MIDTERM
MATH 70200

- You can answer any FOUR of these problems. If you submit five solutions, I'll grade only the first four.
- Lebesgue measure is denoted by m.
- You can use any result from chapters 1 and 2 of the textbook, but not homework exercises.

Problem 1. Let $(X, \mathcal{M})$ be a measurable space. Show that the characteristic function $\chi_E \colon X \to \mathbb{R}$ is measurable if and only if $E \in \mathcal{M}$.

Problem 2. Let $\{f_n\}$ be a sequence of measurable functions on measure space $(X, \mathcal{M}, \mu)$ and suppose that $f_n \to f$ pointwise. Suppose that $\int |f_n| \, d\mu \leq 1$ for all n.

(a) Show that $\int |f| \, d\mu \leq 1$.
(b) Give an example of such a sequence $\{f_n\}$ on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}}, m)$ where $f_n \not\to f$ in L^1.
(c) Suppose that $|f_n| \leq |f|$ for all n. Show that $f_n \to f$ in L^1.

Problem 3. Let μ be a measure on X. Let $E_1, E_2, \ldots$ be measurable sets such that $\sum_{n=1}^{\infty} \mu(E_n) < \infty$. Let

$$F = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} E_n.$$

Prove that $\mu(F) = 0$.

Problem 4. Let $f \colon \mathbb{R} \to [0, \infty]$ be a Borel measurable function, and for $t \in \mathbb{R}$ define $f_t \colon \mathbb{R} \to [0, \infty]$ by $f_t(x) = f(t + x)$. (You do not need to show that f_t is measurable.) Show that for any t,

$$\int f \, dm = \int f_t \, dm,$$

where m is Lebesgue measure on $\mathbb{R}$.

Problem 5. Let m^* denote Lebesgue outer measure on $\mathbb{R}$, and let E be a subset of $\mathbb{R}$, not necessarily Lebesgue measurable, with $m^*(E) < \infty$. Let $E_n = E \cap [-n, n]$. Prove that $m^*(E_n) \to m^*(E)$ as $n \to \infty$.

Date: March 19, 2024.