved by nonlinear compact forward regularizing dissipative semigroups $S(t)$ in adequate Hilbert spaces H. They possess global attractors—compact invariant sets consisting of all omega-limit sets, which can be characterized by $A = \{u_0 \in H : S(t)u_0$ extends to, and is bounded for all negative times$\}$. Attempts to develop practical algorithms for the study of the long time behaviour of those equations, such as various interpolation techniques to determine whether a solution is on the global attractor, have led to the study of solutions for negative times (see [12–14]). There has been a series of results in that respect, and a wide range of phenomena was observed. For the Kuramoto–Sivashinsky equations and the complex Ginzburg–Landau equations, it was shown that all global solutions are bounded (see [22, 9]), and thus contained in the attractor. In contrast, the global solutions to the two-dimensional periodic NSEs, which exhibit exponential backwards growth, exist, and are, in fact, quite rich (see [6]). In this paper, we will show that this is the case for two-dimensional periodic KFNSEs, in general.

Bardos and Tartar (1973) conjectured that the set of the initial data for which the solution exists backwards for a given interval of time $(t_0, 0]$ is dense in the phase space (see [1]). In [6], the authors proved that for the periodic two-dimensional NSE, the set of initial data for which the solutions extend to all negative times and grow backwards at most exponentially are dense in the phase space, but with respect to a weaker norm. In this paper, we will show that the same is generally the case also for the two-dimensional periodic KFNSEs. However, if the filter satisfies the condition $\limsup_{n\to\infty} \phi(\Lambda_{n+1})/\phi(\Lambda_n) = \infty$ (Λ_i being the ith distinct eigenvalue of $-\Delta$), the density follows with respect to the strong topology induced by the natural energy norm of the phase space $(\mathcal{F}v, v)^{1/2}$.

2. Kelvin-filtered NSEs

2.1. *Filtered viscous fluid equations*

We consider the NSEs and related equations supplemented with space-periodic boundary conditions: the velocity $v(x,t)$ and the pressure $p(x,t)$ will be required to be L-periodic in every space coordinate x_i, $i = 1, \dots, d$. We denote $\Omega = [0, L]^d$. Let $u \in H^1_{\mathrm{per}}(\Omega \times (0, T))^d$ be a strong solution of the NSE on some interval $[0, T)$. Two important quantities for the solution are the total kinetic energy $e(t) := \frac{1}{2}\int_\Omega |v(x,t)|^2\, \mathrm{d}x$, and the enstrophy $E(t) := \int_\Omega |\nabla v(x,t)|^2\, \mathrm{d}x$. The strongest source of quantitative information in the study of the NSE is the energy balance equation

$$\frac{\mathrm{d}e}{\mathrm{d}t} + \nu E = \int v \cdot f\, \mathrm{d}x, \tag{2.1}$$

which we obtain after multiplying the NSE by v, and integrating by parts. The nonlinear term cancels out because of the divergence-free condition and the identity $2u_i(\partial_i v_j)v_j + (\partial_i u_i)v_j^2 = \partial_i(u_i v_j^2)$. Another important tool is the vorticity equation, obtained by taking the curl in the NSE. Denoting the vorticity by $q = \operatorname{curl} v$, it reads

$$\partial_t q - \nu\Delta q + (v\cdot\nabla)q - (q\cdot\nabla)v = \operatorname{curl} f. \tag{2.2}$$

The existence of solutions of NSE is generally proved by constructing regular approximate solutions, and passing to the limit as the approximation parameter tends to zero (or infinity). Approximations of particular interest are the ones that preserve a certain property of the NSE, for example, the energy balance equation or the vorticity equation. Here, we describe two ways of constructing regular approximate solutions, the Leray regularization and the KFNSEs. In both cases we modify the nonlinear term of the NSE by introducing a spatially filtered Eulerian fluid velocity, while the linear term remains unchanged. The velocity can be filtered by mollifying: $u = \mathcal{F}v = J_\delta * v$, where J is a positive, symmetric, normalized, smooth kernel that decays sufficiently fast at infinity, and $J_\delta(\xi) = \delta^d J(\xi/\delta)$, or using the Fourier multipliers: $u = \mathcal{F}v = \phi^{-1}(-\delta\Delta)v$, where $\phi : [0,\infty) \to [1,\infty)$ is a nondecreasing convex function with $\phi(0) = 1$. The Leray regularization of the NSE is given by

$$\partial_t v - \nu\Delta v + (u\cdot\nabla)v + \nabla p = f, \qquad u = \mathcal{F}v, \quad \nabla\cdot v = 0. \tag{2.3}$$

A solution of the initial boundary value problem for (2.3) with the initial condition $v(0) = v_0$ exists for all positive times, and is real analytic and unique (see [4]). One can easily see that solutions satisfy the same energy balance equation (2.1). The KFNSEs read

$$\partial_t v - \nu\Delta v + (u\cdot\nabla)v + (\nabla u)^T\cdot v + \nabla Q = f, \qquad u = \mathcal{F}v, \quad \nabla\cdot v = 0. \tag{2.4}$$

The solution to the initial value problem is smooth and global (see [3, 4, 11]). Taking the curl in (2.4) yields the vorticity equation

$$\partial_t q - \nu\Delta q + (u\cdot\nabla)q - (q\cdot\nabla)u = \operatorname{curl} f,$$

where $q = \operatorname{curl} v$ is the vorticity. Observe that it has the same structure as the vorticity equation of NSE. Another important analogy to the NSEs is that KFNSEs have a Kelvin circulation theorem

$$\frac{\mathrm{d}}{\mathrm{d}t}\oint_{\gamma(u)} v\cdot\mathrm{d}x = \oint_{\gamma(u)} (\nu\Delta v + f)\cdot\mathrm{d}x,$$

where $\gamma(u)$ is a closed path moving with the spatially filtered velocity u. In addition, in the two-dimensional case, the enstrophy balance equation for the NSE is preserved exactly:

$$\frac{\mathrm{d}E}{\mathrm{d}t} + 2\nu\int_\Omega |\Delta v|^2\,\mathrm{d}x = -2\int_\Omega \Delta v\cdot f\,\mathrm{d}x.$$

In contrast, the Leray regularization does not have a Kelvin circulation theorem, does not preserve the structure of the vorticity equation, or the enstrophy balance equation in two dimensions. However, KFNSEs do not have the same energy balance as the NSEs. Instead,we have an approximate equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{1}{2}u\cdot v\,\mathrm{d}x + \nu\int \nabla u_i\cdot\nabla v_i\,\mathrm{d}x = \int u\cdot f\,\mathrm{d}x.$$

2.2. *Mathematical framework for two-dimensional periodic KFNSEs*

2.2.1. Function spaces, Helmholtz–Leray decomposition, and the Stokes operator. In order to study the periodic KFNSEs in a functional setting, we introduce suitable function spaces. By $L^2_{\mathrm{per}}(\Omega)^2$ and $H^l_{\mathrm{per}}(\Omega)^2$ we denote the spaces of vector fields $u:\mathbb{R}^2\to\mathbb{R}^2$, which are L-periodic in each space variable, have zero space average $\int_\Omega u(x)\,\mathrm{d}x = 0$ and belong, respectively, to $L^2(\mathcal{O})^2$ and $H^l(\mathcal{O})^2$ for every bounded open set $\mathcal{O}\subset\mathbb{R}^2$. Let

$$\mathcal{V} = \left\{u\in\mathcal{C}^\infty_{\mathrm{per}}(\Omega)^2:\ \nabla\cdot u = 0,\ \int_\Omega u = 0\right\}$$

be the set of test functions. We introduce the classical spaces H and V to be closures of $\mathcal{V}$ in the (real) Hilbert spaces $L^2(\Omega)^2$ and $H^1(\Omega)^2$, respectively. The spaces H and V are also (real) Hilbert spaces with respective scalar products

$$(u,v) = \sum_{j=1}^2\int_\Omega u_j v_j \qquad\text{and}\qquad ((u,v)) = \sum_{j,k=1}^2\int_\Omega \frac{\partial u_j}{\partial x_k}\frac{\partial v_j}{\partial x_k},\qquad u,v\in V.$$

The corresponding norms are denoted by $|u| = (u,u)^{1/2}$ and $\|u\| = ((u,u))^{1/2}$, respectively. We denote the dual of V by V'. By the Rellich embedding theorem, the natural inclusions $i_1: V\hookrightarrow H$ and $i_2: H\hookrightarrow V'$ are compact.

The Helmholtz–Leray decomposition resolves any vector field $w\in L^2_{\mathrm{per}}(\Omega)^2$ uniquely into the sum of a curl and a gradient vector field $w = u + \nabla p$ with $u\in H$, and $p\in H^1_{\mathrm{per}}(\Omega)$. The map $w\mapsto u$ is well defined, and it is the orthogonal projection (called the Helmholtz–Leray projector) on the space H in $L^2_{\mathrm{per}}(\Omega)^2$. We denote it by $P_L: L^2_{\mathrm{per}}(\Omega)^2\to H$; observe that $H^\perp = \{\nabla p : p\in H^1_{\mathrm{per}}(\Omega)\}$ is the orthogonal complement of the space H in $L^2_{\mathrm{per}}(\Omega)^2$.

By $A = -P_L\Delta$ we define the Stokes operator with domain $D(A) = H^2_{\mathrm{per}}(\Omega)^2\cap V$. In the space periodic case $Au = -P_L\Delta u = -\Delta u$, $u\in D(A)$. The Stokes operator A is one-to-one from $D(A)$ onto H. The inverse $A^{-1}: H\to D(A)$ is compact. Integration by parts verifies that the Stokes operator is self-adjoint $(Au, v) = (u, Av)$, and positive $(Au, u) = \|u\|^2 > 0$, $u \in D(A)\backslash\{\mathbf{0}\}$. Thus, A^{-1} is also self-adjoint and positive. From the elementary spectral theory of compact, self-adjoint operators on Hilbert spaces, there exists a sequence of eigenvectors $w_m,\ m = 1, 2, \dots$ of A^{-1} forming an orthonormal basis in H, such that the corresponding sequence of eigenvalues forms a nonincreasing sequence $\sigma_m,\ m = 1, 2, \dots$ of real, positive numbers accumulating at 0. Let $\lambda_m := 1/\sigma_m$. Note that the Stokes operator A can be extended to an isomorphism between V and V'. Then, $D(A) = \{u \in V : Au \in H\}$.

We expand each vector field $u \in H$ in a Fourier or spectral expansion $u = \sum_{m=1}^{\infty} \hat{u}_m w_m$, where $\hat{u}_m := (u, w_m)$. Parseval's identity reads $|u|^2 = \sum_{m=1}^{\infty} |\hat{u}_m|^2$. For a function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ let us define

$$H_\phi := \left\{ u = \sum_{m=1}^{\infty} \hat{u}_m w_m : \sum_{m=1}^{\infty} \phi(\lambda_m) |\hat{u}_m|^2 < \infty \right\}$$

and

$$V_\phi := \left\{ u = \sum_{m=1}^{\infty} \hat{u}_m w_m : \sum_{m=1}^{\infty} \lambda_m \phi(\lambda_m) |\hat{u}_m|^2 < \infty \right\},$$

endowed with inner products

$$(u, v)_\phi = \sum_{m=1}^{\infty} \phi(\lambda_m) \hat{u}_m \cdot \hat{v}_m \qquad \text{and} \qquad ((u, v))_\phi = \sum_{m=1}^{\infty} \lambda_m \phi(\lambda_m) \hat{u}_m \cdot \hat{v}_m,$$

respectively. The corresponding norms are $|u|_\phi^2 = \sum_{m=1}^{\infty} \phi(\lambda_m) |\hat{u}_m|^2$ and $\|u\|_\phi^2 = \sum_{m=1}^{\infty} \lambda_m \phi(\lambda_m) |\hat{u}_m|^2$. Let us also define Fourier multipliers $\phi(A)u = \sum_{m=1}^{\infty} \phi(\lambda_m) \hat{u}_m w_m$, $u \in H_\phi$. One can verify that $D(A^{1/2}) = V$ and $D(A^{-1/2}) = V'$. For $u \in V$, $Au = \sum_{m=1}^{\infty} \lambda_m \hat{u}_m w_m$ and, thus, $\|u\|^2 = (Au, u) = |A^{1/2}u|^2 = \sum_{m=1}^{\infty} \lambda_m |\hat{u}_m|^2$.

We arrange the eigenvalues $\lambda_1, \lambda_2, \dots$ of the Stokes operator in strictly increasing order $(2\pi/L)^2 = \Lambda_1 < \Lambda_2 < \cdots$. It is known that $\Lambda_n \sim n$, as $n \to \infty$, and $\Lambda_{n+1} - \Lambda_n \geqslant \Lambda_1$. An important spectral gap property in the two-dimensional case is

$$\limsup_{n\to\infty} \frac{\Lambda_{n+1} - \Lambda_n}{\log n} > 0 \tag{2.5}$$

(see [23]). This implies, in particular, $\lim_{n\to\infty} \Lambda_{n+1}/\Lambda_n = 1$.

We define P_n to be the orthogonal projection in H on the spectral space of A corresponding to the eigenvalues $\Lambda_1, \Lambda_2, \dots, \Lambda_n$, respectively; also let $Q_n := I - P_n$. We have several variations of the Poincaré inequality

$$\|Q_n u\|^2 \geqslant \Lambda_{n+1} |Q_n u|^2, \qquad u \in V$$

and, if ϕ is increasing,

$$|Q_n u|_\phi^2 \geqslant \phi(\Lambda_{n+1}) |Q_n u|^2, \qquad u \in H_\phi.$$

2.2.2. Nonlinear terms. Regarding the nonlinear terms that appear in the NSEs and the KFNSEs, we introduce the following trilinear forms:

$$b(u, v, w) = \int_\Omega u_j \frac{\partial v_i}{\partial_{x_j}} w_i \, \mathrm{d}x = \int_\Omega (u \cdot \nabla) v \cdot w \, \mathrm{d}x$$

and

$$b^*(u, v, w) = -\int_\Omega v_j \frac{\partial u_j}{\partial_{x_i}} w_i \, \mathrm{d}x = -\int_\Omega ((\nabla u)^T \cdot v) \cdot w \, \mathrm{d}x,$$

where $u, v, w \in \mathcal{C}^{\infty}_{\text{per}}(\Omega)^2$ are divergence-free. Using the fact that u is divergence-free, the product rule and the partial integration give $b(u, v, w) = -b(u, w, v)$, and in particular $b(u, v, v) = 0$. The two trilinear forms are related through the identity:

$$\begin{aligned} b^*(u, v, w) = & -\int_\Omega v_j \frac{\partial u_j}{\partial_{x_i}} w_i \, \mathrm{d}x = -\int_\Omega w_i \frac{\partial u_j}{\partial_{x_i}} v_j \, \mathrm{d}x \\ = & -b(w, u, v) = b(w, v, u). \end{aligned}$$

In the two-dimensional case we have the inequality

$$|b(u, v, w)| \leqslant c_1 |u|^{1/2} \|u\|^{1/2} |v|^{1/2} \|v\|^{1/2} \|w\|,$$

which allows us to extend b and b^* on $V \times V \times V$. In particular, this allows us to define $B(u, v)$ and $B^*(u, v) \in V'$ for $u, v \in V$ through $(B(u, v), w) = b(u, v, w)$ and $(B^*(u, v), w) = b^*(u, v, w) = b(w, v, u)$, respectively. If u or v is in $D(A)$, one can prove that $B(u, v)$, $B^*(u, v) \in H$. In this case, we have $B(u, v) = P_L((u \cdot \nabla)v)$, and $B^*(u, v) = -P_L(v_j \nabla u_j)$. We have the following orthogonality property:

$$(B(u, v), v) = 0, \qquad u, v \in V. \tag{2.6}$$

In the two-dimensional periodic case with zero space average, the inertial term satisfies a further important orthogonality property, known as the enstrophy invariance

$$(B(u, u), Au) = 0, \qquad u \in D(A).$$

Actually, an even stronger form of enstrophy invariance holds:

$$(B(u, v), Av) = (B(Av, v), u), \qquad u \in V, \quad v \in D(A). \tag{2.7}$$

Recall that the connection between B and B^* is given by the identity

$$(B^*(u, v), w) = (B(w, v), u), \qquad u, v, w \in V. \tag{2.8}$$

Using identities (2.6)–(2.8), we obtain the following important orthogonality properties for the nonlinear term $B - B^*$ of the KFNSEs

$$((B - B^*)(u, v), u) = (B(u, v), u) - (B(u, v), u) = 0, \qquad u, v \in V, \tag{2.9}$$

and for $u \in V$, $v \in D(A)$

$$((B - B^*)(u, v), Av) = (B(u, v), Av) - (B(Av, v), u) = 0. \tag{2.10}$$

2.3. Functional form of two-dimensional periodic KFNSEs and some elementary facts

2.3.1. Functional form. Let $\phi : [0, \infty) \to [1, \infty)$ be a twice differentiable function such that $\phi' \geqslant 0$, $\phi'' \geqslant 0$ and $\phi(0) = 1$. Let $\psi(\xi) = \xi\phi(\xi)$, $\upsilon(\xi) = \phi(\xi)/\xi$ and $\chi(\xi) = \xi\phi(\xi)^2$. Applying the Helmholtz–Leray projector P_L on the KFNSEs (2.4), we obtain the following functional form of the equations:

$$\begin{aligned} &\dot{v} + \nu A v + (B - B^*)(u, v) = f, \\ &v = \phi(A)u, \end{aligned} \tag{2.11}$$

where without loss of generality we assume $f \in H$. Observe that the term containing modified pressure does not occur in this equation since $P_L(\nabla Q) = 0$. This new functional version of the KFNSE is understood in V'. Classical theorems imply that, for every $u_0 \in H_\phi$, there exists a unique solution $u(t) = S(t)u_0$ for $t \geqslant 0$ of (2.11), which satisfies $u(0) = u_0$ and $u \in \mathcal{C}_b([0, \infty), H_\phi) \cap \mathcal{C}((0, \infty), V_{\phi^2})$. If the solution $u(t)$ also exists for $t \in [t_0, 0]$ for some $t_0 < 0$, then it is still uniquely determined by u_0, and therefore we may denote the solution by $S(t)u_0$ wherever it is defined. Also, for any $t_0 > 0$, the solution operator $S(t_0) : H_\phi \to H_\phi$ is continuous.

2.3.2. Energy and enstrophy inequalities. Our aim is now to find balance equations for the KFNSE that do not involve terms in which trilinear forms b and b^* appear. Therefore, keeping (2.9) and (2.10) in mind, if we multiply the KFNSE by u, we obtain the so-called energy balance equation

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|u|_\phi^2 + \nu(Av, u) = (f, u) \tag{2.12}$$

and if we multiply it by Av, we obtain the so-called enstrophy balance equation

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v\|^2 + \nu(Av, Av) = (f, Av). \tag{2.13}$$

Using Young's inequality, these equations yield, respectively, the following useful inequalities:

$$\frac{\mathrm{d}}{\mathrm{d}t}|u|_\phi^2 + \nu\Lambda_1|u|_\phi^2 \leqslant \frac{|f|^2}{\nu\Lambda_1} \tag{2.14}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_{\phi^2}^2 + \nu\Lambda_1\|u\|_{\phi^2}^2 \leqslant \frac{|f|^2}{\nu}. \tag{2.15}$$

Observe that relation (2.14) yields that $t \mapsto |u(t)|_\phi$ is a decreasing function as long as $|u(t)|_\phi > |f|/(\nu\Lambda_1)$, and (2.15) yields that $t \mapsto \|u(t)\|_{\phi^2}$ is decreasing as long as $\|u(t)\|_{\phi^2} > |f|/(\nu\Lambda_1^{1/2})$. If u is a solution of the KFNSE defined on some interval $[t_0, \infty)$, the Gronwall lemma gives

$$\begin{aligned}|u(t)|_\phi^2 &\leqslant |u(t_0)|_\phi^2\,\mathrm{e}^{-\nu\Lambda_1(t-t_0)} + \frac{|f|^2}{\nu^2\Lambda_1^2}\left(1 - \mathrm{e}^{-\nu\Lambda_1(t-t_0)}\right)\\ &= \left(|u(t_0)|_\phi^2 - \frac{|f|^2}{\nu^2\Lambda_1^2}\right)\mathrm{e}^{-\nu\Lambda_1(t-t_0)} + \frac{|f|^2}{\nu^2\Lambda_1^2}, \qquad t \geqslant t_0.\end{aligned} \tag{2.16}$$

Similarly,

$$\|u(t)\|_{\phi^2}^2 \leqslant \|u(t_0)\|_{\phi^2}^2\,\mathrm{e}^{-\nu\Lambda_1(t-t_0)} + \frac{|f|^2}{\nu^2\Lambda_1}\left(1 - \mathrm{e}^{-\nu\Lambda_1(t-t_0)}\right), \qquad t \geqslant t_0. \tag{2.17}$$

Also, if the solution is defined on $[t, t_0]$ we have

$$\begin{aligned}|u(t)|_\phi^2 &\geqslant |u(t_0)|_\phi^2\,\mathrm{e}^{\nu\Lambda_1(t_0-t)} - \frac{|f|^2}{\nu^2\Lambda_1^2}\left(\mathrm{e}^{\nu\Lambda_1(t_0-t)} - 1\right)\\ &= \left(|u(t_0)|_\phi^2 - \frac{|f|^2}{\nu^2\Lambda_1^2}\right)\mathrm{e}^{\nu\Lambda_1(t_0-t)} + \frac{|f|^2}{\nu^2\Lambda_1^2}.\end{aligned} \tag{2.18}$$

Observe that, if the solution is defined on $(-\infty, t_0]$, and if $|u(t_0)|_\phi > |f|/\nu\Lambda_1$ the integral $\int_{-\infty}^{t_0}(1/|u(t)|_\phi^2)\,\mathrm{d}t < \infty$.

2.3.3. Global solutions and attractor. Every solution $u(t) = S(t)u_0$ of the KFNSE, which can be extended for all $t < 0$, is called a global solution. The set of initial data, for which there exists a global solution, can be defined as

$$\mathcal{G} := \bigcap_{t\geqslant 0} S(t)H_\phi. \tag{2.19}$$

Let $\rho_0 := |f|/\nu\Lambda_1$. Inequality (2.16) implies

$$\limsup_{t\to\infty}|u(t)|_\phi \leqslant \rho_0.$$

Moreover, the balls $B_{H_\phi}(0,\rho)$ in H_ϕ with $\rho \geqslant \rho_0$ are positively invariant for the semigroup $S(t)$, and for $\rho > \rho_0$ they are absorbing balls. Let us define

$$\mathcal{A} := \bigcap_{t\geqslant 0} S(t)\left\{u_0 \in H_\phi : |u_0|_\phi \leqslant \rho_0\right\}.$$

This set is the global attractor for the KFNSE. It is the smallest compact, invariant set that attracts all the solutions. Even more, it attracts all the bounded sets in H_ϕ uniformly. It can be characterized in the following way:

$$\begin{aligned}\mathcal{A} &= \left\{u_0 \in \mathcal{G} \,:\, \limsup_{t\to-\infty} |S(t)u_0|_\phi < \infty\right\}\\ &= \left\{u_0 \in \mathcal{G} \,:\, |S(t)u_0|_\phi \leqslant \frac{|f|}{\nu\Lambda_1},\ t\in\mathbb{R}\right\}\\ &= \left\{u_0 \in \mathcal{G} \,:\, \|S(t)u_0\|_{\phi^2} \leqslant \frac{|f|}{\nu\Lambda_1^{1/2}},\ t\in\mathbb{R}\right\}.\end{aligned}$$

The following properties will be needed (see [5]):

- $\mathcal{A}$ is a nonempty compact connected subset of H_ϕ.
- $\mathcal{A}$ is positively and negatively invariant, i.e. $S(t)\mathcal{A} = \mathcal{A}$ for $t \in \mathbb{R}$. Moreover, any positively invariant set containing $\mathcal{A}$ is connected.
- $d_{\mathrm{F}}(\mathcal{A}) < \infty$, where d_{F} denotes the fractal dimension.

3. Backward asymptotic behaviour of the global solutions

3.1. Dirichlet quotients. Definition and some properties of the sets $\mathcal{M}_n$

3.1.1. 'Filtered' Dirichlet quotients. The quotients

$$\frac{(\nabla v, \nabla v)}{(u,v)} = \frac{(Av, v)}{(u,v)} = \frac{\|u\|_{\phi^2}^2}{|u|_\phi^2}$$

are referred to as 'filtered' Dirichlet quotients. Since both energy and enstrophy satisfy differential equations that do not involve the nonlinear term $B - B^*$ (see (2.12) and (2.13)), the Dirichlet quotients do as well. The asymptotic behaviour of the Dirichlet quotients for the NSE when $t \to \infty$ has been studied extensively (see [17, 18]). In [6], the asymptotic behaviour of the Dirichlet quotients for the two-dimensional periodic NSE when $t \to -\infty$ was examined. Here, we will concentrate on the asymptotic behaviour of the 'filtered' Dirichlet quotients for the two-dimensional periodic KFNSE when $t \to -\infty$.

Let us first define the following locally compact cones.

Definition 1. *Let $n \in \mathbb{N}$, and $0 < \kappa \leqslant (\Lambda_{n+1} - \Lambda_n)/2$. Let*

$$\mathcal{C}_{n,\kappa} := \left\{u_0 \in V_{\phi^2} \,:\, \frac{\|u_0\|_{\phi^2}^2}{|u_0|_\phi^2} \leqslant \psi(\Lambda_n + \kappa)\right\}.$$

The following theorem on the behaviour of the Dirichlet quotients will be the main technical tool in the study of the backwards behaviour of the two-dimensional periodic KFNSE.

Theorem 1. *There exists $c_\phi > 0$ such that for any $n \in \mathbb{N}$, $0 < \kappa \leqslant (\Lambda_{n+1} - \Lambda_n)/2$, $u_0 \in \mathcal{C}_{n,\kappa}$, and $T > 0$:*

$$|S(t)u_0|_\phi > \frac{c_\phi |f|}{\nu\kappa}, \qquad t \in [0,T) \Rightarrow S(t)u_0 \in \mathcal{C}_{n,\kappa}, \quad t \in [0,T].$$

Remark 1. Observe that in the case $f = 0$, the last theorem states that the cones $\mathcal{C}_{n,\kappa}$ are positively invariant under the solution operator S.

Proof. Let us denote $\tilde{u} := u/|u|_\phi$, $v = \phi(A)u$ and $\tilde{v} := v/|u|_\phi$. Also, let $f = \sum_{m=1}^{\infty} \hat{f}_m w_m$ and $\tilde{u} = \sum_{m=1}^{\infty} \hat{\tilde{u}}_m w_m$. Obviously, $|\tilde{u}|_\phi^2 = 1$. Let us first assume that $v(0) \in D(A)$. Applying (2.12) and (2.13) we obtain

$$
\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\frac{\|v\|^2}{|u|_\phi^2} &= \frac{1}{|u|_\phi^4}\left(\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v\|^2 \cdot |u|_\phi^2 - \|v\|^2 \cdot \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|u|_\phi^2\right) \\
&= \frac{1}{|u|_\phi^2}[(-\nu(Av, Av) + (f, Av)) - \|\tilde{v}\|^2(-\nu(Av, u) + (f, u))] \\
&= -\nu(A\tilde{v}, A\tilde{v} - \|\tilde{v}\|^2\tilde{u}) + \left(f/|u|_\phi, A\tilde{v} - \|\tilde{v}\|^2\tilde{u}\right).
\end{aligned}
$$

Thus, we obtain

$$
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\tilde{v}\|^2 + \nu(A\tilde{v}, A\tilde{v} - \|\tilde{v}\|^2\tilde{u}) = (f/|u|_\phi, A\tilde{v} - \|\tilde{v}\|^2\tilde{u}). \tag{3.1}
$$

For a given $t \geqslant 0$ let us define $\mu(t)$ to be the solution of the equation

$$
\psi(\mu(t)) = \mu(t)\phi(\mu(t)) = \|\tilde{v}(t)\|^2. \tag{3.2}
$$

Since $(\tilde{v}, A\tilde{v} - \|\tilde{v}\|^2\tilde{u}) = \|\tilde{v}\|^2 - \|\tilde{v}\|^2|\tilde{u}|_\phi^2 = 0$, we have

$$
\begin{aligned}
\left(A\tilde{v}, A\tilde{v} - \|\tilde{v}\|^2\tilde{u}\right) &= \left(A\tilde{v} - \mu\tilde{v}, A\tilde{v} - \|\tilde{v}\|^2\tilde{u}\right) \\
&= \sum_{m=1}^{\infty}(\lambda_m - \mu)\phi(\lambda_m)(\lambda_m\phi(\lambda_m) - \|\tilde{v}\|^2)|\hat{\tilde{u}}_m|^2 \\
&= \sum_{m=1}^{\infty}(\lambda_m - \mu)(\psi(\lambda_m) - \psi(\mu))\phi(\lambda_m)|\hat{\tilde{u}}_m|^2.
\end{aligned}
$$

The latter is a positive quantity, since ψ is increasing. With

$$
\gamma(x, y) := \begin{cases} \dfrac{\psi(x) - \psi(y)}{x - y}, & x \neq y, \\ \psi'(x), & x = y, \end{cases}
$$

$$
(A\tilde{v}, A\tilde{v} - \|\tilde{v}\|^2\tilde{u}) = \sum_{m=1}^{\infty}(\lambda_m - \mu)^2\gamma(\mu, \lambda_m)\phi(\lambda_m)|\hat{\tilde{u}}_m|^2.
$$

On the other hand, applying Young's inequality, and assuming for a moment that $\mu \notin \{\lambda_1, \lambda_2, \ldots\}$, we obtain the following inequality for the term involving the force f:

$$
\begin{aligned}
\left(f/|u|_\phi, A\tilde{v} - \|\tilde{v}\|^2\tilde{u}\right) &= \sum_{m=1}^{\infty}(\psi(\lambda_m) - \psi(\mu))\frac{\hat{f}_m \cdot \hat{\tilde{u}}_m}{|u|_\phi} \\
&\leqslant \frac{\nu}{2}\sum_{m=1}^{\infty}(\lambda_m - \mu)(\psi(\lambda_m) - \psi(\mu))\phi(\lambda_m)|\hat{\tilde{u}}_m|^2 \\
&\quad + \frac{1}{2\nu|u|_\phi^2}\sum_{m=1}^{\infty}\frac{\psi(\lambda_m) - \psi(\mu)}{\phi(\lambda_m)(\lambda_m - \mu)}|\hat{f}_m|^2.
\end{aligned}
$$

Thus,

$$
\frac{\mathrm{d}}{\mathrm{d}t}\|\tilde{v}\|^2 + \nu\sum_{m=1}^{\infty}(\lambda_m - \mu)^2\gamma(\mu, \lambda_m)\phi(\lambda_m)|\hat{\tilde{u}}_m|^2 \leqslant \frac{1}{\nu|u|_\phi^2}\sum_{m=1}^{\infty}\frac{\gamma(\mu, \lambda_m)}{\phi(\lambda_m)}|\hat{f}_m|^2.
$$

Since

$$\sum_{m=1}^{\infty}(\lambda_m-\mu)^2\gamma(\mu,\lambda_m)\phi(\lambda_m)|\hat{\tilde{u}}_m|^2 \geqslant \min_{m\in\mathbb{N}}\left[(\lambda_m-\mu)^2\gamma(\mu,\lambda_m)\right],$$

we also obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\tilde{v}\|^2+\nu\min_{m\in\mathbb{N}}\left[(\lambda_m-\mu)^2\gamma(\mu,\lambda_m)\right]\leqslant\frac{1}{\nu|u|_\phi^2}\sum_{m=1}^{\infty}\frac{\gamma(\mu,\lambda_m)}{\phi(\lambda_m)}|\hat{f}_m|^2. \tag{3.3}$$

If for some $t_0\in[0,T]$: $\|\tilde{u}(t_0)\|_{\phi^2}=\|\tilde{v}(t_0)\|^2=\psi(\Lambda_n+\kappa)$, then $\mu(t_0)=\Lambda_n+\kappa$. Since ψ is convex, we have

$$\min_{m\in\mathbb{N}}[(\lambda_m-\mu)^2\gamma(\mu,\lambda_m)]=(\mu-\Lambda_n)^2\gamma(\mu,\Lambda_n)=\kappa^2\gamma(\Lambda_n+\kappa,\Lambda_n).$$

Let for $m\in\mathbb{N}$

$$c(m,n,\kappa):=\frac{\gamma(\Lambda_n+\kappa,\lambda_m)}{\phi(\lambda_m)\gamma(\Lambda_n+\kappa,\Lambda_n)}.$$

In the Navier–Stokes case, $\phi\equiv 1$, so $c(m,n,\kappa)=1$. Otherwise, because of the convexity of ψ, if $\lambda_m\leqslant\Lambda_n$

$$\frac{\psi(\Lambda_n+\kappa)-\psi(\lambda_m)}{\Lambda_n+\kappa-\lambda_m}\leqslant\frac{\psi(\Lambda_n+\kappa)-\psi(\Lambda_n)}{\kappa},$$

so $c(m,n,\kappa)\leqslant 1$. If $\lambda_m\geqslant\Lambda_{n+1}$, we have

$$c(m,n,\kappa)\leqslant\frac{\lambda_m/(\lambda_m-(\Lambda_n+\kappa))}{(\psi(\Lambda_n+\kappa)-\psi(\Lambda_n))/\kappa}.$$

Since $\lambda\to\lambda/(\lambda-(\Lambda_n+\kappa))$ is decreasing,

$$c(m,n,\kappa)\leqslant\frac{\Lambda_{n+1}}{(\Lambda_{n+1}-(\Lambda_n+\kappa))\psi'(\Lambda_n)}\leqslant\frac{2\Lambda_{n+1}}{(\Lambda_{n+1}-\Lambda_n)(1+\phi'(0)\Lambda_n)}.$$

Therefore,

$$c_\phi:=\sup\left\{c(m,n,\kappa)^{1/2}:m\in\mathbb{N},\ n\in\mathbb{N},\ \kappa\in(0,(\Lambda_{n+1}-\Lambda_n)/2)\right\}<\infty.$$

Now, (3.3) implies

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|\tilde{v}(t_0)\|^2&\leqslant\gamma(\Lambda_n+\kappa,\Lambda_n)\left(\frac{c(m,n,\kappa)|f|^2}{\nu|u(t_0)|_\phi^2}-\nu\kappa^2\right)\\&\leqslant\gamma(\Lambda_n+\kappa,\Lambda_n)\left(\frac{c_\phi^2|f|^2}{\nu|u(t_0)|_\phi^2}-\nu\kappa^2\right).\end{aligned}$$

The right-hand side is negative by one of our assumptions. Therefore,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\tilde{v}(t_0)\|^2<0,$$

so $\|\tilde{v}(t)\|$ decreases in a neighbourhood of t_0. Since $\|\tilde{v}(0)\|\leqslant\psi(\Lambda_n+\kappa)$, and because of the latter, $\|\tilde{v}(t)\|$ cannot exceed $\psi(\Lambda_n+\kappa)$ on the interval $[0,T]$. This proves the theorem in the case $v(0)\in D(A)$. Otherwise, we fix $\epsilon\in(0,T)$. There exist $0<\epsilon',\epsilon''<\epsilon$ such that $v(\epsilon')\in D(A)$, and

$$\|\tilde{v}(\epsilon')\|\leqslant\psi(\Lambda_n+\kappa+\epsilon'').$$

Applying the first part of the proof and then letting $\epsilon\to 0$ completes the proof. □

3.1.2. Definition and properties of sets $\mathcal{M}_n$.

Definition 2. *For $n \in \mathbb{N}$ and $0 < \kappa \leqslant (\Lambda_{n+1} - \Lambda_n)/2$, let us define the set*

$$\mathcal{M}_{n,\kappa} := \mathcal{A} \bigcup \left\{ u_0 \in \mathcal{G} \backslash \mathcal{A} : \limsup_{t \to -\infty} \frac{\|S(t)u_0\|^2_{\phi^2}}{|S(t)u_0|^2_\phi} \leqslant \psi(\Lambda_n + \kappa) \right\}.$$

The set $\mathcal{M}_{n,\kappa}$ is positively and negatively invariant: $S(t)\mathcal{M}_{n,\kappa} = \mathcal{M}_{n,\kappa}$ for $t \in \mathbb{R}$. As we will see, it does not really depend on κ. This enables us to define the set $\mathcal{M}_n$ as $\mathcal{M}_{n,\kappa}$ for an arbitrary choice of $0 < \kappa \leqslant (\Lambda_{n+1} - \Lambda_n)/2$. Let us begin with two corollaries of theorem 1.

Corollary 1. *Let $n \in \mathbb{N}$ and $0 < \kappa \leqslant \Lambda_1/2$. Let $u_0 \in \mathcal{M}_{n,\kappa}$. If $|u_0|_\phi \geqslant c_\phi |f|/\nu\kappa$, then there exists a unique $t_0 \geqslant 0$ such that*

$$|S(t)u_0|_\phi > \frac{c_\phi |f|}{\nu\kappa}, \qquad t \in (-\infty, t_0),$$

$$|S(t)u_0|_\phi \leqslant \frac{c_\phi |f|}{\nu\kappa}, \qquad t \in [t_0, \infty),$$

and

$$\frac{\|S(t)u_0\|^2_{\phi^2}}{|S(t)u_0|^2_\phi} \leqslant \psi(\Lambda_n + \kappa), \qquad t \in (-\infty, t_0].$$

Proof. By our previous remarks, $t \mapsto |S(t)|_\phi$ is decreasing as long as $|S(t)u_0|_\phi > \rho_0$, and $\limsup_{t\to\infty} |S(t)u_0|_\phi \leqslant \rho_0$. Since $c_\phi |f|/\nu\kappa > \rho_0$, there exists $t_0 > 0$ such that $S(t_0)u_0 = c_\phi|f|/\nu\kappa$. The first two inequalities follow immediately. Now let $t \in (-\infty, t_0]$. Since $\limsup_{t\to-\infty} \|S(t)u_0\|^2_{\phi^2}/|S(t)u_0|^2_\phi \leqslant \psi(\Lambda_n + \kappa)$, there exists $t_1 \leqslant t$ such that $\|S(t_1)u_0\|^2_{\phi^2}/|S(t_1)u_0|^2_\phi \leqslant \psi(\Lambda_n + \kappa)$. By virtue of theorem 1, the last inequality follows. □

Corollary 2. *For each $n \in \mathbb{N}$, $0 < \kappa \leqslant \Lambda_1/2$,*

$$\mathcal{M}_{n,\kappa} \subset \left\{ u_0 \in V \; : \; |u_0|_\phi < \frac{c_\phi |f|}{\nu\kappa} \right\} \bigcup \mathcal{C}_{n,\kappa}.$$

Proof. This follows trivially from the last corollary. □

Theorem 2. *We can characterize the sets $\mathcal{M}_{n,\kappa}$ in the following way:*

$$\mathcal{M}_{n,\kappa} = \mathcal{A} \bigcup \left\{ u_0 \in \mathcal{G}\backslash\mathcal{A} \; : \; \lim_{t\to-\infty} \frac{\|S(t)u_0\|^2_{\phi^2}}{|S(t)u_0|^2_\phi} \in \{\psi(\Lambda_1), \psi(\Lambda_2), \ldots, \psi(\Lambda_n)\} \right\}.$$

In particular, $\mathcal{M}_{n,\kappa}$ does not depend on the choice of κ. Moreover, for every $u_0 \in \mathcal{G}\backslash\mathcal{A}$,

$$\lim_{t\to-\infty} \frac{\|S(t)u_0\|_{\phi^2}}{|S(t)u_0|_\phi} \in \{\psi(\Lambda_1), \psi(\Lambda_2), \ldots\} \cup \{\infty\}.$$

Proof. Inequality (3.3) implies

$$\frac{\mathrm{d}}{\mathrm{d}t} \|\tilde{v}\|^2 \leqslant \frac{1}{\nu |u|^2_\phi} \sum_{m=1}^{\infty} \frac{\gamma(\mu, \lambda_m)}{\phi(\lambda_m)} |\hat{f}_m|^2.$$

Since

$$\frac{\mathrm{d}}{\mathrm{d}t} \|\tilde{v}(t)\|^2 = \frac{\mathrm{d}}{\mathrm{d}t} \psi(\mu(t)) = \psi'(\mu(t))\mu'(t),$$

we obtain the inequality

$$\mu'(t) \leqslant \frac{1}{\nu|u|_\phi^2}\sum_{m=1}^{\infty}\frac{\gamma(\mu,\lambda_m)}{\psi'(\mu)\phi(\lambda_m)}|\hat{f}_m|^2 \leqslant \frac{c_\phi^2|f|^2}{\nu|u|_\phi^2}. \tag{3.4}$$

Let $u_0 \in \mathcal{M}_{n,\kappa}\backslash\mathcal{A}$ and $u(t) = S(t)u_0$. Let $\{t_n\}$ and $\{T_n\}$ be two sequences of real numbers such that $t_n < T_n$ and $\lim_{n\to\infty} t_n = \lim_{n\to\infty} T_n = -\infty$, and so that

$$\lim_{n\to\infty}\frac{\|u(T_n)\|_{\phi^2}^2}{|u(T_n)|_\phi^2} = \limsup_{t\to-\infty}\frac{\|u(t)\|_{\phi^2}^2}{|u(t)|_\phi^2}$$

and

$$\lim_{n\to\infty}\frac{\|u(t_n)\|_{\phi^2}^2}{|u(t_n)|_\phi^2} = \liminf_{t\to-\infty}\frac{\|u(t)\|_{\phi^2}^2}{|u(t)|_\phi^2}.$$

In particular,

$$\lim_{n\to\infty}\mu(T_n) = \limsup_{t\to-\infty}\ \mu(t), \qquad \lim_{n\to\infty}\mu(t_n) = \liminf_{t\to-\infty}\ \mu(t).$$

Integrating inequality (3.4) on the interval $[t_n, T_n]$, we obtain

$$\mu(T_n) - \mu(t_n) \leqslant \frac{c_\phi^2|f|^2}{\nu}\int_{t_n}^{T_n}\frac{1}{|u(t)|_\phi^2}\,\mathrm{d}t \to 0, \qquad n\to\infty.$$

Letting $n\to\infty$, we conclude

$$\limsup_{t\to-\infty}\mu(t) \leqslant \liminf_{t\to-\infty}\mu(t),$$

so $\mu_\infty = \lim_{t\to-\infty}\mu(t) < \infty$ exists. Therefore, by (3.3), since otherwise the limit would not exist, we have

$$\begin{aligned} 0 \leqslant \limsup_{t\to-\infty}\mu'(t) &\leqslant -\nu\liminf_{t\to-\infty}\left[\min_{m\in\mathbb{N}}\frac{(\mu(t)-\lambda_m)(\psi(\mu(t))-\psi(\lambda_m))}{\psi'(\mu(t))}\right] + \limsup_{t\to-\infty}\frac{c_\phi^2|f|^2}{\nu|u(t)|_\phi^2} \\ &= -\nu\min_{m\in\mathbb{N}}\frac{(\mu_\infty-\lambda_m)(\psi(\mu_\infty)-\psi(\lambda_m))}{\psi'(\mu_\infty)}. \end{aligned}$$

Since ψ is increasing, the latter expression is equal to zero. Therefore, $\mu_\infty \in \{\Lambda_1,\Lambda_2,\ldots,\Lambda_n\}$ and

$$\lim_{t\to-\infty}\frac{\|u(t)\|_{\phi^2}}{|u(t)|_\phi} \in \{\psi(\Lambda_1),\psi(\Lambda_2),\ldots,\psi(\Lambda_n)\}.$$

If $\limsup_{t\to-\infty}\|u(t)\|_{\phi^2}^2/|u(t)|_\phi^2 = \infty$, repeating the above argument as in the first case we conclude that $\lim_{t\to-\infty}\|u(t)\|_{\phi^2}^2/|u(t)|_\phi^2 = \infty$. This proves the theorem. □

Definition 3. *We define* $\mathcal{M}_n := \mathcal{M}_{n,\kappa}$ *for any* $\kappa \in]0, (\Lambda_{n+1}-\Lambda_n)/2]$. *Let also* $\tilde{\mathcal{M}}_n = \{u/|u|_\phi : u \in \mathcal{M}_n\}$.

Corollary 3. *For* $u_0 \in \mathcal{M}_n$ *such that* $|u_0|_\phi > 2c_\phi|f|/\nu\Lambda_1$, *we have*

$$\frac{\|u_0\|_{\phi^2}^2}{|u_0|_\phi^2} \leqslant \psi\left(\Lambda_n + \frac{c_\phi|f|}{\nu|u_0|_\phi}\right). \tag{3.5}$$

Proof. This follows trivially from corollary 2 taking $\kappa = c_\phi|f|/\nu|u_0|_\phi$. □

Remark 2. For $u \in V_{\phi^2}$, we have because of the convexity of ψ

$$\psi\left(\frac{\|u\|_\phi^2}{|u|_\phi^2}\right) \leqslant \frac{\|u\|_{\phi^2}^2}{|u|_\phi^2}.$$

In particular, since ψ is increasing,

$$\frac{\|u\|_{\phi^2}^2}{|u|_\phi^2} \leqslant \psi(\mu) \Rightarrow \frac{\|u\|_\phi^2}{|u|_\phi^2} \leqslant \mu.$$

Remark 3. For the solution u of the KFNSE that satisfies the conditions in theorem 1, we obtain the estimate

$$|u(t)|_\phi^2 \geqslant \left(|u(0)|_\phi^2 + \frac{|f|^2}{8\nu^2(\Lambda_n+\kappa)^2}\right) \mathrm{e}^{-4\nu(\Lambda_n+\kappa)t} - \frac{|f|^2}{8\nu^2(\Lambda_n+\kappa)^2} \tag{3.6}$$

for $t \in [0, T]$.

Proof. From the last remark we have $\|u(t)\|_\phi^2/|u(t)|_\phi^2 \leqslant \Lambda_n + \kappa$, $t \in [0, T]$. Using Young's inequality we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}|u|_\phi^2 &= -2\nu\|u\|_\phi^2 + 2(f, u)\\ &\geqslant -2\nu(\Lambda_n+\kappa)|u|_\phi^2 - 2\nu(\Lambda_n+\kappa)|u|^2 - \frac{|f|^2}{2\nu(\Lambda_n+\kappa)}\\ &\geqslant -4\nu(\Lambda_n+\kappa)|u|_\phi^2 - \frac{|f|^2}{2\nu(\Lambda_n+\kappa)}.\end{aligned}$$

By the Gronwall inequality, we then obtain (3.6). □

Corollary 4. *We have*

$$\mathcal{M}_n \subset \left\{u_0 \in \mathcal{G} \;:\; |S(t)u_0|_\phi = \mathcal{O}\left(\mathrm{e}^{(1+\epsilon)\nu\Lambda_n|t|}\right) \qquad \text{as } t \to -\infty\right\}$$

for any $\epsilon > 0$.

The next result provides us with a method for producing elements of the sets $\mathcal{M}_n$. It will be used for all further results.

Theorem 3. *Let* $u_1, u_2, \ldots \in H_\phi$, *and let* $t_1 > t_2 > \cdots > t_j \to -\infty$, $j \to \infty$. *Suppose that* $S(t)u_j$ *exists on the interval* $[t_j, \infty)$. *Let us also assume*

$$|u_k|_\phi \leqslant M, \qquad k \in \mathbb{N} \tag{3.7}$$

for some constant $M > 0$, *and*

$$|S(t_k)u_k|_\phi \geqslant \frac{c_\phi|f|}{\nu\kappa}, \qquad k \in \mathbb{N}. \tag{3.8}$$

Let there exist some $n \in \mathbb{N}$ *such that*

$$\frac{\|S(t_k)u_k\|_{\phi^2}^2}{|S(t_k)u_k|_\phi^2} \leqslant \psi(\Lambda_n+\kappa), \qquad k \in \mathbb{N}. \tag{3.9}$$

Then, there exist $u_\infty \in \mathcal{M}_n$ *and a subsequence* $\{u_{k_j}\}$ *of* $\{u_k\}$ *such that*

$$\lim_{j\to\infty} |S(t)u_{k_j} - S(t)u_\infty|_\phi = 0, \qquad t \in \mathbb{R}. \tag{3.10}$$

Proof. We first want to prove that, for every $t \in \mathbb{R}$, there exists a constant $C(t) > 0$ such that

$$\|S(t)u_k\|_{\phi^2}^2 \leqslant C(t) \tag{3.11}$$

for k large enough. Without loss of generality, we may assume $M > |f|/\nu\Lambda_1$. First, we fix $k \in \mathbb{N}$. By lemma 1, there exists a unique $\beta_k \geqslant t_k$ such that

$$|S(t)u_k|_\phi \geqslant \frac{c_\phi|f|}{\nu\kappa}, \qquad t \in [t_k, \beta_k], \quad |S(t)u_k|_\phi \leqslant \frac{c_\phi|f|}{\nu\kappa}, \ t \geqslant \beta_k, \tag{3.12}$$

$$\frac{\|S(t)u_k\|_{\phi^2}^2}{|S(t)u_k|_\phi^2} \leqslant \psi(\Lambda_n + \kappa), \qquad t \in [t_k, \beta_k]. \tag{3.13}$$

Since we are interested in extracting a subsequence, without loss of generality we may assume that either $\beta_k \leqslant 0$ for all $k \in \mathbb{N}$, or that $\beta_k > 0$ for all $k \in \mathbb{N}$. Assuming that the latter is true, by (2.16) and (3.7), we obtain

$$\begin{aligned}
\frac{c_\phi^2|f|^2}{\nu^2\kappa^2} &= |S(\beta_k)u_k|_\phi^2 \\
&\leqslant |u_k|_\phi^2 \,\mathrm{e}^{-\nu\Lambda_1\beta_k} + \frac{|f|^2}{\nu^2\Lambda_1^2}(1 - \mathrm{e}^{-\nu\Lambda_1\beta_k}) \\
&\leqslant \mathrm{e}^{-\nu\Lambda_1\beta_k}\left(M^2 - \frac{|f|^2}{\nu^2\Lambda_1^2}\right) + \frac{|f|^2}{\nu^2\Lambda_1^2}.
\end{aligned}$$

We obtain an upper bound on β_k, which covers both of the cases

$$\beta_k \leqslant \frac{1}{\nu\Lambda_1}\log\frac{\kappa^2(\nu^2\Lambda_1^2M^2 - |f|^2)}{(c_\phi^2\Lambda_1^2 - \kappa^2)|f|^2} =: t_M. \tag{3.14}$$

With $C_1, C_2, \ldots$ being various constants, (3.6) implies now

$$\begin{aligned}
|S(t)u_k|_\phi^2 &\leqslant \left(|S(\beta_k)u_k|_\phi^2 + \frac{|f|^2}{8\nu^2\Lambda_1^2}\right)\mathrm{e}^{4\nu\Lambda_{n+1}(\beta_k - t)} \\
&= \left(\frac{c_\phi^2|f|^2}{\nu^2\kappa^2} + \frac{|f|^2}{8\nu^2\Lambda_1^2}\right)\mathrm{e}^{4\nu\Lambda_{n+1}(\beta_k - t)} \\
&= C_1\,\mathrm{e}^{4\nu\Lambda_{n+1}(\beta_k - t)}, \qquad t \in [t_k, \beta_k].
\end{aligned}$$

From here, using (3.13) and (3.14), we conclude

$$\begin{aligned}
\|S(t)u_k\|_{\phi^2}^2 &\leqslant \psi(\Lambda_n + \kappa)C_1\,\mathrm{e}^{4\nu\Lambda_{n+1}(\beta_k - t)} \\
&\leqslant C_2\,\mathrm{e}^{4\nu\Lambda_{n+1}(t_M - t)}, \qquad t \in [t_k, \beta_k].
\end{aligned} \tag{3.15}$$

On the other hand, for $t \geqslant \beta_k$ from (2.18) and the previous calculations, we obtain

$$\begin{aligned}
\|S(t)u_k\|_{\phi^2}^2 &\leqslant \|S(\beta_k)u_k\|_{\phi^2}^2\,\mathrm{e}^{-\nu\Lambda_1(t-\beta_k)} + \frac{|f|^2}{\nu^2\Lambda_1}(1 - \mathrm{e}^{-\nu\Lambda_1(t-\beta_k)}) \\
&\leqslant \psi(\Lambda_n + \kappa)|S(\beta_k)u_k|_\phi^2\,\mathrm{e}^{-\nu\Lambda_1(t-\beta_k)} + \frac{|f|^2}{\nu^2\Lambda_1}(1 - \mathrm{e}^{-\nu\Lambda_1(t-\beta_k)}) \\
&\leqslant \psi(\Lambda_n + \kappa)\frac{c_\phi^2|f|^2}{\nu^2\kappa^2} + \frac{|f|^2}{\nu^2\Lambda_1} =: C_3.
\end{aligned}$$

This and (3.15) together imply

$$\|S(t)u_k\|_{\phi^2}^2 \leqslant \max\left\{C_2\,\mathrm{e}^{4\nu\Lambda_{n+1}(t_M - t)}, C_3\right\} =: C(t) \qquad t \geqslant t_k. \tag{3.16}$$

Since H_ϕ is compactly imbedded in V_{ϕ^2}, for every $i \in \mathbb{N}$, we can extract a subsequence $\{S(t_i)u_{k^i_j}\}_{j\in\mathbb{N}}$ of $\{S(t_i)u_k\}_{k\geqslant i}$ which converges in H_ϕ. Without loss of generality, we may assume that $\{k^i_j\} \subset \{k^l_j\}$ for $i \geqslant l$. Now we may use the Cantor diagonal process to extract a subsequence $\{u_{k_j}\}$ of $\{u_k\}$ such that $\lim_{j\to\infty} S(t_{k_i})u_{k_j} =: w_i \in H_\phi$, $i \in \mathbb{N}$ exists. Actually, $w_i \in V_{\phi^2}$. Since $S(t) : H_\phi \to H_\phi$ is a continuous mapping for $t \geqslant 0$, we obtain

$$w_i = S(t_{k_i} - t_{k_j})w_j, \qquad j \leqslant i, \quad i, j \in \mathbb{N},$$

which means that w_i belongs to the trajectory of a solution. Letting $u_\infty := S(-t_{k_1})w_1$, we obtain $u_\infty = S(-t_{k_j})w_j$, $j \in \mathbb{N}$. Therefore, $u_\infty \in \mathcal{G}$. Again, by the continuity of $S(t) : H_\phi \to H_\phi$, we obtain

$$\lim_{j\to\infty} |S(t)u_{k_j} - S(t)u_\infty|_\phi = 0, \qquad t \in \mathbb{R}.$$

It remains to prove that $u_\infty \in \mathcal{M}_n$. To this end, we consider two cases. If $\liminf_{k\to\infty} \beta_k = -\infty$, then (3.10) and (3.12) imply

$$|S(t)u_\infty|_\phi \leqslant \frac{c_\phi |f|}{\nu\kappa}, \qquad t \in \mathbb{R},$$

and, thus, $u_\infty \in \mathcal{A}$. If, on the other hand, $\liminf_{k\to\infty} \beta_k = \beta_\infty > -\infty$, (3.10) and (3.13), give

$$\frac{\|S(t)u_\infty\|^2_{\phi^2}}{|S(t)u_\infty|^2_\phi} \leqslant \psi(\Lambda_n + \kappa), \qquad t \leqslant \beta_\infty.$$

In both cases $u_\infty \in \mathcal{M}_n$. □

Theorem 4. *For each $n \in \mathbb{N}$, $\mathcal{M}_n$ is a connected, locally compact, both positively and negatively invariant subset of H_ϕ.*

Proof. As an invariant set containing $\mathcal{A}$, $\mathcal{M}_n$ is connected. In order to prove the local compactness of $\mathcal{M}_n$, because of the compactness of $\mathcal{A}$, it suffices to check that every sequence $u_1, u_2, \ldots \in \mathcal{M}_n\backslash\mathcal{A}$, which is bounded in H_ϕ, has a subsequence converging to an element of $\mathcal{M}_n$. Since $\lim_{t\to-\infty} |S(t)u_k|_\phi = \infty$ for $k \in \mathbb{N}$, there exist $t_1 > t_2 > \cdots$ such that $\lim_{t\to-\infty} t_j = -\infty$ and $|S(t_k)u_k|_\phi \geqslant c_\phi|f|/\Lambda_1\kappa$. By virtue of corollary 1 we have $\|S(t_k)u_k\|^2_{\phi^2}/|S(t_k)u_k|^2_\phi \leqslant \psi(\Lambda_n + \kappa)$, $k \in \mathbb{N}$. Applying theorem 3, there exists a subsequence of $\{u_k\}$ which converges to an element of $\mathcal{M}_n$ in H_ϕ. □

3.2. Richness of the sets $\mathcal{M}_n$

One of the main results of this paper is the following theorem on the richness of the sets $\mathcal{M}_n$.

Theorem 5. *Let $n \in \mathbb{N}$. For every $p_0 \in P_nH$, there exists $u_\infty \in \mathcal{M}_n$ such that $P_nu_\infty = p_0$. In other words, $P_nH = P_n\mathcal{M}_n$.*

First, we need to prove a series of lemmas.

Lemma 1. *Let $u \in \mathcal{C}_{n,\kappa}$ for some $n \in \mathbb{N}$, and $0 < \kappa \leqslant (\Lambda_{n+1} - \Lambda_n)/2$. Then,*

$$|Q_nu|^2_\phi \leqslant \frac{\psi(\Lambda_n + \kappa)}{\psi(\Lambda_{n+1}) - \psi(\Lambda_n + \kappa)}|P_nu|^2_\phi. \tag{3.17}$$

Also,

$$|Q_nu|^2_\phi \leqslant \frac{1}{\Lambda_{n+1} - \Lambda_n - \kappa}((\Lambda_n + \kappa)|P_nu|^2_\phi - \|P_nu\|^2_\phi). \tag{3.18}$$

In particular,

$$|Q_n u|_\phi^2 \leqslant \frac{\Lambda_n + \Lambda_{n+1}}{\Lambda_{n+1} - \Lambda_n} |P_n u|_\phi^2 \tag{3.19}$$

and

$$|u|_\phi^2 \leqslant \frac{2\Lambda_{n+1}}{\Lambda_{n+1} - \Lambda_n} |P_n u|_\phi^2. \tag{3.20}$$

Proof. Inequality (3.17) follows from

$$|Q_n u|_\phi^2 \leqslant \frac{1}{\psi(\Lambda_{n+1})} \|Q_n u\|_{\phi^2}^2 \leqslant \frac{\psi(\Lambda_n + \kappa)}{\psi(\Lambda_{n+1})} \left(|P_n u|_\phi^2 + |Q_n u|_\phi^2\right).$$

Observe that by lemma 2 $\|u\|_\phi^2 / |u|_\phi^2 \leqslant \Lambda_n + \kappa$. Therefore,

$$\begin{aligned} |Q_n u|_\phi^2 &\leqslant \frac{1}{\Lambda_{n+1}} (\|u\|_\phi^2 - \|P_n u\|_\phi^2) \\ &\leqslant \frac{\Lambda_n + \kappa}{\Lambda_{n+1}} |Q_n u|_\phi^2 + \frac{1}{\Lambda_{n+1}} ((\Lambda_n + \kappa)|P_n u|_\phi^2 - \|P_n u\|_\phi^2), \end{aligned}$$

and (3.18) follows. The other two inequalities follow from (3.18) and the fact that $\kappa \leqslant (\Lambda_{n+1} - \Lambda_n)/2$. □

Lemma 2. *Let $u \in \mathcal{M}_n$, for some $n \in \mathbb{N}$. Let $0 < \kappa \leqslant \Lambda_1/2$. Then, the following estimates hold:*

$$|Q_n u|_\phi \leqslant \max\left\{\frac{c_\phi |f|}{\nu\kappa}, \left(\frac{\psi(\Lambda_n + \kappa)}{\psi(\Lambda_{n+1}) - \psi(\Lambda_n + \kappa))}\right)^{1/2} |P_n u|_\phi\right\} \tag{3.21}$$

and

$$|Q_n u|_\phi \leqslant \max\left\{\frac{2c_\phi |f|}{\nu\Lambda_1}, \left(\frac{\Lambda_n + \Lambda_{n+1}}{\Lambda_{n+1} - \Lambda_n}\right)^{1/2} |P_n u|_\phi\right\}. \tag{3.22}$$

Proof. If $|u|_\phi \leqslant c_\phi |f|/\nu\kappa$, the statement is obvious. If, on the other hand, $|u|_\phi > c_\phi |f|/\nu\kappa$, we get $\|u\|_{\phi^2}^2 \leqslant \psi((\Lambda_n + \Lambda_{n+1})/2)|u|_\phi^2$ by corollary 1. Applying lemma 1, we obtain (3.21) and (3.22). □

In order to prove the next important step towards the proof of theorem 5 we will need the well-known theorem by Brouwer, which we will state here.

Theorem 6. *Let $\mathbb{R}^n$ be endowed with a norm, and let $B(r) \subset \mathbb{R}^n$ be a closed ball with radius $r > 0$. If $g : B(r) \to B(r)$ is a continuous mapping and if $g(x) = x$ for all $x \in \partial B(r)$, then g is onto.*

Lemma 3. *Let $p_0 \in P_n H$ for some $n \in \mathbb{N}$. Then, for every $t_0 > 0$, there exists $w_0 \in P_n H$ such that $P_n S(t_0) w_0 = p_0$. In other words, the operator $P_n S(t_0) : P_n H \to P_n H$ is onto.*

Proof. For $r > 0$ we define $B^{H_\phi}(r) := \{u_0 \in H_\phi : |u_0|_\phi \leqslant r\}$, and let $B_n(r) = B^{H_\phi}(r) \cap P_n H$. Let us fix a $r_0 > c_\phi |f|/\nu\kappa$ large enough that $p_0 \in B_n(2r_0)$. In order to prove the lemma, let us choose a continuous function $\theta : \mathbb{R} \to \mathbb{R}$ such that $\theta(x) = 1$ for $x \leqslant r_0$ and $\theta(x) = 0$ for $x \geqslant 2r_0$. Define

$$g(u_0) = P_n S(\theta(|u_0|_\phi) t_0) u_0, \qquad u_0 \in P_n H.$$

By (2.16), we have $S(t)B(r) \subset B(r)$, $r \geqslant |f|/\nu\Lambda_1$, $t \geqslant 0$. Therefore, $g(B_n(2r_0)) \subset B_n(2r_0)$. Also, g is continuous, and it satisfies $g(u_0) = u_0$ for $|u_0|_\phi = 2r_0$. By the previous Brouwer's theorem, there exists $w_0 \in B_n(2r_0)$ such that $g(w_0) = p_0$. Let us assume for a moment that r_0 was chosen in such a way that $|w_0|_\phi \leqslant r_0$. Then, by the definition of g, we would have $P_n S(t_0)w_0 = g(w_0) = p_0$, and this is exactly the claim of this lemma.

In order to complete the proof, let $0 < \kappa \leqslant \Lambda_1/2$. For $u_0 \in P_n H$ such that $|u_0|_\phi > c_\phi|f|/\nu\kappa$ we have $\|u_0\|^2_{\phi^2}/|u_0|^2_\phi \leqslant \psi(\Lambda_n) \leqslant \psi(\Lambda_n + \kappa)$. By corollary 1, there exists a unique $\tau_{u_0} > 0$, such that $|S(t)u_0|_\phi > c_\phi|f|/\nu\kappa$, $t \in [0, \tau_{u_0})$, $|S(t)u_0|_\phi \leqslant c_\phi|f|/\nu\kappa$, $t \in (\tau_{u_0}, \infty)$, and for $t \in [0, \tau_{u_0}]$ we have $\|S(t)u_0\|^2_{\phi^2}/|S(t)u_0|^2_\phi \leqslant \psi(\Lambda_n + \kappa)$ and $|S(t)u_0|^2_\phi \geqslant |u_0|^2_\phi\,\mathrm{e}^{-4\nu\Lambda_{n+1}t} - |f|^2/8\nu^2\Lambda_n^2$. Lemma 1 now implies

$$|S(t)u_0|^2_\phi \leqslant \frac{2\Lambda_{n+1}}{\Lambda_{n+1} - \Lambda_n}|P_n S(t)u_0|^2_\phi$$

for $t \in [0, \tau_{u_0}]$. Therefore,

$$\begin{aligned} |P_n S(t)u_0|^2_\phi &\geqslant \frac{\Lambda_{n+1} - \Lambda_n}{2\Lambda_{n+1}}|S(t)u_0|^2_\phi \\ &\geqslant \frac{\Lambda_{n+1} - \Lambda_n}{2\Lambda_{n+1}}\left(|u_0|^2_\phi\,\mathrm{e}^{-4\nu\Lambda_{n+1}t} - \frac{|f|^2}{8\nu^2\Lambda_n^2}\right), \qquad t \in [0, \tau_{u_0}]. \end{aligned} \tag{3.23}$$

We now want to prove that there exists $r_0 > c_\phi|f|/\nu\kappa$ such that $|P_n S(t)u_0|_\phi > |p_0|_\phi$, $t \in [0, t_0]$, $u_0 \in P_n H\backslash B_n(r_0)$. Let

$$l = \max\left\{|p_0|_\phi, \frac{c_\phi|f|}{\nu\kappa}\right\}. \tag{3.24}$$

We fix any $r_0 > c_\phi|f|/\nu\kappa$ such that

$$\frac{\Lambda_{n+1} - \Lambda_n}{2\Lambda_{n+1}}\left(r_0^2\,\mathrm{e}^{-4\nu\Lambda_{n+1}t_0} - \frac{|f|^2}{8\nu^2\Lambda_n^2}\right) > l^2. \tag{3.25}$$

For $|u_0|_\phi > r_0$, we obtain

$$\begin{aligned} l^2 \geqslant \left(\frac{c_\phi|f|}{\nu\kappa}\right)^2 &= |S(\tau_{u_0})u_0|^2_\phi \\ &\geqslant |P_n S(\tau_{u_0})u_0|^2_\phi \\ &\geqslant \frac{\Lambda_{n+1} - \Lambda_n}{2\Lambda_{n+1}}\left(r_0^2\,\mathrm{e}^{-4\nu\Lambda_{n+1}\tau_{u_0}} - \frac{|f|^2}{8\nu^2\Lambda_n^2}\right). \end{aligned} \tag{3.26}$$

Combining (3.25) and (3.26), we get $t_0 < \tau_{u_0}$. In particular, (3.23) holds for $t \in [0, t_0]$, and so does (3.25). From (3.23), (3.25) and (3.24), we obtain

$$|P_n S(t)u_0|_\phi > |p_0|_\phi, \qquad t \in [0, t_0] \tag{3.27}$$

for all $u_0 \in P_n H\backslash B_n(r_0)$. In order to complete the proof it remains to show that $|w_0|_\phi \leqslant r_0$. Assuming the opposite, we would have $|g(w_0)|_\phi = |P_n S(\theta(|w_0|_\phi)t_0)w_0|_\phi > |p_0|_\phi$. This is a contradiction, since $g(w_0) = p_0$. The lemma is proven. □

Now we have the tools to prove theorem 5.

Proof of theorem 5. Let $p_0 \in P_n H$. Choose a sequence $0 > t_1 > t_2 > t_3 > \cdots$ such that $\lim_{k\to\infty} t_k = -\infty$. By lemma 3, there exist $p_1, p_2, \ldots \in P_n H$ such that $P_n S(-t_k)p_k = p_0$, $k \in \mathbb{N}$. Let $u_k := S(-t_k)p_k$, $k \in \mathbb{N}$. Therefore, $P_n u_k = p_0$, $k \in \mathbb{N}$. Let us define $U_k(t) := S(t)u_k$, $t \geqslant t_k$, and $V_k(t) := \phi(A)U_k(t)$, $t \geqslant t_k$. Let us assume first that

$$|p_k|_\phi \leqslant \frac{2c_\phi|f|}{\nu\Lambda_1} \tag{3.28}$$

for infinitely many $k \in \mathbb{N}$. Without loss of generality, we may assume that this is true for all $k \in \mathbb{N}$. By the Poincaré inequality, we have

$$\begin{aligned}\|V_k(t_k)\| = \|\phi(A)p_k\| &\leqslant \psi(\Lambda_n)^{1/2}|p_k|_\phi \\ &\leqslant \psi(\Lambda_n)^{1/2}\frac{2c_\phi|f|}{\nu\Lambda_1} \\ &= (2c_\phi)\phi(\Lambda_n)^{1/2}\left(\frac{\Lambda_n}{\Lambda_1}\right)^{1/2}\frac{|f|}{\nu\Lambda_1^{1/2}}.\end{aligned}$$

Since $(2c_\phi)\phi(\Lambda_n)^{1/2}(\Lambda_n/\Lambda_1)^{1/2} \geqslant 1$, inequality (2.17) gives

$$\|V_k(t)\| \leqslant (2c_\phi)\phi(\Lambda_n)^{1/2}\left(\frac{\Lambda_n}{\Lambda_1}\right)^{1/2}\frac{|f|}{\nu\Lambda_1^{1/2}}, \qquad t \geqslant t_k.$$

Also,

$$|U_k(t)|_\phi \leqslant \frac{\psi(\Lambda_n)^{1/2}}{\psi(\Lambda_1)^{1/2}}\frac{2c_\phi|f|}{\nu\Lambda_1}, \qquad t \geqslant t_k. \tag{3.29}$$

Repeating the arguments from theorem 3, passing to a subsequence, and using Cantor's diagonal process, we can conclude that

$$\lim_{k\to\infty}|U_k(t) - S(t)u_\infty|_\phi = 0, \qquad t \in \mathbb{R}$$

for some $u_\infty \in \mathcal{G}$. From here because of the continuity of the projection P_n, it follows that

$$P_n u_\infty = P_n u_k = p_0$$

and, using (3.29), $u_\infty \in \mathcal{A} \subset \mathcal{M}_n$. This proves the lemma under assumption (3.28). Let us now assume that $|p_k|_\phi \geqslant 2c_\phi|f|/\nu\Lambda_1$ for infinitely many $k \in \mathbb{N}$. Again, by passing to a subsequence, we may assume that this is true for all $k \in \mathbb{N}$. Again, for each $k \in \mathbb{N}$, either $|u_k|_\phi \leqslant 2c_\phi|f|/\nu\Lambda_1$ or $|u_k|_\phi > 2c_\phi|f|/\nu\Lambda_1$. If the latter is true, since $u_k = S(-t_k)p_k$, we have

$$|U_k(t)|_\phi = |S(t)u_k|_\phi > \frac{2c_\phi|f|}{\nu\Lambda_1}, \qquad t \in [t_k, 0]$$

and since

$$\frac{\|p_k\|_{\phi^2}^2}{|p_k|_\phi^2} \leqslant \psi(\Lambda_n) \leqslant \psi\left(\frac{\Lambda_n + \Lambda_{n+1}}{2}\right),$$

by theorem 1, we have

$$\frac{\|u_0\|_{\phi^2}^2}{|u_0|_\phi^2} = \frac{\|U_k(0)\|_{\phi^2}^2}{|U_k(0)|_\phi^2} \leqslant \psi\left(\frac{\Lambda_n + \Lambda_{n+1}}{2}\right).$$

In any case, by lemma 1, since $P_n u_k = p_0$,

$$|u_k|_\phi \leqslant \max\left\{\frac{2c_\phi|f|}{\nu\Lambda_1}, \left(\frac{2\Lambda_{n+1}}{\Lambda_{n+1} - \Lambda_n}\right)^{1/2}|p_0|_\phi\right\}, \qquad k \in \mathbb{N}.$$

Therefore, the sequence $\{u_k\}$ satisfies the assumptions of theorem 3. We conclude that there exists a $u_\infty \in \mathcal{M}_n$, so that

$$\lim_{k\to\infty}|U_k(t) - S(t)u_\infty|_\phi = 0, \qquad t \in \mathbb{R}.$$

In particular, $\lim_{k\to\infty}|u_k - u_\infty|_\phi = 0$ and, as in the previous case, $P_n u_\infty = p_0$. This completes the proof of the theorem. □

4. Density properties of the sets $\mathcal{M}_n$

This section is devoted to the density properties of the sets $\mathcal{M}_n$ and $\mathcal{G}$. In [1], the authors postulated that for the solution operator S of the NSEs, the set $S(t)H$ for $t > 0$ is dense in the phase space H. This conjecture was partly proven in [6]. It was shown that the set of initial data for which there exists a global solution, the set $\mathcal{G}$, is dense in the phase space H, but with respect to the weaker norm of the space V'. Here, we will prove that a similar result holds for two-dimensional periodic KFNSEs for any choice of the nondecreasing convex ϕ. However, if ϕ demonstrates exponential asymptotic behaviour, the density is shown with respect to the norm of the phase space H_ϕ.

Theorem 7. *The set $\bigcup_{n\in\mathbb{N}} \mathcal{M}_n (\subset \mathcal{G})$ is dense in H_ϕ with respect to the norm $|\cdot|_\upsilon$.*

Proof. Let $u_0 \in H_\phi$ be arbitrary. By theorem 5, for each $n \in \mathbb{N}$ there exists $u_n \in \mathcal{M}_n$ such that $P_n u_n = P_n u_0$, and from (3.22) we have

$$|Q_n u_n|_\phi \leqslant \max\left\{\frac{2c_\phi|f|}{\nu\Lambda_1}, \left(\frac{\Lambda_n+\Lambda_{n+1}}{\Lambda_{n+1}-\Lambda_n}\right)^{1/2}|P_n u_0|_\phi\right\}.$$

This implies

$$\begin{aligned}
|u_n - u_0|_\upsilon &= |Q_n(u_n-u_0)|_\upsilon \leqslant \Lambda_{n+1}^{-1/2}|Q_n(u_n-u_0)|_\phi \\
&\leqslant \Lambda_{n+1}^{-1/2}\left(|Q_n u_n|_\phi + |Q_n u_0|_\phi\right) \\
&\leqslant \Lambda_{n+1}^{-1/2}\left(\max\left\{\frac{2c_\phi|f|}{\nu\Lambda_1}, \left(\frac{\Lambda_n+\Lambda_{n+1}}{\Lambda_{n+1}-\Lambda_n}\right)^{1/2}|P_n u_0|_\phi\right\} + |Q_n u_0|_\phi\right) \\
&\leqslant \max\left\{\frac{2c_\phi|f|}{\nu\Lambda_1\Lambda_{n+1}^{1/2}}, \frac{\sqrt{2}}{(\Lambda_{n+1}-\Lambda_n)^{1/2}}|P_n u_0|_\phi\right\} + \Lambda_{n+1}^{-1/2}|Q_n u_0|_\phi.
\end{aligned}$$

By virtue of (2.5), we obtain $\liminf_{n\to\infty}|u_n - u_0|_\upsilon = 0$. □

Theorem 8. *Let $\phi(\xi) \leqslant C\phi'(\xi)$, $\xi > 0$ for some positive constant C. The set $\bigcup_{n\in\mathbb{N}} \mathcal{M}_n (\subset \mathcal{G})$ is dense in H_ϕ with respect to the norm $|\cdot|_\phi$.*

Proof. Let $u_0 \in H_\phi$ be arbitrary. By theorem 5, for each $n \in \mathbb{N}$ there exists $u_n \in \mathcal{M}_n$ such that $P_n u_n = P_n u_0$. By (2.5) there exists an increasing sequence of integers $\{n_k\}_{k\in\mathbb{N}}$ such that $\lim_{k\to\infty}(\Lambda_{n_k+1} - \Lambda_{n_k}) = \infty$. Let $0 < \kappa \leqslant \Lambda_1/2$.

First, let us assume that for infinitely many $k \in \mathbb{N}$, $u_{n_k} \in \mathcal{A}$. Since $\mathcal{A}$ is compact, there exist a subsequence of $\{u_{n_k}\}$ that converges in H_ϕ to some $\tilde{u}_0 \in \mathcal{A}$. Without loss of generality, we can assume that $|u_{n_k} - \tilde{u}_0|_\phi \to 0$, as $k \to \infty$. Also, $|u_{n_k} - \tilde{u}_0|_\upsilon \to 0$, as $k \to \infty$. From the last theorem we know that $u_{n_k} \to u_0$ in the norm $|\cdot|_\upsilon$, so the two limits have to coincide. Thus, $u_0 \in \mathcal{A}$ and $u_{n_k} \to u_0$ in the $|\cdot|_\phi$ norm. This shows the density in this case.

Let us now assume that for infinitely many $k \in \mathbb{N}$, $u_{n_k} \in \mathcal{M}_n \backslash \mathcal{A}$. Without loss of generality we may assume that this is the case for all $k \in \mathbb{N}$. We consider two cases. First, let us assume that for infinitely many $k \in \mathbb{N}$, $\|u_{n_k}\|^2_{\phi^2}/|u_{n_k}|^2_\phi \leqslant \psi(\Lambda_{n_k} + \kappa)$. Without loss of generality, let us assume that this is the case for all $k \in \mathbb{N}$. Then, by (3.17) we have

$$|Q_{n_k} u_{n_k}|^2_\phi \leqslant \frac{\psi(\Lambda_{n_k}+\kappa)}{\psi(\Lambda_{n_k+1}) - \psi(\Lambda_{n_k}+\kappa)}|P_{n_k} u_{n_k}|^2_\phi.$$

By our assumptions, we have now

$$
\begin{aligned}
|Q_{n_k}u_{n_k}|_\phi^2 &\leqslant \frac{\psi(\Lambda_{n_k}+\kappa)}{(\Lambda_{n_k+1}-\Lambda_{n_k}-\kappa)\psi'(\Lambda_{n_k}+\kappa)}|P_{n_k}u_{n_k}|_\phi^2 \\
&\leqslant \frac{\phi(\Lambda_{n_k}+\kappa)}{(\Lambda_{n_k+1}-\Lambda_{n_k}-\kappa)\phi'(\Lambda_{n_k}+\kappa)}|P_{n_k}u_{n_k}|_\phi^2 \\
&\leqslant \frac{2C}{(\Lambda_{n_k+1}-\Lambda_{n_k})}|u_0|_\phi^2 \to 0, \qquad k\to\infty.
\end{aligned}
$$

Let us now assume that $\|u_{n_k}\|_{\phi^2}^2/|u_{n_k}|_\phi^2 > \psi(\Lambda_{n_k}+\kappa)$ for infinitely many $k\in\mathbb{N}$. By corollary 2, we know that $|u_{n_k}|_\phi < c_\phi|f|/\nu\kappa$. Since $\lim_{t\to-\infty}S(t)u_{n_k}=\infty$, there exists $t_k<0$ such that $\|S(t_k)u_{n_k}\|_{\phi^2}^2/|S(t_k)u_{n_k}|_\phi^2 \leqslant \psi(\Lambda_{n_k}+\kappa)$ and $|S(t_k)u_{n_k}|_\phi = c_\phi|f|/\nu\kappa$, so $\|S(t_k)u_{n_k}\|_{\phi^2}^2 \leqslant \psi(\Lambda_{n_k}+\kappa)\cdot c_\phi^2|f|^2/\nu^2\kappa^2$. Now,

$$
\begin{aligned}
|Q_{n_k}u_{n_k}|_\phi^2 &\leqslant \frac{1}{\psi(\Lambda_{n_k+1})}\|Q_{n_k}u_{n_k}\|_{\phi^2}^2 \\
&\leqslant \frac{1}{\psi(\Lambda_{n_k+1})}\left(\|S(t_k)u_{n_k}\|_{\phi^2}^2\,\mathrm{e}^{\nu\Lambda_1 t_k}+\frac{|f|^2}{\nu^2\Lambda_1}(1-\mathrm{e}^{\nu\Lambda_1 t_k})\right) \\
&\leqslant \frac{\psi(\Lambda_{n_k}+\kappa)}{\psi(\Lambda_{n_k+1})}\frac{c_\phi^2|f|^2}{\nu^2\kappa^2}+\frac{1}{\psi(\Lambda_{n_k+1})}\frac{|f|^2}{\nu^2\Lambda_1} \\
&\leqslant \frac{\psi(\Lambda_{n_k}+\kappa)}{\psi(\Lambda_{n_k}+\kappa)+\psi'(\Lambda_{n_k}+\kappa)(\Lambda_{n_k+1}-\Lambda_{n_k}-\kappa)}\frac{c_\phi^2|f|^2}{\nu^2\kappa^2}+\frac{1}{\psi(\Lambda_{n_k+1})}\frac{|f|^2}{\nu^2\Lambda_1} \\
&\leqslant \frac{2C}{\Lambda_{n_k+1}-\Lambda_{n_k}}\frac{c_\phi^2|f|^2}{\nu^2\kappa^2}+\frac{1}{\psi(\Lambda_{n_k+1})}\frac{|f|^2}{\nu^2\Lambda_1}.
\end{aligned}
$$

Therefore, in both cases $|Q_{n_k}u_{n_k}|_\phi\to 0$ when $k\to\infty$. Thus,

$$
|u_{n_k}-u_0|_\phi = |Q_{n_k}(u_{n_k}-u_0)|_\phi \leqslant |Q_{n_k}u_{n_k}|_\phi+|Q_{n_k}u_0|_\phi \to 0, \tag{4.1}
$$

when $k\to\infty$. This completes the proof. □

Remark 4. The condition on ϕ in theorem 8 can be replaced by the following property:

$$
\limsup_{n\to\infty}\frac{\phi(\Lambda_{n+1})}{\phi(\Lambda_n)}=\infty. \tag{4.2}
$$

Because of the asymptotic behaviour of the spectral gaps $\limsup_{n\to\infty}(\Lambda_{n+1}-\Lambda_n)/\log n>0$, the proof of theorem 8 does not work for any polynomial choice of the function ϕ, and in particular for the two-dimensional periodic NSEs, $\phi=1$, and the two-dimensional periodic Navier–Stokes-α model, $\phi(\xi)=1+\alpha^2\xi$. In those cases we only have the weaker result from theorem 7. In order for (4.2) to be satisfied, we need exponential asymptotic behaviour of the function ϕ, for example, $\phi(\xi)=\mathrm{e}^{\alpha^2\xi}$.

The next theorem, which resulted from a private communication with Ciprian Foias and Igor Kukavica, is a density result in the energy Norm $|\cdot|_\phi$ for any choice of nondecreasing, convex ϕ.

Theorem 9. *Let $W_n:=\{w: Aw=\Lambda_n w, |w|_\phi=1\}$. Then, $W_n\subset\tilde{\mathcal{M}}_n^-$ and*

$$
P_nH\subset\left(\sum_{k=1}^{n}\mathbb{R}\mathcal{M}_k\right)^-,
$$

where the closure is taken in H_ϕ.

Proof. Let $r > c_\phi|f|/\nu\Lambda_1$. By theorem 5, there exists $u_r \in \mathcal{M}_n$ such that $P_n u_r = rw$. Also, by lemma 1 and corollary 3, we have

$$|Q_n u_r|_\phi^2 \leqslant \frac{(c_\phi|f|/\nu|u_r|_\phi)|P_n u_r|_\phi^2}{\Lambda_{n+1} - \Lambda_n - c_\phi|f|/\nu|u_r|_\phi} \leqslant \frac{c_\phi|f|r^2}{\nu r(\Lambda_{n+1} - \Lambda_n) - c_\phi|f|}.$$

Thus,

$$\frac{|Q_n u_r|_\phi^2}{r^2} \leqslant \frac{c_\phi|f|}{\nu r(\Lambda_{n+1} - \Lambda_n) - c_\phi|f|} \to 0$$

and

$$\frac{|u_r|_\phi^2}{r^2} = \frac{|P_n u_r|_\phi^2 + |Q_n u_r|_\phi^2}{r^2} \to 1,$$

when we let $r \to \infty$. Finally,

$$\left|\frac{u_r}{|u_r|_\phi} - w\right|_\phi = \left|\frac{u_r - rw}{|u_r|_\phi} - w\left(1 - \frac{r}{|u_r|_\phi}\right)\right|_\phi \leqslant \frac{|Q_n u_r|_\phi}{r} + \left|1 - \frac{r}{|u_r|_\phi}\right| \to 0,$$

when $r \to \infty$. This proves the theorem. □

References

[1] Bardos C and Tartar L 1973 Sur l'unicité rétrograde des équations paraboliques et quelques questions voisines *Arch. Ration. Mech. Anal.* **50** 10–25

[2] Chen S, Foias C, Holm D D, Olson E, Titi E S and Wynne S 1998 A connection between the Camassa–Holm equations and turbulence flows in pipes and channels *Phys. Fluids* **11** 2343–53

[3] Constantin P 2003 Filtered viscous fluid equations *Comput. Math. Appl.* **46** 537–46

[4] Constantin P 2003 Near identity transformations for the Navier–Stokes equations *Handbook of Mathematical Fluid Dynamics* vol 2 (Amsterdam: Elsevier)

[5] Constantin P and Foias C 1988 *Navier–Stokes Equations* (Chicago: Lectures in Mathematics) (Chicago, IL: Chicago University Press)

[6] Constantin P, Foias C, Kukavica I and Majda A J 1997 Dirichlet quotients and 2D periodic Navier–Stokes equations *J. Math. Pure. Appl.* **76** 125–53

[7] Constantin P, Foias C, Nicolaenko B and Temam R 1988 Spectral barriers and inertial manifolds for dissipative partial differential equations *J. Dyn. Diff. Eqns* **1** 45–73

[8] Constantin P, Foias C and Temam R 1988 On the dimension of attractors in two-dimensional turbulence *Physica* D **30** 284–96

[9] Doering C, Gibbon J D, Holm D D and Nicolaenko B 1988 Low-dimensional behavior in the complex Ginzburg–Landau equation *Nonlinearity* **1** 279–309

[10] Dascaliuc R 2003 On backward-time behavior of Burger's original model for turbulence *Nonlinearity* **16** 1945–65

[11] Foias C, Holm D D and Titi E S 2001 The three-dimensional viscous Camassa–Holm equations, and their relation to the Navier–Stokes equations and turbulence theory *J. Dyn. Diff. Eqns* **14**

[12] Foias C, Jolly M S and Kukavica I 1996 Localisation of attractors *Nonlinearity* **9** 1565–81

[13] Foais C, Jolly M S, Kukavica I and Titi E S 2001 The Lorenz equation as a methafor for some analytic, geometric, and statistical properties of the Navier–Stokes equations *Descrete Continuous Dyn. Syst.* **7** 403–30

[14] Foais C, Jolly M S and Lee W S 2002 Nevanlinna–Pick interpolation of global attractors *Nonlinearity* **15** 1881–904

[15] Foias C, Holm D D and Titi E S 2001 The Navier–Stokes-alpha model of fluid turbulence *Physica* D **152–153** 505–19

[16] Foias C and Kukavica I 2001 private communication

[17] Foias C and Saut J C 1981 Limite du rapport de l'enstrophie sur l'énergie pour une solution faible des équations de Navier–Stokes *C. R. Acad. Sci. Paris, Sér I Math.* **298** 241–44

[18] Foias C and Saut J C 1984 Asymptotic behavior, as $t \to \infty$, of solutions of Navier–Stokes equations and nonlinear spectral manifolds *Indiana Univ. Math. J.* **33** 459–77

[19] Foias C, Sell G R and Temam R 1988 Inertial manifolds for nonlinear evolutionary equations *J. Diff. Eqns* **73** 309–53

[20] Holm D D, Marsden J E and Ratiu T S 1998 Euler–Poincaré models of ideal fluids with nonlinear dispersion *Phys. Rev. Lett.* **80** 4173–77

[21] Holm D D, Marsden J E and Ratiu T S 1998 Euler–Poincaré equations and semidirect products with applications to continuum theories *Adv. Math.* **137** 1–81

[22] Kukavica I 1992 On the behavior of the solutions of the Kuramoto–Sivashinsky equations for negative time *J. Math. Anal. Appl.* **166** 601–6

[23] Richards I 1982 On the gaps between numbers that are sums of two squares *Adv. Math.* **46** 1–2

[24] Vukadinovic J 2002 On the backwards behavior of the solutions of the 2D periodic viscous Camassa–Holm equations *J. Dyn. Diff. Eqns* **14**