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NONLINEAR STABILITY OF STATIONARY PLASMAS—AN EXTENSION OF THE ENERGY-CASIMIR METHOD*

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Abstract. We describe the time evolution of a nonrelativistic, collisionless plasma by the Vlasov–Poisson system. In [G. Rein, *Math. Methods Appl. Sci.*, 17 (1994), pp. 1129–1140], the energy-Casimir method was used to prove nonlinear stability of steady states where the phase-space density of the particles is a decreasing function of the particle energy. In the present paper we extend this method to steady states with phase-space density depending on additional invariants of the particle motion. The existence of such steady states is established as well.

Key words. Vlasov–Poisson system, nonlinear stability, stationary plasmas, energy-Casimir method

AMS subject classifications. 82D10, 76X05, 35B35

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1. Introduction. The time evolution of a nonrelativistic, collisionless plasma consisting of ions and electrons can be modeled by the Vlasov–Poisson system

$$\partial_t f^\pm + v\cdot\partial_x f^\pm \mp \partial_x U\cdot\partial_v f^\pm = 0,$$
$$\triangle U = -4\,\pi\,(\rho^+ - \rho^-),\ \ \rho^\pm(t,x) = \int f^\pm(t,x,v)\,dv;\ \ t\geq 0,\ \ x\in\Omega,\ \ v\in\mathbb{R}^3.$$

Here $f^\pm = f^\pm(t,x,v)\geq 0$ denotes the phase-space density of the positively charged ions and of the electrons, respectively, $t\geq 0$ the time, $x\in\Omega$ the spatial coordinate, and $v\in\mathbb{R}^3$ the velocity coordinate; for the moment Ω can be either $\mathbb{R}^3$ or a bounded domain in $\mathbb{R}^3$. The particles interact only by the electrostatic potential $U = U(t,x)$ which they create collectively, and $\rho^\pm = \rho^\pm(t,x)$ denotes the spatial charge density of the ions and electrons, respectively. All physical constants are set equal to one for simplicity. To obtain a well-posed initial value problem, boundary conditions at spatial infinity or at $\partial\Omega$ will have to be specified as well.

The present paper is concerned with stationary solutions of this system and in particular with their stability properties. We shall later distinguish the following two cases: Either $\Omega = \mathbb{R}^3$, the ions are given by a fixed, time-independent density ρ^+, and U and f^- vanish at spatial infinity, or $\Omega\subset\mathbb{R}^3$ is a bounded domain with smooth boundary, the particles are specularly reflected at the boundary $\partial\Omega$, and U vanishes on $\partial\Omega$. The precise statements are found in sections 2 and 3, respectively; for the purpose of this introduction we need not distinguish the two cases. In the first case we can alternatively think of the potential generated by ρ^+ as the potential of a given, external force field which keeps the electrons in a steady state distribution. Note that on $\mathbb{R}^3$ and without an external force there exist no nontrivial steady states with finite charge where both ions and electrons can move; see [10].

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In the following, quantities referring to a steady state are denoted with a subscript 0. Since U_0 is time independent, the energy

$$E_0^{\pm}(x,v) = \frac{1}{2}\,v^2 \pm U_0(x)$$

of an ion or electron with coordinates $(x,v) \in \Omega \times \mathbb{R}^3$ is constant along the particle trajectory, which is given by

$$\dot{x} = v, \quad \dot{v} = \mp \partial_x U_0,$$

the characteristic system of the corresponding Vlasov equation. Assume that $I = I(x,v)$ is an additional invariant of the particle motion; the examples we have in mind are $I(x,v) = F(x,v) := |x \times v|^2$, the square of the modulus of angular momentum, which is a conserved quantity if U_0 is spherically symmetric; and $I(x,v) = F_3(x,v) := x_1\,v_2 - x_2\,v_1$, the third component of angular momentum, which is a conserved quantity if U_0 is axially symmetric with respect to the x_3-axis. Note that in both cases I does not depend on the particle species but only on the particle coordinates. The ansatz

$$f_0^{\pm}(x,v) = \varphi^{\pm}(E^{\pm}(x,v))\,\psi^{\pm}(I(x,v)),\ (x,v) \in \Omega \times \mathbb{R}^3, \tag{1.1}$$

automatically satisfies the Vlasov equation and reduces the Vlasov–Poisson system to a semilinear elliptic equation for the potential U_0, which is obtained by inserting the ansatz into the definition of $\rho^{\pm}$. In [18] it has been shown that if—up to some technical assumptions—$\psi = 1$ and $\varphi^{\pm}$ are both monotonically decreasing, then the corresponding steady state is nonlinearly stable. The method of proof is as follows: First we note that the total energy

$$H(f) := \frac{1}{2}\int_{\Omega\times\mathbb{R}^3} v^2\,(f^+ + f^-)(x,v)\,dv\,dx + \frac{1}{8\,\pi}\int_{\Omega} |\partial_x U_{\rho_f}(x)|^2\,dx$$

is conserved along solutions of the Vlasov–Poisson system; here U_{ρ_f} denotes the potential which is induced by the spatial charge density ρ_f corresponding to $f = (f^+, f^-)$. We then construct an additional conserved quantity C, the so-called Casimir functional, in such a way that the steady state (f_0^+, f_0^-) is a critical point of the energy-Casimir functional $H_C := H + C$; i.e., the first variation of H_C vanishes at (f_0^+, f_0^-). If the second variation of H_C at (f_0^+, f_0^-) is positive definite, this defines a norm on an appropriately chosen state space, with respect to which we obtain nonlinear stability of the steady state (f_0^+, f_0^-). More background on this method can be found in [8] and [18].

It turns out that the above procedure can also be used if—as in (1.1)—(f_0^+, f_0^-) depend on an additional invariant I of the particle motion. We show that such a steady state is stable provided $\varphi^{\pm}$ are both decreasing functions of the particle energy. The energy-Casimir method is thus extended to more general steady states. On the other hand, the price to pay is a restriction on the admissible perturbations of the steady state: The additional invariant I must also be an invariant for the time-dependent problem in order that the Casimir functional we need to make the first variation of H_C vanish at the steady state is actually a conserved quantity along solutions of the Vlasov–Poisson system. For the two examples we mentioned above, F or F_3, this restricts our stability result to spherically symmetric or axially symmetric perturbations. While it is worth noticing that this restricted stability property of the steady state is not affected by the way in which the steady state depends on I, it would also be interesting to know whether such steady states are stable with respect to not spherically symmetric or not axially symmetric perturbations respectively.

Our paper proceeds as follows: In the next section we carry out the stability analysis for the case $\Omega = \mathbb{R}^3$ and a fixed ion background. In section 3 the same is done for a bounded domain $\Omega \subset \mathbb{R}^3$ with smooth boundary and specularly reflecting boundary conditions. Here a technical difficulty enters: As opposed to the case in section 2, there is no existence result—not even locally in time—for classical solutions to the corresponding initial value problem, and we have to work with weak solutions for which a global-in-time existence result is available; see [21]. The results of both sections work for quite general steady states: up to mild technical assumptions we only need that $(\varphi^{\pm})' < 0$ on the support of $\varphi^{\pm}$. In section 4 we establish the existence of steady states which satisfy the assumptions of our stability results.

We conclude this introduction with a brief review of the relevant literature. Global existence of classical solutions to the corresponding initial value problem on $\mathbb{R}^3$ was established in [14]; see also [9, 12, 19]. A corresponding result for weak solutions in the case of a bounded domain $\Omega \subset \mathbb{R}^3$ and reflecting boundary conditions was proven in [21]. We should also mention that for the much more difficult case of the Vlasov–Maxwell system, global weak solutions on a bounded domain with reflecting boundary conditions were established in [4]. Stationary plasmas are constructed for example in [17] and [3, Kap. 4, Abschn. 4.2]. In addition to [18], the stability of stationary plasmas is investigated in [1, 2, 5, 11]. First results which rigorously show that steady states are nonlinearly unstable if the monotonicity of $\varphi^{\pm}$ is violated sufficiently strongly are given in [6, 7]. We also mention the diploma thesis [20] of the third author, where the stability results reported in the present paper were obtained in a more special situation.

2. The case of a plasma on the whole space with fixed ion background. In this section $\Omega = \mathbb{R}^3$, the ions are described by a fixed ion background with density $\rho^+ \in C^1_c(\mathbb{R}^3)^+$, and $f := f^-$ denotes the electron density on the phase space; $C^1_c(\mathbb{R}^3)^+$ denotes the set of nonnegative, compactly supported C^1 functions on $\mathbb{R}^3$. The system then becomes

$$\partial_t f + v \cdot \partial_x f + \partial_x U \cdot \partial_v f = 0,$$
$$\triangle U = -4\,\pi\,(\rho^+ - \rho),$$
$$\rho(t,x) := \int f(t,x,v)\,dv,\ \ t \geq 0,\ \ x, v \in \mathbb{R}^3,$$

which in the following is denoted by $(\mathrm{VP})_\infty$. The reason for this terminology is that if the ratio of the ion mass to the electron mass is sent to infinity—the case of heavy ions—then the solutions of the system stated in the introduction converge to solutions of $(\mathrm{VP})_\infty$; see [16]. As is shown in [19], for every initial datum $\mathring{f} \in C^1_c(\mathbb{R}^6)^+$ there exists a unique classical solution of $(\mathrm{VP})_\infty$, and this solution has the property that $f(t) \in C^1_c(\mathbb{R}^6)^+$ for $t \geq 0$. In the present section we are always working with this classical solution. In particular, if we denote by $Z(s,t,z) = (X,V)(s,t,x,v)$ the solution of the characteristic system

$$\dot{x} = v,\ \ \dot{v} = \partial_x U(s,x), \tag{2.1}$$

with $Z(t,t,z) = z$, $t \geq 0$, $z = (x,v) \in \mathbb{R}^6$, then

$$f(t,x,v) = \mathring{f}((X,V)(0,t,x,v)). \tag{2.2}$$

Let

$$I : \mathbb{R}^6 \to \mathbb{R},\ (x, v) \mapsto I(x, v)$$

be continuously differentiable and such that $N := I^{-1}(\{0\})$ has Lebesgue measure zero. Clearly, $I = F$ and $I = F_3$, which were defined in the introduction, have these properties. For $\varphi,\ \psi : \mathbb{R} \to [0, \infty[$ we consider the following assumptions:

(φ1) $\varphi \in C^1(\mathbb{R})$, and there exists a constant $E_{\max} \in \mathbb{R}$ such that $\varphi(s) = 0$ for $s \geq E_{\max}$ and $\varphi'(s) < 0$ for $s < E_{\max}$.

(ψ1) $\psi \in C^1(\mathbb{R})$, and $\psi(s) > 0$ for $s \neq 0$.

The perturbations of our steady state will be taken from the following state space:

$$X := \Big\{ g \in C^1_c(\mathbb{R}^6)^+ \mid g/(\psi \circ I) \in L^1(\mathbb{R}^6), \text{ and for the solution } f \text{ of } (\mathrm{VP})_\infty \text{ with } f(0) = g \text{ the quantity } I \text{ is conserved along solutions of (2.1)} \Big\}. \tag{2.3}$$

The quantities $I = F$ and $I = F_3$ are conserved along solutions of (2.1) provided U is spherically symmetric and axially symmetric with respect to the x_3-axis, respectively. By uniqueness, spherically symmetric or axially symmetric initial data lead to classical solutions of $(\mathrm{VP})_\infty$ with the same symmetry property, provided ρ^+ has the corresponding symmetry property. Our general setup therefore includes the above two special cases. Note also that since I is conserved along solutions of (2.1) and since the characteristic flow is measure preserving, (2.2) implies that the condition $g/(\psi \circ I) \in L^1(\mathbb{R}^6)$ propagates along solutions of $(\mathrm{VP})_\infty$. The following nonlinear stability result is the main result of the present section; here and in the following $\|\cdot\|_p$ denotes the usual L^p-norm, where the integral always extends over the whole domain of the function considered.

THEOREM 1. *Assume that φ and ψ satisfy the conditions (φ1) and (ψ1) respectively, let $\rho^+ \in C^1_c(\mathbb{R}^3)^+$, and let (f_0, U_0) be a stationary solution of $(\mathrm{VP})_\infty$, where f_0 has the form*

$$f_0(x, v) = \varphi\left(\frac{1}{2}v^2 - U_0(x)\right)\, \psi(I(x, v))$$

and $U_0 \in C^2(\mathbb{R}^3)$ with $\lim_{|x|\to\infty} U_0(x) = 0$. Then (f_0, U_0) is nonlinearly stable in the following sense: For every $C_1 > 0$ there exists $C_2 > 0$ such that every solution f of $(\mathrm{VP})_\infty$ with initial condition $\mathring{f} \in X$ and $\mathring{f} \leq C_1$ satisfies the following estimate:

$$\int |f(t, z) - f_0(z)|^2 \frac{dz}{\psi(I(z))} \leq C_2 \left(\int \left(1 + v^2 + \frac{1}{\psi(I(z))}\right) |\mathring{f}(z) - f_0(z)|\, dz + \|\mathring{\rho} - \rho_0\|^2_{6/5} \right),\ t \geq 0.$$

Proof. For $f \in X$ define the kinetic energy

$$E_{\mathrm{kin}}(f) := \frac{1}{2} \int v^2 f(z)\, dz$$

and the potential energy

$$E_{\mathrm{pot}}(f) := \frac{1}{8\pi} \int |\partial_x U_{\rho^+ - \rho_f}(x)|^2\, dx.$$

Here,

$$U_\rho(x) := \int \frac{\rho(y)}{|x-y|}\,dy,\ x \in \mathbb{R}^3$$

denotes the Coulomb potential, which is generated by the charge density $\rho \in C_c^1(\mathbb{R}^3)$ and vanishes at infinity, and ρ_f is the spatial charge density generated by f. It is well known that the total energy

$$H(f) := E_{\mathrm{kin}}(f) + E_{\mathrm{pot}}(f)$$

is conserved along classical solutions of $(\mathrm{VP})_\infty$. Now

$$E_{\mathrm{kin}}(f) = E_{\mathrm{kin}}(f_0) + \frac{1}{2}\int v^2\,(f(z) - f_0(z))\,dz$$

and

$$E_{\mathrm{pot}}(f) = E_{\mathrm{pot}}(f_0) - \int U_0(x)\,(\rho_f - \rho_0)(x)\,dx + \frac{1}{8\,\pi}\int |\partial_x U_{\rho_0} - \partial_x U_{\rho_f}|^2(x)\,dx.$$

Hence

$$H(f) = H(f_0) + \int E_0(z)\,(f-f_0)(z)\,dz + \frac{1}{8\,\pi}\int |\partial_x U_{\rho_0-\rho_f}|^2(x)\,dx,$$

where

$$E_0(z) = \frac{1}{2}v^2 - U_0(x),\ z = (x,v) \in \mathbb{R}^6.$$

We now construct an additional conserved quantity C in such a way that its derivative at f_0 will—up to the sign—equal the linear part in the above expansion of the total energy. To this end, let

$$E_{\min} := \inf\{E_0(z)\,|\,z \in \mathbb{R}^6\} = -\sup\{U_0(x)\,|\,x \in \mathbb{R}^3\}$$

if $E_{\min} < E_{\max}$, i.e., the steady state is nontrivial, which is of course the case of interest, and $E_{\min} := E_{\max} - 1$ otherwise. Let $\varphi_{\max} := \varphi(E_{\min})$. The mapping

$$\varphi:\ [E_{\min}, E_{\max}] \to [0, \varphi_{\max}]$$

is strictly decreasing and onto. Define for $\sigma \neq 0$ and $\tau \in [0, \varphi_{\max}\psi(\sigma)]$ the mapping

$$\Phi(\tau,\sigma) := -\psi(\sigma)\int_0^{\tau/\psi(\sigma)} \varphi^{-1}(s)\,ds. \tag{2.4}$$

Then $\Phi(\cdot,\sigma) \in C^1([0,\varphi_{\max}\psi(\sigma)]) \cap C^2(]\,0,\varphi_{\max}\psi(\sigma)])$,

$$\partial_\tau\Phi(\tau,\sigma) = -\varphi^{-1}\left(\frac{\tau}{\psi(\sigma)}\right),\ \tau \in [0,\varphi_{\max}\psi(\sigma)],$$

and

$$\begin{aligned}
\partial_\tau^2\Phi(\tau,\sigma) &= -\frac{1}{\varphi'(\varphi^{-1}(\tau/\psi(\sigma)))\,\psi(\sigma)} \\
&\geq -\frac{1}{\inf\{\varphi'(s)\,|\,s \in [E_{\min},E_{\max}]\}\,\psi(\sigma)} \\
&=: \frac{c_\varphi}{\psi(\sigma)} \in]0,\infty[,\ \tau \in]0,\varphi_{\max}\psi(\sigma)];
\end{aligned}$$

throughout this proof, constants denoted by c may depend on the steady state under consideration and on the given constant C_1 from the formulation of the theorem. For $\tau > \varphi_{\max}\psi(\sigma)$ the formula

$$\Phi(\tau,\sigma) := -\frac{(\tau-\varphi_{\max}\psi(\sigma))^2}{2\,\varphi'(E_{\min})\,\psi(\sigma)} - E_{\min}\,(\tau-\varphi_{\max}\psi(\sigma)) - \psi(\sigma)\int_0^{\varphi(E_{\min})}\varphi^{-1}(s)\,ds$$

extends Φ to a function $\Phi\in C([0,\infty[\times(\mathbb{R}\setminus\{0\}))$ with $\Phi(\cdot,\sigma)\in C^2(]\,0,\infty\,[)$,

$$|\Phi(\tau,\sigma)| \le c\left(\tau+\frac{\tau^2}{\psi(\sigma)}\right),\ \tau\ge 0, \tag{2.5}$$

$$|\partial_\tau\Phi(\tau,\sigma)| \le c\left(1+\frac{\tau}{\psi(\sigma)}\right),\ \tau\ge 0, \text{ and} \tag{2.6}$$

$$\partial_\tau^2\Phi(\tau,\sigma) \ge \frac{c_\varphi}{\psi(\sigma)},\ \tau>0. \tag{2.7}$$

We define the Casimir functional as

$$C : X\to\mathbb{R},\ f\mapsto \int \Phi(f(z),I(z))\,dz. \tag{2.8}$$

This integral exists for all $f\in X$ since by the estimate (2.5),

$$|C(f)| \le c\,\|f\|_1 + c\|f\|_\infty \int \frac{f(z)}{\psi(I(z))}\,dz < \infty.$$

Because I is an integral of the Vlasov equation, C is conserved along classical solutions of $(\mathrm{VP})_\infty$ with initial value $\mathring{f}\in X$. We conclude that the energy-Casimir functional $H_C := H + C$ is a conserved quantity of $(\mathrm{VP})_\infty$.

Next we show that the quadratic part in the expansion of H_C at the steady state is positive definite. Let $f\in X$. Obviously,

$$\begin{aligned}H_C(f)-H_C(f_0) &= \int [\Phi(f(z),I(z)) - \Phi(f_0(z),I(z)) + E_0(z)\,(f-f_0)(z)]\,dz\\ &\quad + \frac{1}{8\,\pi}\int |\partial_x U_{\rho_0-\rho_f}|^2(x)\,dx\\ &\ge \int_{\mathbb{R}^6\setminus N}\Big[\cdots\Big]\,dz.\end{aligned}$$

Now let $z\in\mathbb{R}^6\setminus N$ with $f_0(z)>0$. Then $E_0(z)\in[E_{\min},E_{\max}[$ where φ is invertible, and thus

$$E_0(z) = \varphi^{-1}(\varphi(E_0(z))) = \varphi^{-1}\left(\frac{f_0(z)}{\psi(I(z))}\right) = -\partial_\tau\Phi(f_0(z),I(z)).$$

Since Φ is continuous,

$$\begin{aligned}\Big[\cdots\Big] &= \Phi(f(z),I(z)) - \Phi(f_0(z),I(z)) - \partial_\tau\Phi(f_0(z),I(z))\,(f-f_0)(z)\\ &= \lim_{\epsilon\to 0+}(\Phi(f(z)+\epsilon,I(z)) - \Phi(f_0(z),I(z))\\ &\qquad - \partial_\tau\Phi(f_0(z),I(z))\,(f(z)+\epsilon-f_0(z))).\end{aligned}$$

For any $\epsilon > 0$ there exists ζ between $f_0(z)$ and $f(z)+\epsilon$ such that by (2.7),

$$\Big(\cdots\Big) = \frac{1}{2}\,\partial_\tau^2\Phi(\zeta, I(z))\,(f(z)+\epsilon-f_0(z))^2 \geq \frac{1}{2}\frac{c_\varphi}{\psi(I(z))}(f(z)+\epsilon-f_0(z))^2.$$

In the limit $\epsilon \to 0+$ we obtain the estimate

$$\Big[\cdots\Big] \geq \frac{1}{2}\frac{c_\varphi}{\psi(I(z))}\,|f(z)-f_0(z)|^2.$$

Now consider $z \in \mathbb{R}^6 \setminus N$ with $f_0(z) = 0$. Then

$$E_0(z) \geq E_{\max} = \varphi^{-1}(0) = -\partial_\tau\Phi(0, I(z)).$$

Because $\Phi(\cdot, I(z)) \in C^1([0,\infty[)$,

$$\begin{aligned}\Big[\cdots\Big] &\geq \Phi(f(z), I(z)) - \Phi(0, I(z)) - \partial_\tau\Phi(0, I(z))\,f(z)\\ &= \lim_{\epsilon\to 0+}(\Phi(f(z)+\epsilon, I(z)) - \Phi(\epsilon, I(z)) - \partial_\tau\Phi(\epsilon, I(z))\,f(z)).\end{aligned}$$

For any $\epsilon > 0$ there exists ζ between ϵ and $f(z)+\epsilon$ such that

$$\Big(\cdots\Big) = \frac{1}{2}\,\partial_\tau^2\Phi(\zeta, I(z))\,f(z)^2 \geq \frac{1}{2}\frac{c_\varphi}{\psi(I(z))}f(z)^2.$$

Taking the limit $\epsilon \to 0+$, we obtain again the estimate

$$\Big[\cdots\Big] \geq \frac{1}{2}\frac{c_\varphi}{\psi(I(z))}\,|f(z)-f_0(z)|^2,$$

and we have shown that

$$H_C(f) - H_C(f_0) \geq \frac{c_\varphi}{2}\int |f(z)-f_0(z)|^2\frac{dz}{\psi(I(z))}; \tag{2.9}$$

recall that by assumption $N = I^{-1}(\{0\})$ has measure zero. It remains to check that H_C is continuous at f_0. Let $f \in X$ with $f \leq C_1$. Then

$$\begin{aligned}|H_C(f) - H_C(f_0)| \leq{}& \frac{1}{2}\int v^2\,|f(z)-f_0(z)|\,dz\\ &+ \int_{\mathbb{R}^6\setminus N}|\Phi(f(z), I(z)) - \Phi(f_0(z), I(z))|\,dz\\ &+ \frac{1}{2}\int |U_{\rho^+-\rho_f}(x)\,(\rho^+-\rho_f)(x) - U_{\rho^+-\rho_0}(x)\,(\rho^+-\rho_0)(x)|\,dx.\end{aligned}$$

The estimate (2.6) implies that

$$\begin{aligned}|\Phi(f(z), I(z)) - \Phi(f_0(z), I(z))| &= |\partial_\tau\Phi(\zeta, I(z))|\,|f(z)-f_0(z)|\\ &\leq c\left(1+\frac{1}{\psi(I(z))}\right)|f(z)-f_0(z)|\end{aligned}$$

for all $z \in \mathbb{R}^6\setminus N$, where ζ lies between 0 and $\max\{\|f_0\|_\infty, \|f\|_\infty\} \leq \max\{\varphi_{\max}, C_1\} \leq c$. Furthermore,

$$\begin{aligned}&U_{\rho^+-\rho_f}\,(\rho^+-\rho_f) - U_{\rho^+-\rho_0}\,(\rho^+-\rho_0)\\ &\qquad = U_{\rho^+-\rho_f}\,(\rho_0-\rho_f) - (U_{\rho^+-\rho_f} - U_{\rho^+-\rho_0})\,(\rho^+-\rho_0),\end{aligned}$$

so that

$$\begin{aligned}
&\int |U_{\rho^+-\rho_f}(x)\,(\rho^+-\rho_f)(x) - U_{\rho^+-\rho_0}(x)\,(\rho^+-\rho_0)(x)|\,dx \\
&\quad \le \int \left(U_{|\rho^+-\rho_f|}(x)\,|\rho_0(x)-\rho_f(x)| + U_{|\rho_0-\rho_f|}(x)\,|\rho^+(x)-\rho_0(x)|\right)\,dx \\
&\quad \le 2\int U_{|\rho^+-\rho_0|}(x)\,|\rho_0(x)-\rho_f(x)|\,dx + \int U_{|\rho_0-\rho_f|}(x)\,|\rho_0(x)-\rho_f(x)|\,dx \\
&\quad \le c\,\|f-f_0\|_1 + c\,\|\rho_f-\rho_0\|_{6/5}^2,
\end{aligned}$$

where we have used Sobolev's inequality [15, p. 31] to estimate the second term. Altogether we have shown that

$$|H_C(f)-H_C(f_0)| \le c\left[\int\left(1+v^2+\frac{1}{\psi(I(z))}\right)|f(z)-f_0(z)|\,dz + \|\rho_f-\rho_0\|_{6/5}^2\right].$$

The claim of the theorem now follows if we combine this estimate with (2.9) and the fact that H_C is a conserved quantity. ◻

Remarks.

1. If ψ is bounded, then we can replace the left-hand side of the stability estimate by $\|f(t)-f_0\|_2^2$.
2. Theorem 1 remains valid if we require only that $\varphi \in C^1(]-\infty, E_{\max}])$ and $\varphi'(s) < 0$ for all $s < E_{\max}$. However, in this case the steady state f_0 need not belong to the state space X.
3. Note that if $\varphi(s)$ does not vanish for large values of s and $\psi\circ I$ is not integrable, then the resulting steady state f_0 has infinite charge. Thus the condition that φ vanishes for large values of the particle energy is rather natural.
4. Our assumption that functions in the state space X are compactly supported can be replaced by an appropriate fall-off condition at infinity; see [9].

3. The case of a plasma on a bounded domain. In this section $\Omega \subset \mathbb{R}^3$ is a bounded domain with smooth boundary $\partial\Omega \in C^{2,\mu}$ for $\mu \in]0,1[$. The system under consideration consists of the equations

$$\partial_t f^\pm + v\cdot\partial_x f^\pm \mp \partial_x U \cdot \partial_v f^\pm = 0, \tag{3.1}$$

$$\triangle U = -4\,\pi\,(\rho^+-\rho^-), \tag{3.2}$$

$$\rho^\pm(t,x) = \int f^\pm(t,x,v)\,dv,\ t\ge 0,\ x\in\Omega,\ v\in\mathbb{R}^3, \tag{3.3}$$

and the boundary conditions of specular reflection

$$f^\pm(t,x,v) = f^\pm(t,x,v-2\,v\cdot n(x)\,n(x)) \tag{3.4}$$

for f and of an ideal conductor

$$\partial_x U(t,x)\times n(x) = 0,\ t\ge 0,\ x\in\partial\Omega,\ v\in\mathbb{R}^3 \tag{3.5}$$

for the electrostatic field; n denotes the outward unit normal vector of $\partial\Omega$. In the following we denote this system by (VP). The main difference of the stability analysis for (VP) from the one in the previous section is that for (VP) there is not even a local existence result for classical or strong solutions, and we have to work within the framework of weak solutions. In order to understand and even formulate our stability result in this case we have to discuss briefly how these weak solutions are obtained: Instead of (VP) we consider the system (VPδ), where in the Poisson equation $\rho = \rho^+ - \rho^-$ is replaced by the regularized density $\rho_\delta := \rho * \omega_\delta$. Here ω_δ denotes the usual mollifier which converges to the Dirac distribution as $\delta \to 0$; we take ω_δ to be spherically symmetric. For $\delta > 0$ and appropriate initial data we obtain unique, global, strong solutions which among other things satisfy conservation of energy and phase-space volume, i.e., for (VPδ) the characteristic system together with the boundary condition of specular reflection induces a measure-preserving, almost everywhere (a.e.) defined flow on $\Omega \times \mathbb{R}^3$. Taking a sequence $\delta_n \to 0$, the corresponding solutions $f_n = f_{\delta_n}$ converge by compactness—after extracting a subsequence—to a weak global solution f of the original system (VP). These weak solutions are not known to be unique or to satisfy the usual conservation laws. For precise statements we refer to [21].

We let I be as in the previous section and let $\varphi^\pm, \psi^\pm : \mathbb{R} \to [0, \infty[$ satisfy the assumptions ($\varphi 1$) and ($\psi 1$) respectively. The state space is defined as follows:

$$\begin{aligned} X := \{g = (g^+, g^-) | g^\pm \in L^1(\Omega \times \mathbb{R}^3) \cap L^\infty(\Omega \times \mathbb{R}^3),\ g^\pm \geq 0 \text{ a.e.},\\ (x,v) \mapsto v^2\, g^\pm(x,v) \text{ and } g^\pm/(\psi^\pm \circ I) \text{ are integrable}\}. \end{aligned} \tag{3.6}$$

In addition, we say that $g \in X \cap C(\Omega \times \mathbb{R}^3)^2$ is *I-symmetric* iff for every solution f of (VPδ) with $\delta > 0$ and $f(0) = g$ the quantity I is preserved by the corresponding characteristic flow.

For any $\mathring{f} \in X \cap C(\Omega \times \mathbb{R}^3)^2$ there exists a weak solution $f = (f^+, f^-)$ of (VP) such that $f^\pm(t) \geq 0$ a.e., $f^\pm(t) \in L^1(\Omega \times \mathbb{R}^3) \cap L^\infty(\Omega \times \mathbb{R}^3)$, and $(x,v) \mapsto v^2\, f^\pm(x,v) \in L^1(\Omega \times \mathbb{R}^3)$ for all $t \geq 0$; see [21]. In fact, the differential equations (3.1) and (3.2) hold in the sense of distributions, the boundary conditions (3.4) and (3.5) are satisfied in a generalized and classical pointwise sense, respectively. However, here we need only to know that for $p, p' \in [1, \infty]$ with $1/p + 1/p' = 1$,

$$f^\pm(t) = \lim_{n\to\infty} f_n^\pm(t) \text{ with respect to } \sigma(L^p(\Omega \times \mathbb{R}^3), L^{p'}(\Omega \times \mathbb{R}^3)),\ t \geq 0.$$

Now define

$$\Omega_R := \{z \in \mathbb{R}^6 \,|\, |z| < R \text{ and } I(z) > 1/R\},\ R > 0.$$

Since I is continuous and $\psi^\pm$ both satisfy ($\psi 1$), $1/(\psi^\pm \circ I) \in L^\infty(\Omega_R)$, and hence, if $\mathring{f}$ is I-symmetric,

$$\int_{\Omega_R} \frac{f^\pm(t,z)}{\psi^\pm(I(z))}\, dz = \lim_{n\to\infty} \int_{\Omega_R} \frac{f_n^\pm(t,z)}{\psi^\pm(I(z))}\, dz \leq \int \frac{(\mathring{f}^\pm)(z)}{\psi^\pm(I(z))}\, dz$$

for all $R > 0$. Thus we see that $f(t)$ remains in X for all $t \geq 0$ provided $\mathring{f} \in X \cap C(\Omega \times \mathbb{R}^3)^2$ is I-symmetric. Note that the continuity of $\mathring{f}$ need not be preserved by weak solutions, which is why we cannot include it in the definition of the state space X. If I equals F or F_3 then every spherically or axially symmetric initial datum $\mathring{f} \in X \cap C(\Omega \times \mathbb{R}^3)$ is I-symmetric, provided Ω has the corresponding symmetry property.

THEOREM 2. *Assume that $\varphi^\pm$ and $\psi^\pm$ satisfy the conditions* ($\varphi 1$) *and* ($\psi 1$). *Let* $(f_0^\pm, U_0)$ *be a stationary solution of* (VP) *where* $f_0^\pm$ *have the form* (1.1) *and* $U_0 \in C^2(\overline{\Omega})$. *Then* $(f_0^\pm, U_0)$ *is nonlinearly stable in the following sense: For every* $C_1 > 0$ *there is a* $C_2 > 0$ *such that for* $\mathring{f} \in X \cap C(\Omega \times \mathbb{R}^3)^2$ *which is* I*-symmetric with* $\mathring{f}^\pm \leq C_1$, *any weak solution of* (VP) *obtained as described above with initial value* $f(0) = \mathring{f}$ *satisfies the following estimate:*

$$\begin{aligned}\int |f^+(t,z) - f_0^+(z)|^2 \frac{dz}{\psi^+(I(z))} &+ \int |f^-(t,z) - f_0^-(z)|^2 \frac{dz}{\psi^-(I(z))} \\ \leq C_2 \Bigg[&\int \left(1 + v^2 + \frac{1}{\psi^+(I(z))}\right) |\mathring{f}^+(z) - f_0^+(z)|\, dz \\ &+ \int \left(1 + v^2 + \frac{1}{\psi^-(I(z))}\right) |\mathring{f}^-(z) - f_0^-(z)|\, dz \\ &+ \|\mathring{\rho}^+ - \rho_0^+\|_{6/5}^2 + \|\mathring{\rho}^- - \rho_0^+\|_{6/5}^2 \Bigg],\ t \geq 0.\end{aligned}$$

Proof. For $f \in X$, we define the total energy

$$H(f) := \frac{1}{2} \int_{\Omega \times \mathbb{R}^3} v^2\, (f^+ + f^-)(z)\, dz + \frac{1}{8\,\pi} \int_\Omega |\partial_x U_{\rho_f}(x)|^2\, dx,$$

where

$$U_\rho(x) := \int_\Omega G(x,y)\, \rho(y)\, dy,\ x \in \Omega$$

is the Coulomb potential generated by $\rho \in L^1(\Omega) \cap L^\infty(\Omega)$, G is Green's function for the Laplace operator on Ω with zero boundary condition, and ρ_f is the spatial charge density generated by $f = (f^+, f^-)$. It is shown in [21, sect. 5, Thm. 5.5] that the aforementioned weak solution f of (VP) with initial value $\mathring{f}$ satisfies the inequality $H(f(t)) \leq H(\mathring{f})$ for all $t \geq 0$. Analogously to (2.4), we let

$$E_{\min}^\pm := \inf\{E_0^\pm(z) \,|\, z \in \Omega \times \mathbb{R}^3\} = \inf\{\pm U_0(x) \,|\, x \in \Omega\},$$

define

$$\Phi^\pm(\tau,\sigma) := -\psi^\pm(\sigma) \int_0^{\tau/\psi^\pm(\sigma)} (\varphi^\pm)^{-1}(s)\, ds \tag{3.7}$$

for $\tau \in [0, \varphi^\pm(E_{\min}^\pm)\, \psi^\pm(\sigma)]$ and $\sigma \neq 0$, and extend $\Phi^\pm$ to functions on $[0,\infty[\times (\mathbb{R} \setminus \{0\})$ such that for fixed $\sigma \neq 0$, $\Phi^\pm(\cdot,\sigma) \in C^1([0,\infty[) \cap C^2(]\,0,\infty[)$, and

$$\partial_\tau^2 \Phi^\pm(\tau,\sigma) \geq \frac{c_{\varphi^\pm}}{\psi^\pm(\sigma)} > 0,\ \tau > 0,\ \sigma \neq 0.$$

As in (2.8), we define

$$C : X \to \mathbb{R} : f \mapsto \int \Phi^+(f^+(z), I(z))\, dz + \int \Phi^-(f^-(z), I(z))\, dz \tag{3.8}$$

and observe that C is well defined. As in [21, sect. 6], we show that C is not increasing along weak solutions f of (VP) with initial value $\mathring{f}$ as assumed in the theorem. In order to do this, fix $t \geq 0$ and note that

$$\begin{aligned} C(f(t)) &= C(f_n(t)) + (C(f(t)) - C(f_n(t))) \\ &= C(\mathring{f}) + \int_{(\Omega\times\mathbb{R}^3)\setminus N} (\Phi^+(f^+(t,z), I(z)) - \Phi^+(f_n^+(t,z), I(z)))\, dz \\ &\quad + \int_{(\Omega\times\mathbb{R}^3)\setminus N} (\Phi^-(f^-(t,z), I(z)) - \Phi^-(f_n^-(t,z), I(z)))\, dz \\ &= C(\mathring{f}) + \int_{(\Omega\times\mathbb{R}^3)\setminus N} \partial_\tau \Phi^+(\zeta^+, I(z))\, (f^+ - f_n^+)(t,z)\, dz \\ &\quad + \int_{(\Omega\times\mathbb{R}^3)\setminus N} \partial_\tau \Phi^-(\zeta^-, I(z))\, (f^- - f_n^-)(t,z)\, dz, \end{aligned}$$

where $\zeta^\pm$ lies between $f^\pm(t,z)$ and $f_n^\pm(t,z)$. Since $\partial_\tau^2 \Phi^\pm > 0$ on $]0,\infty[$,

$$\partial_\tau \Phi^\pm(\zeta^\pm, I(z))\, (f^\pm - f_n^\pm)(t,z) \leq \partial_\tau \Phi^\pm(f^\pm(t,z), I(z))\, (f^\pm - f_n^\pm)(t,z)$$

for all $z \in (\Omega \times \mathbb{R}^3) \setminus N$. Furthermore, the functions $z \mapsto \partial_\tau \Phi^\pm(f^\pm(t,z), I(z))$ are essentially bounded on $\Lambda^\pm := \{z \,|\, f^\pm(t,z) \leq \varphi^\pm(E^\pm_{\min})\, \psi^\pm(I(z))\}$ and equal to

$$z \mapsto -\frac{f^\pm(t,z)}{(\varphi^\pm)'(E^\pm_{\min})\, \psi^\pm(I(z))} + \left(\frac{\varphi^\pm(E^\pm_{\min})}{(\varphi^\pm)'(E^\pm_{\min})} - E^\pm_{\min} \right)$$

on $(\Omega \times \mathbb{R}^3) \setminus (N \cup \Lambda^\pm)$. Since the first term on the right-hand side lies in $L^1(\Omega \times \mathbb{R}^3)$ and the second lies in $L^\infty(\Omega \times \mathbb{R}^3)$, this implies that

$$\lim_{n\to\infty} \int_{(\Omega\times\mathbb{R}^3)\setminus N} \partial_\tau \Phi^\pm(f^\pm(t,z), I(z))\, (f^\pm - f_n^\pm)(t,z)\, dz = 0,$$

and hence $C(f(t)) \leq C(\mathring{f})$. We conclude that the energy-Casimir functional $H_C := H + C$ does not increase along solutions of (VP). The assertion now follows by essentially the same arguments as in the proof of Theorem 1. We need only to observe that conservation of H_C along solutions was not really needed and that the fact that this quantity does not increase along solutions is sufficient. ∎

The first two remarks at the end of the previous section apply here as well. We should also mention that the fact that we only have the inequality $H(f(t)) \leq H(\mathring{f})$ in the proof above does not mean that there is dissipation in the system. The concept of weak solutions is just not good enough to retain conservation of energy.

4. Existence of steady states. In this section we show that steady states which satisfy the assumptions of our stability theorems, but not those of previous such results, do exist. This requires the solution of a semilinear elliptic equation, which is obtained by substituting the ansatz for f into the definition of ρ and the Poisson equation. We can relax the assumptions on φ and ψ somewhat:

(φ2) $\varphi \in L^\infty_{\mathrm{loc}}(\mathbb{R})$ with $E_{\max} := \inf\{s \in \mathbb{R} \,|\, \text{for a.e. } \sigma \geq s : \varphi(\sigma) = 0\} < \infty$, and φ is decreasing.

(ψ2) $\psi \in C^1(\mathbb{R})$, and there exists $\epsilon > 0$ such that $\psi(s) > 0$ for $0 < |s| < \epsilon$.

Throughout this section, let $I = F$ or $I = F_3$. The following lemma will ensure sufficient regularity of the right-hand side of the semilinear elliptic equation which we will need to solve.

LEMMA 1. *Let φ and ψ satisfy conditions (φ2) and (ψ2), respectively. Then the function*

$$h_{\varphi,\psi} : \mathbb{R}^3 \times \mathbb{R} \to [0, \infty[,\ (x,u) \mapsto 4\,\pi \int_{\mathbb{R}^3} \varphi\left(\frac{1}{2}\,v^2 + u\right)\,\psi(I(x,v))\,dv$$

is continuously differentiable, and $h_{\varphi,\psi}(x,\cdot)$ is decreasing for fixed $x \in \mathbb{R}^3$. If $I = F$, then $h_{\varphi,\psi}(\cdot,u)$ is spherically symmetric for fixed $u \in \mathbb{R}$; if $I = F_3$ then $h_{\varphi,\psi}(\cdot,u)$ is axially symmetric.

Proof. First consider the case $I = F$. A simple computation using spherical coordinates shows that

$$h_{\varphi,\psi}(x,u) = 8\sqrt{2}\,\pi^2 \int_u^\infty \varphi(s) \int_0^{s-u} \frac{\psi(2\,|x|^2\,\tau)}{\sqrt{s-u-\tau}}\,d\tau\,ds,$$

and this integral converges. That $h_{\varphi,\psi}$ is continuously differentiable is proved by straightforward applications of Lebesgue's dominated convergence theorem. Now consider the case $I = F_3$. Here an analogous computation using cylindrical coordinates yields that

$$h_{\varphi,\psi}(x,u) = 8\,\pi^2 \int_u^\infty \varphi(s) \int_{-\sqrt{2\,(s-u)}}^{\sqrt{2\,(s-u)}} \psi(r(x)\,\tau)\,d\tau\,ds,$$

where

$$r(x) := \sqrt{x_1^2 + x_2^2}.$$

Again, using Lebesgue's dominated convergence theorem we can show that $h_{\varphi,\psi}$ is continuously differentiable. The rest of the claim is obvious. ∎

First we establish the existence of stationary solutions of $(\mathrm{VP})_\infty$.

THEOREM 3. *Let φ and ψ satisfy the conditions (φ2) and (ψ2), respectively. Assume that $\rho^+ \in C_c^\mu(\mathbb{R}^3)^+$ for $\mu \in]\,0,1\,[$ and*

$$E_{\max} < 0 < E_{\max} + \int \frac{\rho^+(y)}{|y|}\,dy.$$

Then there exists a unique classical solution $U_0 \in C^{2,\mu}_{\mathrm{loc}}(\mathbb{R}^3)$ of the problem

$$\begin{cases} \triangle U = -4\,\pi\,\rho^+ + h_{\varphi,\psi}(x,-U) \text{ on } \mathbb{R}^3, \\ \lim_{|x|\to\infty} U(x) = 0, \end{cases} \tag{4.1}$$

where $h_{\varphi,\psi}$ is defined in Lemma 1. If φ is continuously differentiable, then (φ,ψ) induces a stationary solution (f_0,U_0) of $(\mathrm{VP})_\infty$ such that $f_0 \in X \setminus \{0\}$ with X as defined in (2.3)*, and*

$$f_0(x,v) = \varphi\left(\frac{1}{2}v^2 - U_0(x)\right)\,\psi(I(x,v)),\ (x,v) \in \mathbb{R}^6.$$

Proof. Lemma 1 shows that $h_{\varphi,\psi}(\cdot,-\cdot)$ is continuously differentiable and increasing with respect to the second argument. Since $E_{\max} < 0$ we have $h_{\varphi,\psi}(\cdot,0)=0$, and thus 0 is a subsolution for (4.1). Let

$$\overline{u}(x) := \int_{\mathbb{R}^3} \frac{\rho^+(y)}{|x-y|}\,dy,\ x\in\mathbb{R}^3.$$

Then $\overline{u}\geq 0$, $\lim_{|x|\to\infty}\overline{u}(x)=0$, and

$$\triangle\overline{u} = -4\,\pi\,\rho^+ \leq -4\,\pi\,\rho^+ + h_{\varphi,\psi}(\cdot,-\bar{u}),$$

i.e., $\overline{u}$ is a supersolution for (4.1). It follows from the conditional existence theorem [13, ch. 7, Thm. 5.1] that there is a classical solution $U_0 \in C^{2,\mu}_{\mathrm{loc}}(\mathbb{R}^3)$ of (4.1) so that $0\leq U_0\leq \overline{u}$. Let V be another classical solution of (4.1) and $W := U_0 - V$. Then $W \triangle W \geq 0$ because of the monotonicity of $h_{\varphi,\psi}$. Consider a connected component K of the open set $M := \{x\in\mathbb{R}^3 \mid W(x)>0\}$. Then K is a domain, $\triangle W\geq 0$ in K, and $W=0$ on $\partial K\subset\partial M$. Since $W(x)\to 0$ for $|x|\to\infty$, the maximum principle is applicable and yields $W\leq 0$ in K so that $M=\emptyset$. In the same way we prove that $\{x\in\mathbb{R}^3 \mid W(x)<0\}=\emptyset$. Hence $W=0$, i.e., $U_0=V$.

If $\varphi\in C^1(\mathbb{R})$ and f_0 is defined by our ansatz, then (f_0,U_0) is a classical stationary solution of $(\mathrm{VP})_\infty$. Since $E_{\max}<0$ and $\lim_{|x|\to\infty}U_0(x)=0$, we see that f_0 has compact support. It remains to show that $f_0\neq 0$. We know that $-\overline{u}(0) < E_{\max}$. Using ($\psi$2) and that $I^{-1}(\{0\})$ is of measure zero we find that $h_{\varphi,\psi}(\cdot,-\overline{u})\neq 0$. Hence $\overline{u}$ itself is not the solution of (4.1) and thus $h_{\varphi,\psi}(\cdot,-U_0)\neq 0$. This shows that $f_0\neq 0$, and the proof is complete. □

Remarks.

1. The fact that φ is decreasing is merely used to show that U_0 is unique; we do not need this condition for the existence proof.
2. The uniqueness of U_0 immediately implies that U_0 is spherically symmetric or axially symmetric provided ρ^+ and $h_{\varphi,\psi}(\cdot,u)$ have this property.
3. For $\psi=1$ the existence result for U_0 is a supplement to [1, Thm. 6.1]. If ρ^+ is not spherically symmetric, it provides us with examples of nonsymmetric stationary solutions of $(\mathrm{VP})_\infty$ for which the stability results of [1, sect. 5] and [18, sect. 2] are applicable. However, if f_0 does depend on F or F_3, then these results are not applicable.

We now turn to the case of (VP) on a bounded domain $\Omega\subset\mathbb{R}^3$. As far as the existence of steady states is concerned, the situation is simpler in this case.

THEOREM 4. *Let $\varphi^\pm$ and $\psi^\pm$ satisfy the conditions (φ2) and (ψ2), respectively. Assume that $\partial\Omega\in C^{2,\mu}$ for $\mu\in]0,1[$. Then there exists a unique classical solution $U_0\in C^{2,\mu}(\overline{\Omega})$ of the problem*

$$\left\{\begin{array}{l}\triangle U = h_{\varphi^-,\psi^-}(\cdot,-U) - h_{\varphi^+,\psi^+}(\cdot,U)\ \text{on}\ \Omega,\\ U=0\ \text{on}\ \partial\Omega,\end{array}\right. \tag{4.2}$$

where $h_{\varphi^\pm,\psi^\pm}$ are defined in Lemma 1. *The pairs $(\varphi^\pm,\psi^\pm)$ induce a stationary solution $(f_0^\pm,U_0)$ of* (VP) *such that $f_0^\pm$ are in $X^\pm$, where $X^\pm$ are defined via* (3.6), *and $f_0^\pm$ have the form* (1.1). *The solution is nontrivial, i.e., U_0 is not identically zero, provided*

$$\int_0^\infty \varphi_+(s)\int_{-\sqrt{2s}}^{\sqrt{2s}}\psi_+(r(x)\tau)\,d\tau\,ds \neq \int_0^\infty \varphi_-(s)\int_{-\sqrt{2s}}^{\sqrt{2s}}\psi_-(r(x)\tau)\,d\tau\,ds$$

for some $x \in \Omega$, *where* $r(x) = \sqrt{x_1^2 + x_2^2}$.

Proof. Lemma 1 shows that $h_{\varphi^-,\psi^-}(\cdot, -\cdot) - h_{\varphi^+,\psi^+}$ is continuously differentiable and increasing with respect to the second argument. It is well known that this suffices for the existence and uniqueness of a classical solution $U_0 \in C^{2,\mu}(\overline{\Omega})$ of (4.2). If $f_0^\pm$ are defined as in (1.1), then $(f_0^\pm, U_0)$ is a stationary solution of $(\mathrm{VP})_\infty$. Here we have to observe that $E_0^\pm$ as well as F and F_3 satisfy the boundary condition (3.4) provided Ω is spherically and axially symmetric, respectively. Condition (φ1) implies that $f_0^\pm \in X^\pm$. That U_0 cannot vanish identically under the additional assumption on $(\varphi_\pm, \psi_\pm)$ is obvious from the formula for $h_{\varphi_\pm,\psi_\pm}$ obtained in Lemma 1. □

Remark. As far as the symmetry of U_0 is concerned, the second remark after Theorem 3 applies here as well, provided Ω has the desired symmetry property.

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